General Case

$$* : \Lambda^p \longrightarrow \Lambda^{n-p}$$

$$\{\sigma^i\} \text{ orthonormal}$$

$$\omega = \sigma^1 \wedge \cdots \wedge \sigma^n$$

Assume wlog that $\sigma^I = \sigma^1 \wedge \cdots \wedge \sigma^p$

$$\sigma^I \wedge * \sigma^I = g(\sigma^I, \sigma^I)\,\omega$$

$$= g(\sigma^1, \sigma^1) \cdots g(\sigma^p, \sigma^p)\,\omega$$

$$\Rightarrow \boxed{* \sigma^I = g(\sigma^I, \sigma^I)\,\sigma^J}$$

where $\sigma^J = \sigma^{p+1} \wedge \cdots \wedge \sigma^n$

can do any σ^I by using suitable permutations

$$\sigma^J \wedge * \sigma^J = g(\sigma^J, \sigma^J)\,\omega$$

$$= g(\sigma^J, \sigma^J)\,\sigma^I \wedge \sigma^J$$

$$= g(\sigma^J, \sigma^J)\,\sigma^J \wedge \sigma^I\,(-1)^{p(n-p)}$$

$$\Rightarrow * \sigma^J = (-1)^{p(n-p)} g(\sigma^J, \sigma^J)\,\sigma^I$$

$$\Rightarrow ** \sigma^I = g(\sigma^I, \sigma^I) * \sigma^J$$

$$= (-1)^{p(n-p)} g(\sigma^I, \sigma^I)\, g(\sigma^J, \sigma^J)\,\sigma^I$$

$$= (-1)^{p(n-p)} g(\omega, \omega)\,\sigma^I$$

$$= (-1)^{p(n-p)} (-1)^s\,\sigma^I$$

$$\therefore \boxed{** = (-1)^{p(n-p)+s}}$$

Dot Product

$$\alpha \wedge *\beta = g(\alpha,\beta)\,\omega$$

$$\Rightarrow *(\alpha \wedge *\beta) = g(\alpha,\beta) *\omega$$
$$= (-1)^s g(\alpha,\beta)$$

$$\therefore \boxed{g(\alpha,\beta) = (-1)^s *(\alpha \wedge *\beta)}$$

any dimension!
any signature!
any rank!

"$\alpha \cdot \beta = (-1)^s *(\alpha \wedge *\beta)$"

Cross Product

$$\alpha, \beta \in \Lambda^1 \Rightarrow \alpha \wedge \beta \in \Lambda^2$$

<u>if</u> $n=3$: $*(\alpha\wedge\beta) \in \Lambda^1$

"$\alpha \times \beta = *(\alpha \wedge \beta)$"

<u>Pseudovectors</u>: $x^i \longmapsto -x^i$

$\Rightarrow dx^i \longmapsto -dx^i$

$\Rightarrow \vec{v} \longmapsto -\vec{v}$

but $\vec{v}\times\vec{w} \longmapsto +\vec{v}\times\vec{w}$

<u>vector</u> $\leftrightarrow$ 1-form

<u>pseudovector</u> $\leftrightarrow$ 2-form

similarly: $f \longmapsto f$ <u>scalar</u> $\leftrightarrow$ 0-form

$dV \longmapsto -dV$ <u>pseudoscalar</u> $\leftrightarrow$ 3-form