Hodge dual

Idea: in $\mathbb{R}^3$ we have

$$dx \xleftrightarrow{\;*\;} dy \wedge dz$$
$$dy \xleftrightarrow{\;*\;} dz \wedge dx$$
$$dz \xleftrightarrow{\;*\;} dx \wedge dy$$
$$1 \xleftrightarrow{\;*\;} dx \wedge dy \wedge dz = \omega$$

In each case above,

$$\alpha \wedge *\alpha = \omega$$

Not quite good enough —
want linearity, so need
to include scale

$$\therefore \qquad \boxed{\alpha \wedge *\alpha = g(\alpha, \alpha)\,\omega}$$

Polarize:

$$(\alpha + \beta) \wedge *(\alpha + \beta) = g(\alpha + \beta, \alpha + \beta)\,\omega$$
$$\|$$
$$\alpha \wedge *\alpha + \cancel{\beta \wedge *\beta} = \cancel{g(\alpha, \alpha)\omega} + \cancel{g(\beta, \beta)\omega}$$
$$+ \alpha \wedge *\beta + \beta \wedge *\alpha \qquad\qquad + 2\,g(\alpha, \beta)\omega$$

symmetric!
(must check) ↖ symmetric

$$\therefore \qquad \boxed{\alpha \wedge *\beta = g(\alpha, \beta)\,\omega}$$

definition of Hodge dual

Example : $\mathbb{R}^2$

$$dx \wedge *dx = dx \wedge dy$$

$$\Longrightarrow \boxed{*dx = dy}$$

why not $*dx = dy + f dx$?

because $dy \wedge *dx = 0$

$$dy \wedge *dy = dx \wedge dy = -dy \wedge dx$$

$$\Longrightarrow \boxed{*dy = -dx}$$

$$1 \wedge *1 = dx \wedge dy$$

$$\Longrightarrow \boxed{*1 = dx \wedge dy}$$

in general: $*1 = \omega$

$$(dx \wedge dy) \wedge *(dx \wedge dy) = dx \wedge dy$$

$$\Longrightarrow \boxed{*(dx \wedge dy) = 1}$$

Example : $\mathbb{M}^2$

$$\omega = dx \wedge dt$$

$$dx \wedge *dx = g(dx, dx) \omega = dx \wedge dt$$

$$\Rightarrow \boxed{*dx = dt}$$

$$dt \wedge *dt = g(dt, dt) \omega$$
$$= - dx \wedge dt = dt \wedge dx$$

$$\Rightarrow \boxed{*dt = dx}$$

$$1 \wedge *1 = dx \wedge dt$$

$$\Rightarrow \boxed{*1 = dx \wedge dt}$$

in general: $*1 = \omega$

$$(dx \wedge dt) \wedge *(dx \wedge dt) = g(dx \wedge dt, dx \wedge dt) \omega$$
$$= g(dx, dx) \, g(dt, dt) \, \omega$$
$$= - dx \wedge dt$$

$$\Rightarrow \boxed{*(dx \wedge dy) = -1}$$

in general: $*\omega = g(\omega, \omega)$
$$= (-1)^s$$

Example : Polar Courds

$$\omega = dr \wedge r\, d\Phi$$

$$dr \wedge * dr =$$

$$\Rightarrow \boxed{* dr = r\, d\Phi}$$

$$r\, d\Phi \wedge * r\, d\Phi = dr \wedge r\, d\Phi = - r\, d\Phi \wedge dr$$

$$\Rightarrow \boxed{* r\, d\Phi = - dr}$$

$$1 \wedge * 1 = dr \wedge r\, d\Phi$$

$$\Rightarrow \boxed{* 1 = dr \wedge r\, d\Phi}$$

$$(dr \wedge r\, d\Phi) \wedge * (dr \wedge r\, d\Phi) = dr \wedge r\, d\Phi$$

$$\Rightarrow \boxed{* (dr \wedge r\, d\Phi) = 1}$$