Clifford Algebras

I Pauli Matrices

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Properties: Hermitian: $\sigma_p^\dagger = \sigma_p = \bar{\sigma}^T$

linearly indpt: over ~~$\mathbb{C}$~~ $\mathbb{R}$

$\sigma_p^2 = \mathbb{1}$ $\qquad$ $\det \sigma_p = -1$

$\operatorname{tr} \sigma_p = 0$

$$\boxed{\sigma_x \sigma_y = i\sigma_z} = -\sigma_y \sigma_x$$

Clifford identity

$$\{\sigma_p, \sigma_q\} = \sigma_p \sigma_q + \sigma_q \sigma_p = 2\,\delta_{pq}$$

$$[\sigma_p, \sigma_q] = \sigma_p \sigma_q - \sigma_q \sigma_p = \begin{cases} 2i\,\sigma_r & (p,q,r \text{ cyclic}) \\ 0 & (p = q) \end{cases}$$

$$x\sigma_x + y\sigma_y + z\sigma_z = \begin{pmatrix} z & x-iy \\ x+iy & -z \end{pmatrix} = P$$

$$|P| = -(x^2 + y^2 + z^2)$$

4-vector

$$Q = P + t\,\mathbb{1} = \begin{pmatrix} t+z & x-iy \\ x+iy & t-z \end{pmatrix}$$

$$|Q| = -(x^2 + y^2 + z^2 - t^2)$$

$$\mathbb{1} = \sigma_t$$

(II) Quaternions

$$\sigma_x \sigma_y = i \sigma_z$$

$$\underbrace{(-i\sigma_x)}_{=I}\underbrace{(-i\sigma_y)}_{=J} = \underbrace{(-i\sigma_z)}_{=K}$$

$$IJ = K, \quad + \text{cyclic}$$
$$= -JI$$

$$I^2 = (-i\sigma_x)^2 = -\mathbb{1}$$

$$\mathbb{H} = \langle \mathbb{1}, I, J, K \rangle$$

III Differential Forms

$\mathbb{1}$ $\sigma_x, \sigma_y, \sigma_z$

antisymmetric

$$I, J, K \quad\quad i\mathbb{1}$$

dy∧dx

$$I = -i\sigma_x = -\sigma_y\sigma_z = \sigma_z\sigma_y$$

$$-\sigma_x\sigma_y\sigma_z$$

scalar — vector — pseudov — pseudo s

$$\Lambda^0(\mathbb{R}^3) - \Lambda^1(\mathbb{R}^3) - \Lambda^2(\mathbb{R}^3) - \Lambda^3(\mathbb{R}^3)$$

$$\boxed{\begin{array}{ccc} dx & \longleftrightarrow & \sigma_x \\ dy & \longleftrightarrow & \sigma_y \\ dz & \longleftrightarrow & \sigma_z \end{array}}$$

IV Clifford Algebras

any signature

Vector space V with inner product g

$$Cl(V) = \bigcup \underline{\Lambda^p(V)}$$

formal wedge products of vectors

$$\simeq \Lambda^p(\mathbb{R}^n) \quad \left(\begin{array}{c}\text{coeffs} \\ \text{are const}\end{array}\right)$$

union of all (diff) forms!

$$\boxed{dx + dy \wedge dz}$$

$$\leftrightarrow \sigma_x + \sigma_y \sigma_z$$

$$= \sigma_x + i\,\sigma_x$$

Define a new product!

$$(\sigma_x)(\sigma_y) \leftrightarrow dx \wedge dy \quad ✓$$

$$(\sigma_x)(\sigma_x) \leftrightarrow g(dx, dx) \quad ✓$$

$dx \vee dy$

"vee"

wee

"clifford"

$u, w \in V$

$$\boxed{u \vee w := u \wedge w + g(u, w)}$$

def on 1 forms

extend by associative

$$\Rightarrow \boxed{Cl(\mathbb{R}^3) \simeq 2\times 2 \text{ complex matrices}}$$

(V) Minkowski Space

Dirac Equation

$$\gamma_t = \left(\begin{array}{c|c} 0 & \mathbb{1} \\ \hline -\mathbb{1} & 0 \end{array} \right)$$

$Cl(\mathbb{R}^{3,1}) \not\cong Cl(\mathbb{R}^{1,3})$

$$\gamma_p = \left(\begin{array}{c|c} 0 & \sigma_p \\ \hline \mp\sigma_p & 0 \end{array} \right)$$

$+ + + -$?

$- - - +$?

$$\gamma_p \gamma_q + \gamma_q \gamma_p = 2 g_{ij}$$

$\mathbb{H} \leftrightarrow \langle 1, i, j, k \rangle$

$\mathrm{Im}\,\mathbb{H} \ni \vec{v}$

$$\vec{v}\,\vec{w} = -\underline{\vec{v}\cdot\vec{w}} + \underline{\vec{v}\times\vec{w}}$$

$\mathbb{H}$