Coordinates:

In[1]:=
```
x = (R + ρ Cos[ψ]) Cos[ϕ];
y = (R + ρ Cos[ψ]) Sin[ϕ];
z = ρ Sin[ψ];
```

Position vector:

In[4]:=
```
p = {x, y, z};
```

Frame:

In[5]:= $e_\rho = \text{Simplify}\left[\frac{\text{D[p, }\rho\text{]}}{\sqrt{\text{D[p, }\rho\text{].D[p, }\rho\text{]}}}\right]$

Out[5]= $\{\cos(\psi)\cos(\phi), \cos(\psi)\sin(\phi), \sin(\psi)\}$

In[6]:= $e_\phi = \text{PowerExpand}\left[\text{Simplify}\left[\frac{\text{D[p, }\phi\text{]}}{\sqrt{\text{D[p, }\phi\text{].D[p, }\phi\text{]}}}\right]\right]$

Out[6]= $\{-\sin(\phi), \cos(\phi), 0\}$

In[7]:= $e_\psi = \text{PowerExpand}\left[\text{Simplify}\left[\frac{\text{D[p, }\psi\text{]}}{\sqrt{\text{D[p, }\psi\text{].D[p, }\psi\text{]}}}\right]\right]$

Out[7]= $\{\sin(\psi)(-\cos(\phi)), \sin(\psi)(-\sin(\phi)), \cos(\psi)\}$

Check handedness:

In[8]:= $\text{Cross}[e_\rho, e_\phi] == e_\psi$

Out[8]= True

Dual basis (via arclength):

In[9]:=
```
Rrule = {Dt[R] → 0};
```

In[10]:=
```
Collect[FullSimplify[Dt[x]^2 + Dt[y]^2 + Dt[z]^2] /. Rrule, {Dt[ρ], Dt[ϕ], Dt[ψ]}]
```

Out[10]= $\rho^2 (d\psi)^2 + (d\rho)^2 + (d\phi)^2 (\rho\cos(\psi) + R)^2$

Dual basis (via attitude matrix):

In[11]:= $A = \{e_\rho, e_\phi, e_\psi\}$

Out[11]= $$\begin{pmatrix} \cos(\phi)\cos(\psi) & \cos(\psi)\sin(\phi) & \sin(\psi) \\ -\sin(\phi) & \cos(\phi) & 0 \\ -\cos(\phi)\sin(\psi) & -\sin(\phi)\sin(\psi) & \cos(\psi) \end{pmatrix}$$

In[12]:=
```
Simplify[A.Transpose[A]] == IdentityMatrix[3]
```

Out[12]= True

In[13]:=
```
dξ = {{dx}, {dy}, {dz}}
```

Out[13]= $$\begin{pmatrix} dx \\ dy \\ dz \end{pmatrix}$$

In[14]:=
```
Evaluate[Table[{σm}, {m, 1, 3}]] =
  Simplify[A.dξ /. {dx → Dt[x], dy → Dt[y], dz → Dt[z]}] /. Rrule
```

Out[14]=
$$\begin{pmatrix} d\rho \\ (R+\rho\cos(\psi))\, d\phi \\ \rho\, d\psi \end{pmatrix}$$

Connection 1-forms (via attitude matrix):

In[15]:=
```
W = Simplify[Dt[A].Transpose[A] /. Rrule]
```

Out[15]=
$$\begin{pmatrix} 0 & \cos(\psi)\, d\phi & d\psi \\ -\cos(\psi)\, d\phi & 0 & d\phi \sin(\psi) \\ -d\psi & -d\phi \sin(\psi) & 0 \end{pmatrix}$$

Connection 1-forms (direct computation):

In[16]:=
```
tor = {ρ, ϕ, ψ};
```

In[17]:=
```
frame = Table[e_tor〚m〛, {m, 1, 3}];
```

In[18]:=
```
Simplify[
  Table[Sum[D[frame〚m〛, tor〚k〛] Dt[tor〚k〛], {k, 1, 3}].frame〚n〛, {m, 1, 3}, {n, 1, 3}]]
```

Out[18]=
$$\begin{pmatrix} 0 & \cos(\psi)\, d\phi & d\psi \\ -\cos(\psi)\, d\phi & 0 & d\phi \sin(\psi) \\ -d\psi & -d\phi \sin(\psi) & 0 \end{pmatrix}$$