Forms on M

An <u>adapted frame</u> on M is:

$\hat{e}_1, \hat{e}_2 \in TM$ (frame $\Rightarrow$ orthonormal)

$\hat{e}_3 = \hat{e}_1 \times \hat{e}_2 =: \hat{n}$

<u>Ex</u>: plane: $\{\hat{x}, \hat{y}, \hat{z}\}$

cylinder: $\{\hat{\theta}, \hat{z}, \hat{r}\}$ ← note order!

sphere: $\{\hat{\theta}, \hat{\phi}, \hat{r}\}$

$\Rightarrow \nabla_{\vec{v}} \hat{e}_j = \omega_{ji} \hat{e}_i$

$\Rightarrow S(\vec{v}) = -\omega_{3i}(\vec{v})\hat{e}_i = +\omega_{i3}(\vec{v})\hat{e}_i$

$\Rightarrow S = \begin{pmatrix} \omega_{13}(\hat{e}_1) & \omega_{23}(\hat{e}_1) \\ \omega_{13}(\hat{e}_2) & \omega_{23}(\hat{e}_2) \end{pmatrix}$ in this basis

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Forms

$\vec{v} \in TM \Rightarrow \vec{v} = v_i \hat{e}_i$

Dual basis <u>in $\mathbb{R}^3$</u>:

$\sigma_i(\hat{e}_j) = \delta_{ij}$

$\Rightarrow \sigma_3(\vec{v}) = 0 \Rightarrow \sigma_3 = 0$ "on" M!

<u>Idea</u>: $M = \{g = \text{const}\}$

$\Rightarrow \hat{n} = \dfrac{\vec{\nabla} g}{|\vec{\nabla} g|} \Rightarrow \sigma_3 = \lambda\, dg = 0$ for $g = \text{const}$

<u>Formally</u>: The <u>pullback</u> of σ_3 to M is zero

$\therefore$ <u>on M</u>: $\{\hat{e}_1, \hat{e}_2\}$ frame

$\{\sigma_1, \sigma_2\}$ dual basis

$\omega_{12}, \omega_{13}, \omega_{23}$ connection (with σ_3 set to 0)

"rotation" of frame (ω_{12})

shape operator (ω_{13}, ω_{23})

Structure equations

In $\mathbb{R}^3$: $d\sigma_i = \omega_{ij} \wedge \sigma_j$

$d\omega_{ij} = \omega_{ik} \wedge \omega_{kj}$

$\therefore$ on M: $d\sigma_1 = \omega_{12} \wedge \sigma_2 + 0$

$d\sigma_2 = -\omega_{12} \wedge \sigma_1 + 0$

$0 = d\sigma_3 = \omega_{31} \wedge \sigma_1 + \omega_{32} \wedge \sigma_2$ Symmetry of S!

$d\omega_{12} = \omega_{13} \wedge \omega_{32} = -\omega_{13} \wedge \omega_{23}$ Gauss

$d\omega_{13} = \omega_{12} \wedge \omega_{23}$

$d\omega_{23} = \omega_{21} \wedge \omega_{13} = -\omega_{12} \wedge \omega_{13}$

$\Big\}$ Codazzi

Curvature

1-forms: $\sigma_i(\hat{e}_j) = \delta_{ij}$

2-forms: $\sigma_i \wedge \sigma_j \,(\hat{e}_p, \hat{e}_q) = \delta_{ip}\delta_{jq} - \delta_{iq}\delta_{jp}$

$$\Rightarrow \boxed{\alpha \wedge \beta\,(\vec{v}, \vec{w}) = \begin{vmatrix} \alpha(\vec{v}) & \beta(\vec{v}) \\ \alpha(\vec{w}) & \beta(\vec{w}) \end{vmatrix}}$$

Recall: $S = \begin{pmatrix} \omega_{13}(\hat{e}_1) & \omega_{23}(\hat{e}_1) \\ \omega_{13}(\hat{e}_2) & \omega_{23}(\hat{e}_2) \end{pmatrix}$

$$\therefore K = \det S = \omega_{13}(\hat{e}_1)\omega_{23}(\hat{e}_2) - \omega_{23}(\hat{e}_1)\omega_{13}(\hat{e}_2)$$

$$= \omega_{13} \wedge \omega_{23}\,(\hat{e}_1, \hat{e}_2)$$

$//$

$K\,\sigma_1 \wedge \sigma_2\,(\hat{e}_1, \hat{e}_2)$

$$\Longrightarrow \boxed{\omega_{13} \wedge \omega_{23} = K\,\sigma_1 \wedge \sigma_2}$$

Similarly,

$$2H = \omega_{13}(\hat{e}_1) + \omega_{23}(\hat{e}_2)$$

$$= \omega_{13} \wedge \sigma_2\,(\hat{e}_1, \hat{e}_2) + \sigma_1 \wedge \omega_{23}\,(\hat{e}_1, \hat{e}_2)$$

$$\Rightarrow \boxed{\omega_{13} \wedge \sigma_2 + \sigma_1 \wedge \omega_{23} = 2H\,\sigma_1 \wedge \sigma_2}$$

Recall: $d\omega_{12} = -\omega_{13} \wedge \omega_{23}$

$$\Rightarrow \boxed{d\omega_{12} = -K\,\sigma_1 \wedge \sigma_2}$$

$\Rightarrow$ Gaussian curvature is intrinsic !!

Theorema Egregium

(Gauss)