MTH 434/534 **HW #7** Winter 2011

INTEGRATION ON THE SPHERE

1. Working in rectangular coordinates, choose a particular 1-form β in $\mathbb{R}^3$.

(a) Compute $\alpha = d\beta$. (If $\alpha = 0$, start over.)

(b) Evaluate α on the unit sphere $\mathbb{S}^2$, on which $r = 1$.
HINT: First express α in spherical coordinates. Use what you know!

(c) Evaluate β on the unit sphere $\mathbb{S}^2$.

(d) Recompute $\alpha = d\beta$.

(e) Did you get the same answer?

*The result of "evaluating" a differential form on a surface is called the **pullback** of the form to the surface. You have just investigated whether the operations of exterior differentiation (the d operator) and pullback commute.*

2. Now consider the 2-form α you constructed above, evaluated on $\mathbb{S}^2$.

(a) Show that $\int_{\mathbb{S}^2} \alpha = 0$.

(b) Try to repeat the calculation from part (a) knowing only that $\alpha = d\beta$ but *without* knowing explicitly what β is.
You should actually compute the integral if possible. What coordinates should you use?

(c) What does (the differential form version of) Stokes' Theorem say about the integrals you just evaluated?

3. Consider now the volume element of the unit sphere.

(a) The standard orientation on the unit sphere is $\omega = \sin\theta\, d\theta \wedge d\phi$. Determine $\int_{\mathbb{S}^2} \omega$.

(b) It is easy to see that $\omega = d(-\cos\theta\, d\phi)$. Doesn't the previous problem imply that $\int_{\mathbb{S}^2} \omega = 0$? Explain.