INNER PRODUCTS

1. Let $\vec{\boldsymbol{v}} = v^1 \hat{\boldsymbol{i}} + v^2 \hat{\boldsymbol{j}}$ and $\vec{\boldsymbol{w}} = w^1 \hat{\boldsymbol{i}} + w^2 \hat{\boldsymbol{j}}$ be ordinary vectors in \mathbb{R}^2 . Consider the inner product defined by

$$\vec{\boldsymbol{v}}\star\vec{\boldsymbol{w}} = \frac{1}{2}\left(v^1w^2 + v^2w^1\right)$$

- (a) Compute $\vec{v} \star \vec{v}$. If you only knew this (for all \vec{v}), could you compute $\vec{v} \star \vec{w}$?
- (b) Find a vector $\vec{x} \in \mathbb{R}^2$ such that $\vec{x} \star \vec{x} = 1$
- (c) Find a vector $\vec{\boldsymbol{y}} \in \mathbb{R}^2$ such that $\vec{\boldsymbol{y}} \star \vec{\boldsymbol{y}} = -1$
- (d) How many independent vectors $\vec{z} \in \mathbb{R}^2$ are there satisfying $\vec{z} \star \vec{z} = 0$?
- 2. Now work in \mathbb{R}^3 , and suppose there is an inner product such that

$$\vec{v} \star \vec{v} = v^1 v^2 + v^2 v^3 + v^3 v^1$$

- (a) Determine $\vec{\boldsymbol{v}} \star \vec{\boldsymbol{w}}$.
- (b) Find a basis of \mathbb{R}^3 such that each basis vector $\vec{\boldsymbol{u}}$ satisfies either $\vec{\boldsymbol{u}} \star \vec{\boldsymbol{u}} = 1$ or $\vec{\boldsymbol{u}} \star \vec{\boldsymbol{u}} = -1$, and any two basis vectors $\vec{\boldsymbol{u}}$, $\vec{\boldsymbol{v}}$ satisfy $\vec{\boldsymbol{u}} \star \vec{\boldsymbol{v}} = 0$.

EXTRA CREDIT:

Using the above results, or otherwise, rewrite xy as the sum or difference of exactly two squares, and xy + yz + zx as the sum or difference of exactly three squares.

 ${\it HINT: Your previous answers should help.}$