

1. HODGE DUAL IN MINKOWSKI SPACE

4-dimensional Minkowski space has an orthonormal, oriented basis of 1-forms given by

$$\{dx, dy, dz, dt\}$$

with $g(dt, dt) = -1$, $g(dx, dx) = g(dy, dy) = g(dz, dz) = 1$, and all others zero. The “volume element” (choice of orientation) is given by $\omega = dx \wedge dy \wedge dz \wedge dt$.

- (a) Determine the Hodge dual operator $*$ on all forms by computing its action on basis forms at each rank.
- (b) How does your answer change if the opposite orientation is chosen, namely $\omega = dt \wedge dx \wedge dy \wedge dz$?

2. SPHERICAL COORDINATES

Consider spherical coordinates in 3-dimensional Euclidean space with the usual orientation, namely $\omega = r^2 \sin \theta dr \wedge d\theta \wedge d\phi$.

- (a) Determine the Hodge dual operator $*$ on all forms (expressed in spherical coordinates) by computing its action on basis forms at each rank.
- (b) Compute the dot and cross products of 2 arbitrary “vector fields” (really 1-forms) in spherical coordinates using the expressions:

$$\alpha \cdot \beta = *(\alpha \wedge *\beta)$$

$$\alpha \times \beta = *(\alpha \wedge \beta)$$

You may express your results either with respect to an orthonormal basis or with respect to a “coordinate” (non-orthonormal) spherical basis; make sure you know which you’re doing.