INNER PRODUCTS

1. Let $\vec{\boldsymbol{v}} = v^1\,\hat{\boldsymbol{\imath}} + v^2\,\hat{\boldsymbol{\jmath}}$ and $\vec{\boldsymbol{w}} = w^1\,\hat{\boldsymbol{\imath}} + w^2\,\hat{\boldsymbol{\jmath}}$ be ordinary vectors in $\mathbb{R}^2$. Consider the inner product defined by

$$\vec{\boldsymbol{v}} \star \vec{\boldsymbol{w}} = \frac{1}{2}\left(v^1w^2 + v^2w^1\right)$$

(a) Compute $\vec{\boldsymbol{v}} \star \vec{\boldsymbol{v}}$. If you only knew this (for all $\vec{\boldsymbol{v}}$), could you computer $\vec{\boldsymbol{v}} \star \vec{\boldsymbol{w}}$?

(b) Find a vector $\vec{\boldsymbol{x}} \in \mathbb{R}^2$ such that $\vec{\boldsymbol{x}} \star \vec{\boldsymbol{x}} = 1$

(c) Find a vector $\vec{\boldsymbol{y}} \in \mathbb{R}^2$ such that $\vec{\boldsymbol{y}} \star \vec{\boldsymbol{y}} = -1$

(d) How many independent vectors $\vec{\boldsymbol{z}} \in \mathbb{R}^2$ are there satisfying $\vec{\boldsymbol{z}} \star \vec{\boldsymbol{z}} = 0$?

2. Now work in $\mathbb{R}^3$, and suppose there is an inner product such that

$$\vec{\boldsymbol{v}} \star \vec{\boldsymbol{v}} = v^1v^2 + v^2v^3 + v^3v^1$$

(a) Determine $\vec{\boldsymbol{v}} \star \vec{\boldsymbol{w}}$.

(b) Find a basis of $\mathbb{R}^3$ such that each basis vector $\vec{\boldsymbol{u}}$ satisfies either $\vec{\boldsymbol{u}} \star \vec{\boldsymbol{u}} = 1$ or $\vec{\boldsymbol{u}} \star \vec{\boldsymbol{u}} = -1$, and any two basis vectors $\vec{\boldsymbol{u}}$, $\vec{\boldsymbol{v}}$ satisfy $\vec{\boldsymbol{u}} \star \vec{\boldsymbol{v}} = 0$.

EXTRA CREDIT:

Using the above results, or otherwise, rewrite xy as the sum or difference of exactly two squares, and $xy + yz + zx$ as the sum or difference of exactly three squares.

HINT: Your previous answers should help.