MTH 420 – HW #4

Due on Monday, 1 March 1999

1. **DIFFERENTIAL EQUATIONS**

Consider the following system of differential equations:

$$\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z} = P \qquad \frac{\partial f}{\partial z} - \frac{\partial h}{\partial x} = Q \qquad \frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} = R$$

where f, g, h are functions of the variables x, y, z.

(a) What condition must be imposed on P, Q, R for there to exist solutions f, g, h?

(b) Suppose that $P = 2yz$, $Q = 2xz$, $R = 2xy$. Find *a* solution f, g, h.

(c) With P, Q, R as above, find the *general* solution f, g, h.

HINT: Express the problem in terms of differential forms, that is, combine f, g, h into a differential form of appropriate rank, and do the same for P, Q, R.

2. **INTEGRATION ON THE SPHERE**

(a) Let β be any 1-form in $\mathbb{R}^3$, and let $\alpha = d\beta$. Show by actually doing the integration that

$$\int_{\mathbb{S}^2} \alpha = 0$$

where $\mathbb{S}^2$ denotes the unit sphere.

If you are unable to do this for a generic β, at least work out an explicit example.

(b) The volume element on the unit sphere is $\omega = \sin\theta \, d\theta \wedge d\phi$ and clearly $\omega = d(-\cos\theta \, d\phi)$. Why doesn't the integral above imply that the "volume" (really surface area) of the sphere must be zero, that is, why isn't $\int_{\mathbb{S}^2} \omega$ equal to 0?

*HINT: This is trickier than it looks! It would make sense to work in spherical coordinates. Furthermore, since $r = 1$ on the sphere, we also have $dr = 0$ when doing the integration. Thus, it is safe to assume that the "dr" term in β vanishes, and further that the remaining coefficients do not depend on r. In short, you are assuming that β is a 1-form **on the sphere**.*

***HOWEVER:** Care must be taken that β is well-defined! In particular, "$d\theta$" is not well-defined at the poles (the limit is direction-dependent), and "$d\phi$" actually blows up at the poles! For a rigorous solution, you would have to verify that your choice of β is well-defined at the poles, and the easiest way to do this is to write it in rectangular coordinates.*

*SHORTCUT: It turns out, however, that $\sin\theta \, d\phi$ is no worse than $d\theta$. (You are encouraged to verify this by expressing them in a rectangular basis; see me for help.) Furthermore, this behavior is in fact sufficient to do the integral. It is thus sufficient to use this **orthonormal** basis on the unit sphere when defining β, and not worry further about rigor.*