

MTH 420 – HW #2

Due on Wednesday, 3 February 1999

1. HODGE DUAL IN MINKOWSKI SPACE

4-dimensional Minkowski space has an orthonormal, oriented basis of 1-forms given by

$$\{dx, dy, dz, dt\}$$

i.e. $g(dx, dx) = g(dy, dy) = g(dz, dz) = 1$, $g(dt, dt) = -1$, all others zero; “volume element” (choice of orientation) given by $\omega = dx \wedge dy \wedge dz \wedge dt$.

- (a) Determine the Hodge dual operator $*$ on all forms by computing its action on basis forms at each rank.
- (b) How does your answer change if the opposite orientation is chosen, namely $\omega = dt \wedge dx \wedge dy \wedge dz$?

2. SPHERICAL COORDINATES

Consider spherical coordinates in 3-dimensional Euclidean space with the usual orientation, namely $\omega = r^2 \sin \theta dr \wedge d\theta \wedge d\phi$.

- (a) Determine the Hodge dual operator $*$ on all forms (expressed in spherical coordinates) by computing its action on basis forms at each rank.
- (b) Compute the dot and cross products of 2 arbitrary “vector fields” (really 1-forms) in spherical coordinates using the expressions:

$$\alpha \cdot \beta = *(\alpha \wedge *\beta)$$

$$\alpha \times \beta = *(\alpha \wedge \beta)$$

You may express your results either with respect to an orthonormal basis or with respect to a “coordinate” (non-orthonormal) spherical basis; make sure you know which you’re doing. You can check your answer in many places, such as physics textbooks on electrodynamics, but be warned that these are almost always with respect to an orthonormal basis, and that this is not always stated explicitly!