SAMPLE FINAL QUESTION
MTH 420

1. **CURVATURE OF A CURVE**

Let C be a curve in (Euclidean) $\mathbb{R}^3$. Either using the heuristic argument

$$ds^2 = dx^2 + dy^2 + dz^2 \quad\Longrightarrow\quad ds = \frac{dx}{ds}\,dx + \frac{dy}{ds}\,dy + \frac{dz}{ds}\,dz$$

where s is arc length, or by noting that the unit tangent vector $\vec{\boldsymbol{T}}$ to C satisfies

$$\vec{\boldsymbol{T}} \cdot d\vec{\boldsymbol{r}} = \vec{\boldsymbol{T}} \cdot \vec{\boldsymbol{T}}\,ds = ds$$

we see that it is natural to define $T = ds$ to be the unit 1-form tangent to the curve C. In practice, it is common to parameterize the curve using an arbitrary parameter t, not necessarily arc length. In this case, the s derivatives are replaced using chain rule, and we have in general

$$T = \frac{\frac{dx}{dt}\,dx + \frac{dy}{dt}\,dy + \frac{dz}{dt}\,dz}{\sqrt{\left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}}$$

FACT: If $dT \neq 0$, there is a unique positive function κ and a unique unit 1-form N such that N is orthogonal to T and

$$dT = \kappa\, T \wedge N$$

(If $dT = 0$ we define $\kappa = 0$ and N is not defined.) We call κ the *curvature* of C.

(a) Find the curvature of a circle of radius R.

(b) Find the curvature of the x-axis.

(c) Find the curvature of any other curve of your choice. *You are encouraged to discuss your choice with me; overly simple curves may not receive full credit.*