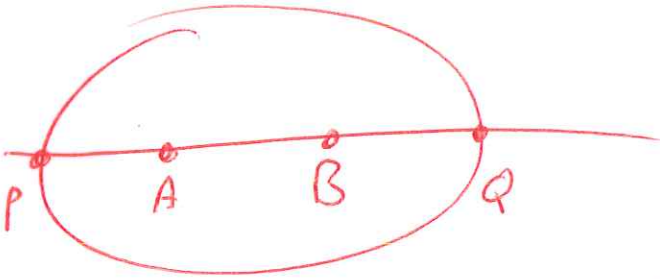
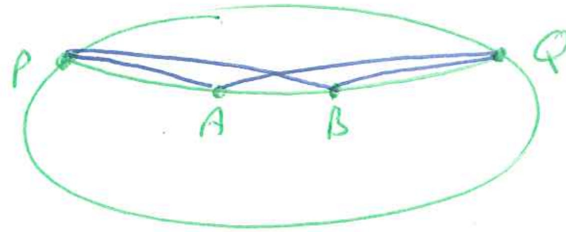
④ <u>distance</u>





<u>Def</u>: $\boxed{d(A,B) = \left| \ln\left(\frac{AQ}{BQ} \cdot \frac{BP}{AP} \right) \right|}$

<u>Case 1</u>: $A = B \implies d(A,B) = |\ln 1| = 0$

<u>Case 2</u>: $A = P \implies d(A,B) = |\ln \infty| = \infty$

<u>Case 3</u>: $A = Q \implies d(A,B) = |\ln 0| = |-\infty| = \infty$

Furthermore, $d(B,A) = \left| \ln\left(\frac{BQ}{AQ} \cdot \frac{AP}{BP} \right) \right|$

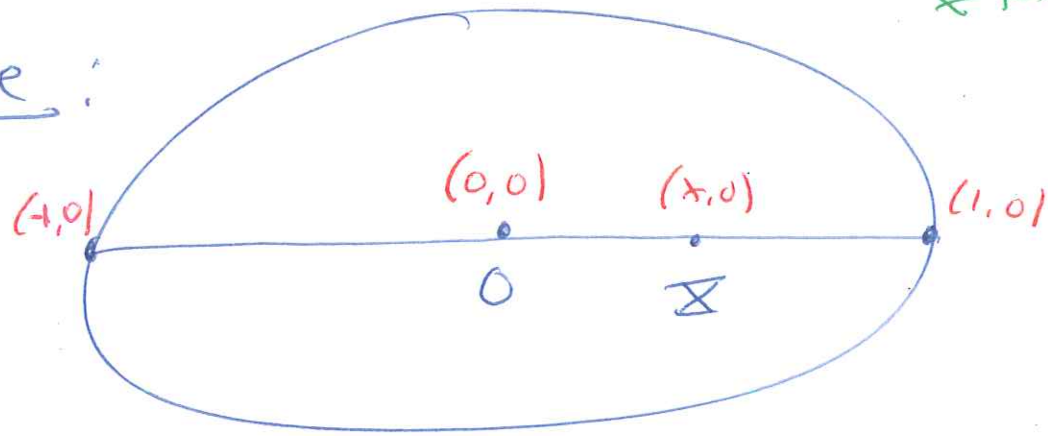
$$= \left| \ln\left(\frac{1}{\frac{AQ}{BQ} \cdot \frac{BP}{AP}} \right) \right|$$

$$= \left| - \ln\left(\frac{AQ}{BQ} \cdot \frac{BP}{AP} \right) \right|$$

$$= |-d(A,B)| = d(A,B) \checkmark$$

book uses K for Euclidean
x for hyperbolic

Example:



$$d(O,X) = \left|\ln\left(\frac{1}{1-x}\cdot\frac{1+x}{1}\right)\right|$$

$$= \left|\ln\left(\frac{1+x}{1-x}\right)\right|$$

We therefore assign X the (signed) hyperbolic coordinate

$$\boxed{K = \ln \frac{1+x}{1-x}}$$

← backwards from text (~~typo in Fig~~ 6.6.5)

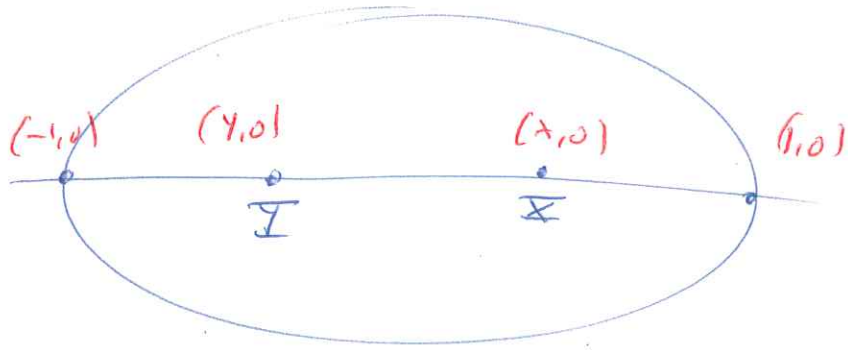
$$\Rightarrow e^K = \frac{1+x}{1-x}$$

$$\Rightarrow e^K(1-x) = 1+x$$

$$\Rightarrow \boxed{x = \frac{e^K-1}{e^K+1}}$$

1-1 correspondence

Now, if



Y has hyperbolic coordinate

$$\boxed{l = \ln \frac{1+y}{1-y}}$$

then $$|k - l| = \left| \ln \frac{1+x}{1-x} - \ln \frac{1+y}{1-y} \right|$$

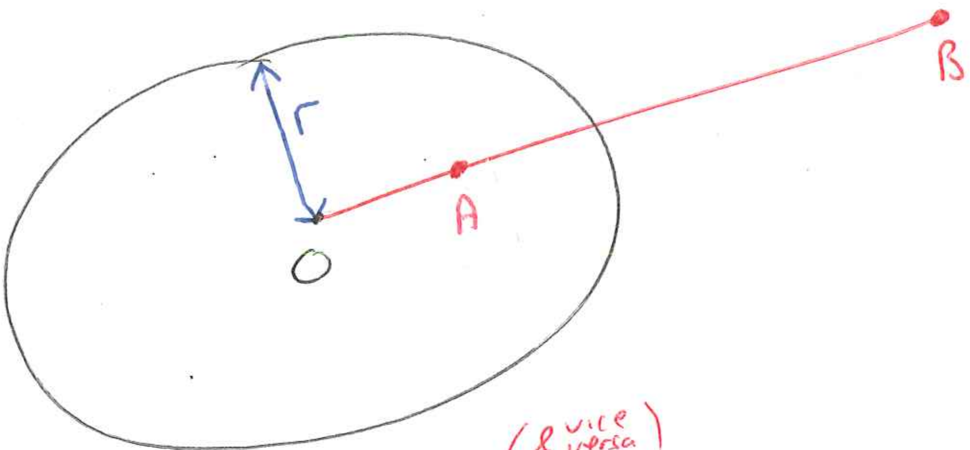
$$= \left| \ln \left(\frac{1+x}{1-x} \cdot \frac{1-y}{1+y} \right) \right|$$

But $$d(X, Y) = d(Y, X)$$

$$= \left| \ln \left(\frac{1-y}{1-x} \cdot \frac{1+x}{1+y} \right) \right|$$

distance = difference in hyperbolic coordinate

Def: Inversion



A is obtained from B (& vice versa) by <u>inversion</u> through the circle at O with radius r if

$$\boxed{(OA)\ (OB) = r^2}$$

<u>Eg</u>: if $O = (0,0)$

& $A = (x,y)$

then $B = \left(\dfrac{xr^2}{x^2+y^2}, \dfrac{yr^2}{x^2+y^2} \right)$

Fact (See §5.5 - good essay topic?)

Any circle containing both A & it's inverse B is invariant under the inversion

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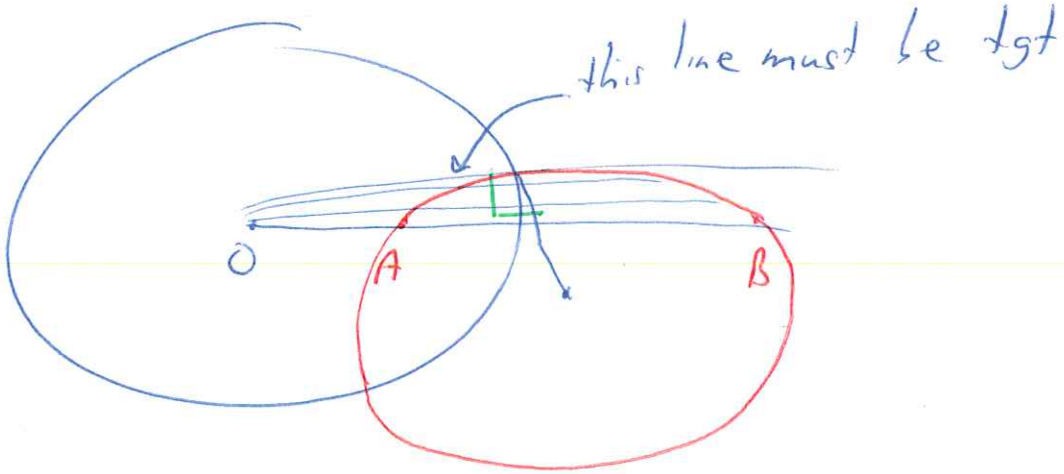
Corollary

Such circles are $\perp$ the circle of inversion

Idea:



because inverses lie on radii from O

$\Rightarrow$ tgt line yields point which = inverse

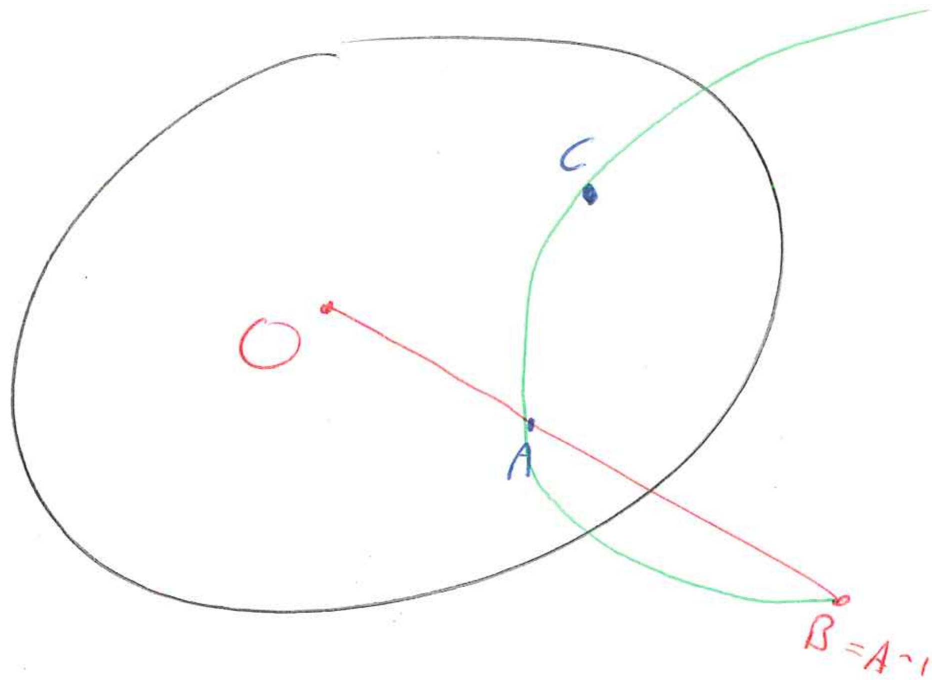
$\Rightarrow$ this pt also on blue circle

i.e. = intersection pt

$\Rightarrow$ radii $\perp$

$\Rightarrow$ circles $\perp$

Application: 2 pts determine a line in Poincaré disk



Given A & C, first construct A^{-1}, then draw circle determined by A, A^{-1}, C

(if C on $\overleftrightarrow{OA}$ then we're done)

Fact: $\exists!$ Euclidean circle through 3 distinct pts

Does Poincaré disk satisfy neutral axioms?

Incidence : Yes – use Euclidean geometry to construct unique diameter or $\perp$ arc

Ruler : Yes – cf example

Plane Separation : Yes – since true for Euclidean geometry

Protracter : Yes – since angles Euclidean

SAS : Yes – tricky – roughly lab 2

$\therefore$ hyperbolic geometry consistent
$\Leftrightarrow$ Euclidean geometry consistent!

(& parallel postulate indpt of neutral postulates)