# Defects

**Recall**: angle sum for any hyperbolic triangle is $< 180^\circ$ $(S < 180)$

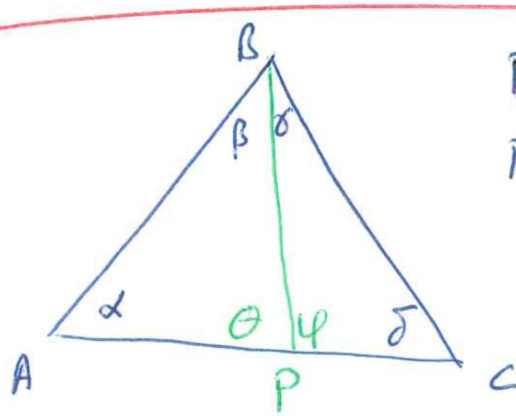
**Def**: defect of a triangle is how much less

$$(D = 180 - S)$$

Excess: (elliptic) $E = S - 180 = -D$

**Thm**: $\boxed{\text{defects are additive}}$

e.g.



$$D(\triangle ABP) = 180 - (\alpha + \beta + \theta)$$
$$D(\triangle BCP) = 180 - (\gamma + \delta + \varphi)$$
$$360 - (\alpha + \beta + \gamma + \delta + \underset{180}{\cancel{\theta + \varphi}})$$
$$= 180 - (\alpha + \beta + \gamma + \delta)$$
$$= D(\triangle ABC)$$

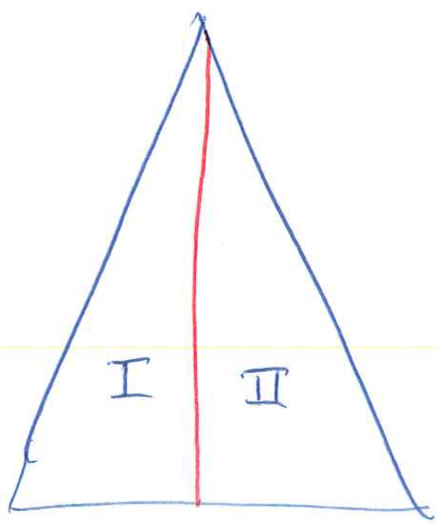
in particular, $D(\triangle ABC) > D(\triangle ABP)$, so not all defects are equal.

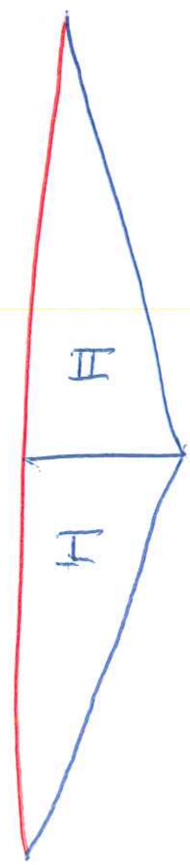
True for any (convex) polygon: defect is sum of defects of <u>any</u> partition into triangles

# Equivalence

**Def:** Two polygons are equivalent $\Longleftrightarrow$ they can be partitioned into congruent sets of triangles

**Idea:** equivalent $\Longleftrightarrow$ same "area"



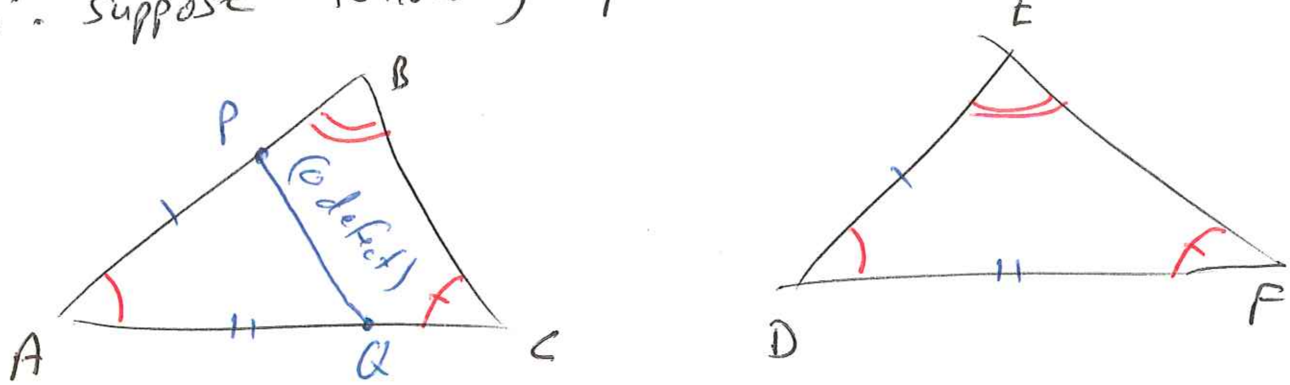



Thm: (AAA) In hyperbolic geometry, 2 triangles are congruent if their angles are =

(no similar triangles!)

Pf: Suppose not. $\Rightarrow$ no corresponding sides = (else ASA)

$\therefore$ suppose following picture:




with $AB > DE$ & $AC > DF$. Construct

$P$ & $Q$ : $AP = DE$ & $AQ = DF$

$\Longrightarrow \triangle APQ = \triangle DEF$

$\Rightarrow D(\triangle APQ) = D(\triangle DEF) \equiv D(\triangle ABC)$

But $D(\triangle APQ) < D(\triangle ABC)$ $\boxed{*}$

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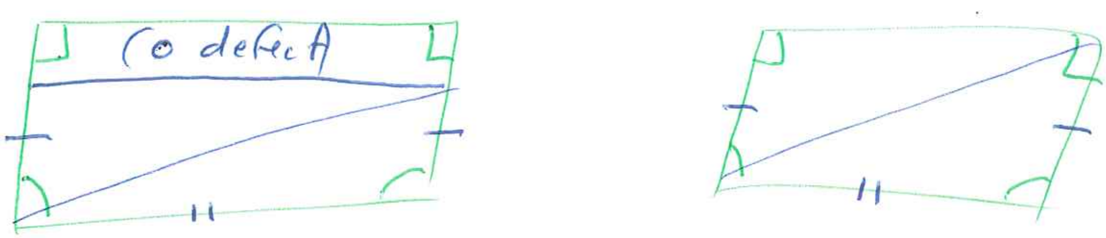
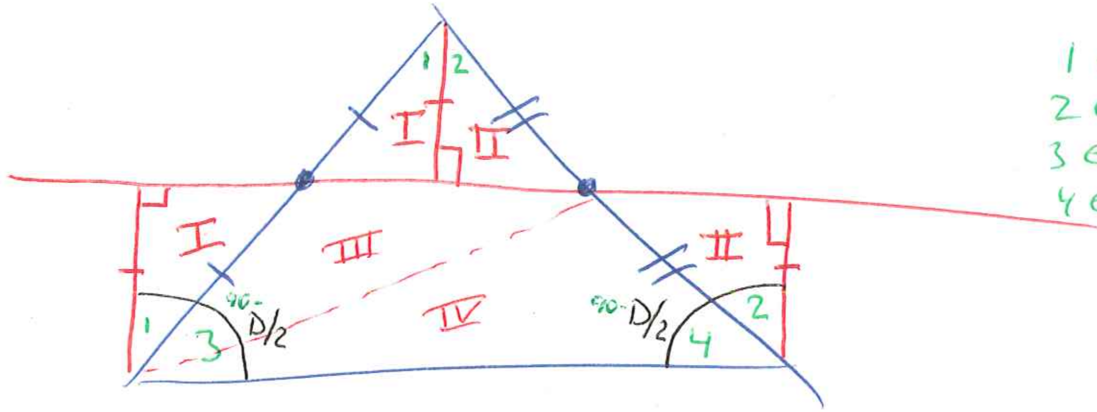
Similarly, Saccheri quadrilaterals congruent if same angles + "summit": (Pf ~~wrong~~ different in book? p. 277/8) (upside down)



# More Interesting Example

Thm 6.4.6 — pf ~~wrong~~ incomplete!




$1 \longleftrightarrow \alpha$
$2 \longleftrightarrow \beta$
$3 \longleftrightarrow \gamma$
$4 \longleftrightarrow \delta$

~~need AAS~~ yes!

$$D(\Delta) = 180 - (\alpha + \beta + \gamma + \delta) =: D$$

$$D(\square) = 360 - (90 + 90 + \alpha + \gamma + \beta + \delta) \equiv D$$

But summit angles of Saccheri $\square$ =

$$\Rightarrow \alpha + \gamma = 90 - D/2 = \delta + \beta$$

(depends only on defect!)

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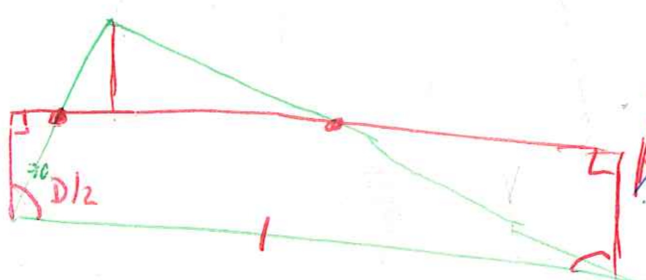
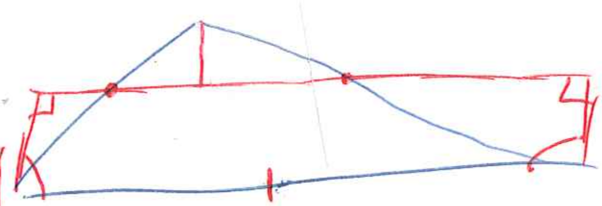
$\Rightarrow$ Thm: Two triangles with same defect & 1 pair of congruent sides are equivalent

Pf: Corresponding Saccheri quadrilaterals have same angles & summit!

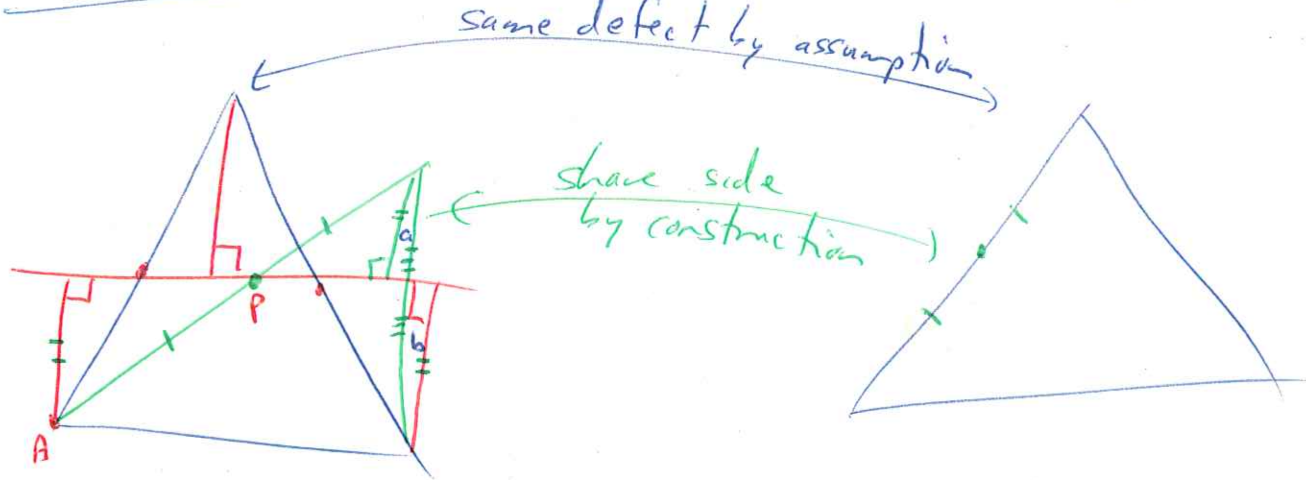


**Thm:** triangles are equivalent $\iff$ same defect

**Pf:** $\Rightarrow$: add defects of partitions

$\Leftarrow$: Construct $3^{rd}$ $\Delta$ with same defect which shares a side with each of the two.




These 2 have same defect since same Saccheri $\square$ & share "summit"

$\Rightarrow$ all 3 have same defect

Find P by drawing circle with center at A

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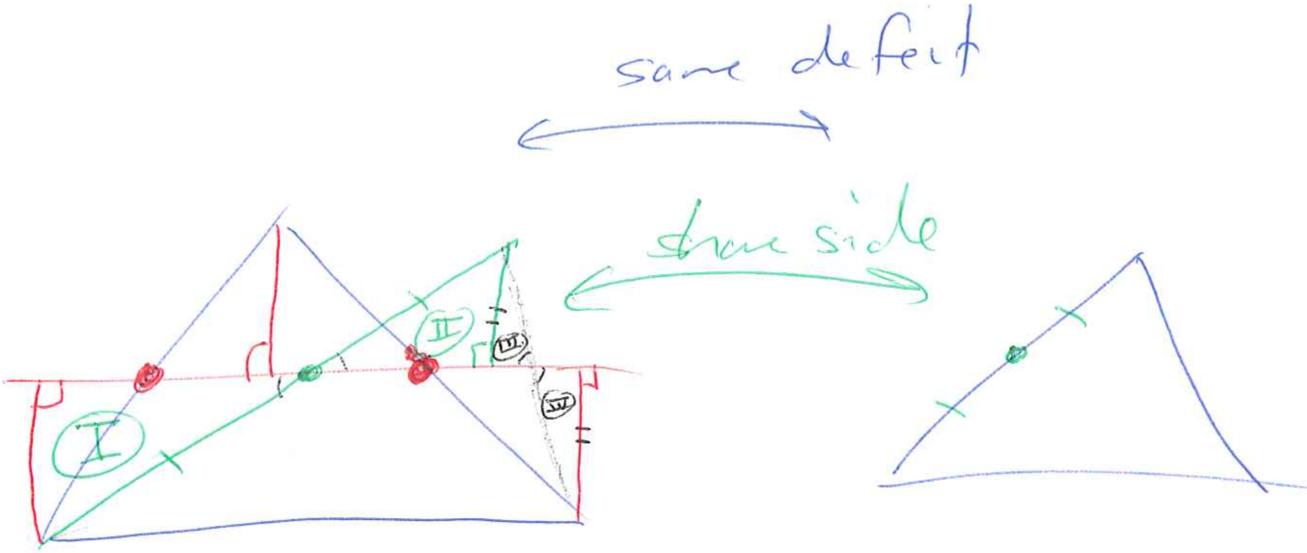
must establish congruence of a & b:

all 3 (actually 4) vertical lines are =

then use AAS




I $\cong$ II by AAS

III $\cong$ IV by AAS

$\Rightarrow$ same associated Saccheri quadrilateral

$\Rightarrow$ same defect

$\Rightarrow$ equivalent

# Area

We conclude that defect is a measure of equivalence!

Def: The area of any polygon is proportional to its defect

$$\boxed{A = K \cdot D}$$

$K$ = given positive constant
e.g. ~~$\pi/180$~~

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Thm: for any triangle, $\boxed{A < K \cdot 180}$

Pf: $A = K \cdot D = K \cdot (180 - S) < K \cdot 180$

Triangles can not be arbitrarily large!