The real projective plane

All lines in Euclidean plane have the form

$$\boxed{ax + by + c = 0}$$

and can be labelled by their slope

$$\boxed{m = -\frac{a}{b}}$$

with the convention that $\pm\infty$ are allowed but label the same line. To each Euclidean line of slope m add the ideal point I_m

different notation than book

Idea: What used to be Euclidean lines with slope m now all intersect at I_m

The ideal line L consists of all the ideal points

Def: The real projective plane is the geometric model whose
points are all points of $\mathbb{R}^2$ + all points of L
and whose
lines are all lines of $\mathbb{R}^2$ with their ideal points added together with L

[no angles]

skim

Claim: $\boxed{\text{The real projective plane is a projective geometry}}$

Pf: Axioms 2 (3 non-colinear pts)
& 3 (≥ 3 pts / line)
follow immediately since they hold in $\mathbb{R}^2$

$\therefore$ Axiom 4 (2 lines intersect in exactly 1 pt):

- 2 non-parallel Euclidean lines intersect in 1 Euclidean point and their ideal points differ
- 2 parallel Euclidean lines intersect in their ideal points
- the ideal line intersects each Euclidean line in its ideal point

Axiom 1 (2 pts determine exactly 1 line)

- 2 Euclidean pts determine a unique Euclidean line, and are not on the ideal line
- 2 ideal pts lie on the ideal line by definition, and cannot both be on any Euclidean line.
- 1 Euclidean pt P and the ideal pt I_m determine the unique Euclidean line through P with slope m

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$\therefore$ dual to itself (points $\longleftrightarrow$ lines)

SKim

<u>Recall</u>: A <u>quadrilateral</u> in neutral geometry is an ordered collection of 4 distinct points, no 3 of which are collinear, together with the 4 line segments connecting the points <u>in the given order</u>, such that any 2 of the line segments either intersect at an endpoint or not at all.

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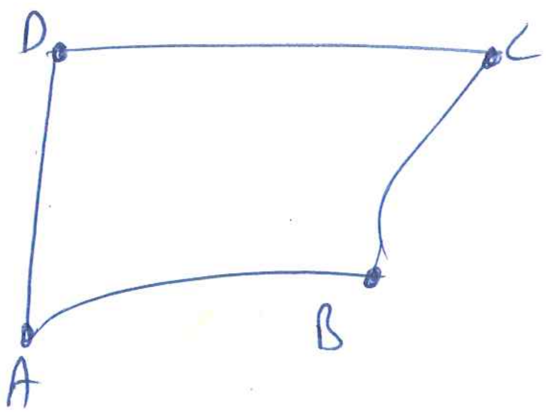
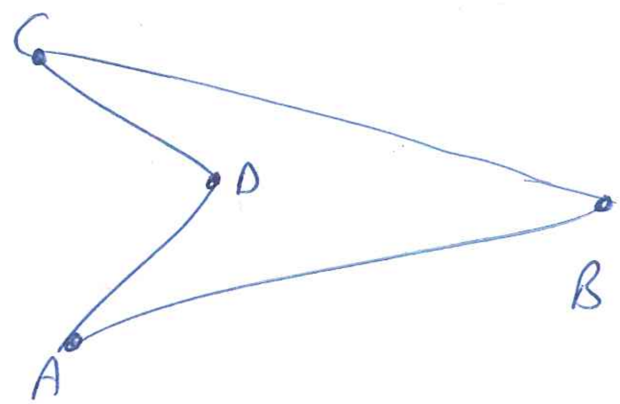
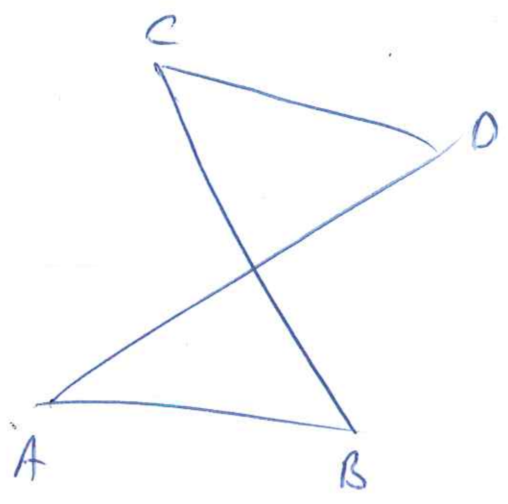
e.g.





but not



~~SKP~~

(A similar definition holds in elliptic geometry, although care must be taken when defining "the" line segment connecting 2 points)

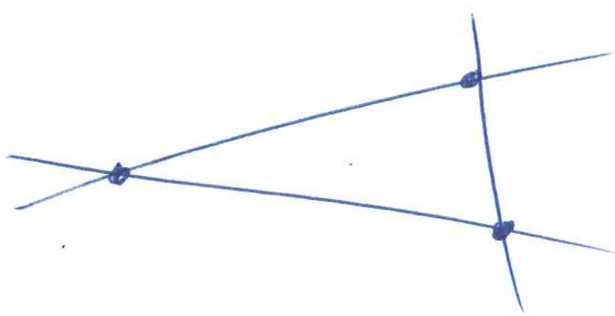
In projective geometry, $\nexists$ line segments!

$\therefore$ start again:

<u>Triangle</u>: 3 distinct pts ("vertices") & all lines through them ("sides")

the dual of this is a

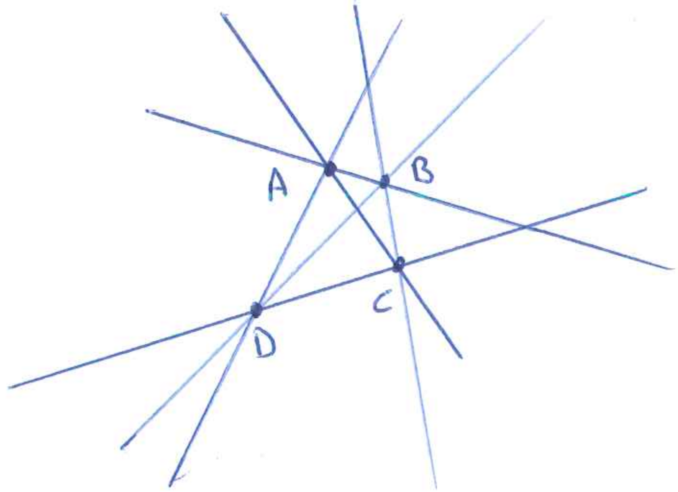
<u>Trilateral</u>: 3 distinct lines ("sides") & all pts of intersection



(an assumption is made that the pts aren't collinear & the lines not concurrent)

(don't all meet in 1 pt)

Complete quadrangle: 4 pts & all 6 lines connecting pairs of pts



Complete quadrilateral: 4 lines & all 6 intersection pts