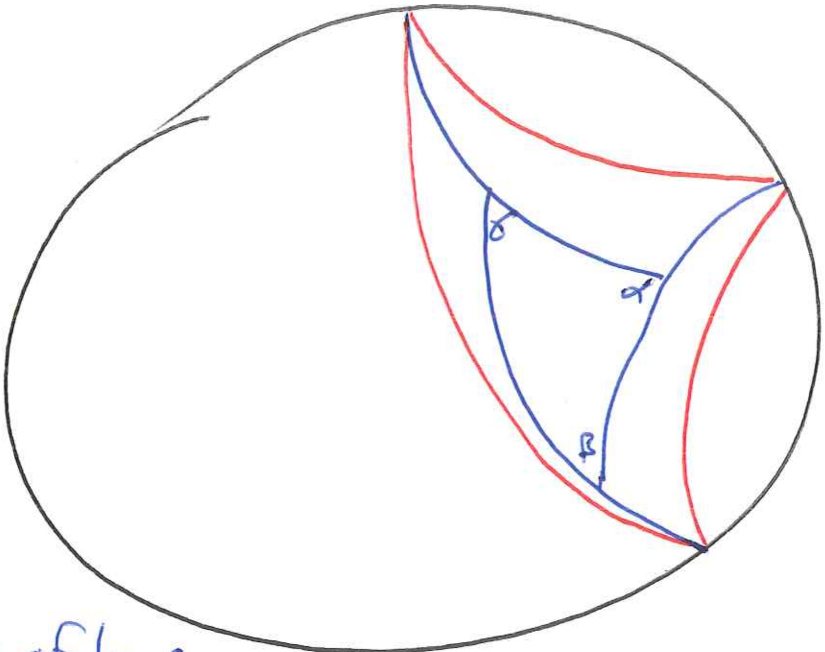
Hyperbolic Lunes

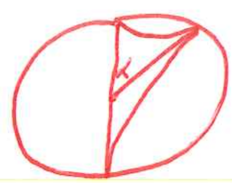


area of lune

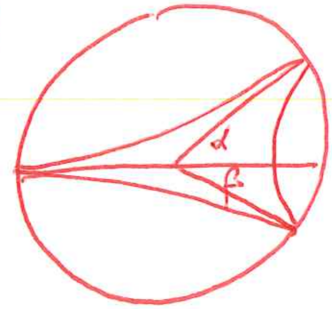
① $A(\theta)$ depends only on θ ~~SAS~~! AAA!

② A_I = const ~~SSS~~! AAA!

③ $A(\theta) + A(\pi-\theta) = A_I$



④ $A(\pi-\alpha) + A(\pi-\beta) + A(\alpha+\beta) = A_I$



$$\Longrightarrow A(\pi-\alpha) + A(\pi-\beta) = A(\pi-\alpha-\beta)$$

$\Longrightarrow f(\alpha)+f(\beta)=f(\alpha+\beta)$

$\Longrightarrow f$ linear

$$\Longrightarrow A(\pi-\theta) = K\theta$$

$$\Longrightarrow A_I = K\pi$$

But $A + A(\pi-\alpha) + A(\pi-\beta) + A(\pi-\gamma) = A_I$

$$\Longrightarrow A = K(\pi-\alpha-\beta-\gamma)$$

$$= K\delta$$

Hyperbolic lunes

- rotational & translational symmetry! (tractrix pseudosphere)
- lunes at origin do give fraction of area but that area is ∞ unless cut off

$\therefore$ introduce ideal pts "at ∞"
& define lunes to be "2/3 ideal triangles"
i.e. with 2 pts at ∞
& define ideal triangle I to be 3 pts at ∞ (can also be thought of as 0° lune!)

- $A(\theta)$ depends only on θ SAS!
- $A(I)$ same $\forall I$ SSS! (sides must be congruent — ∞ ray $\neq$ ∞ line!)

<u>Facts</u>: $A(\theta) + A(\pi - \theta) = A(I)$

$$A(\alpha+\beta) + A(\pi-\alpha) + A(\pi-\beta) = A(I)$$

$$\implies A(\pi-\alpha) + A(\pi-\beta) = A(\pi-\alpha-\beta)$$

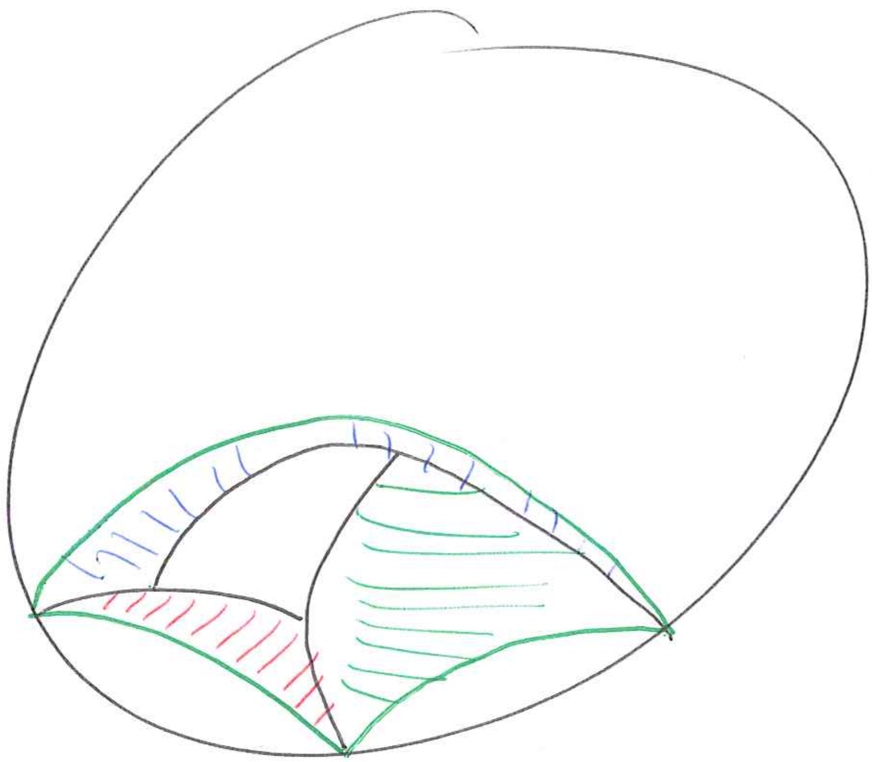
$$\implies A(\pi-\theta) = K\theta \; !$$

$$\implies A(I) = K\theta + K(\pi-\theta) = K\pi \; !$$

But $A(I) = A(\Delta) + A(\pi-\alpha) + A(\pi-\beta) + A(\pi-\gamma)$

$$\implies K\pi = A(\Delta) + K(\alpha+\beta+\gamma)$$

✓



$$\text{Lunes} \longmapsto A = er^2 \cdot \frac{\pi}{180}$$

<u>sphere</u>: $0 < e < 4\pi$

<u>Klein</u>: $0 < e < 2\pi$

<u>Poincaré</u> $A = d\,\rho^2 \cdot \frac{\pi}{180}$

$$0 < d < \pi$$