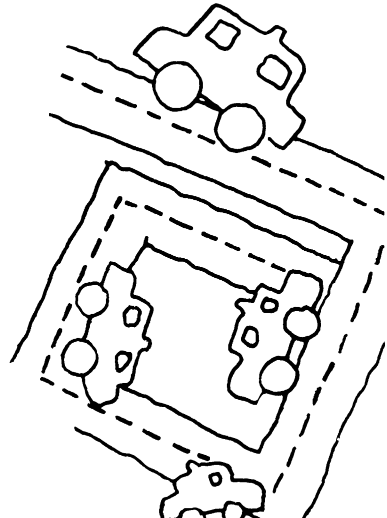
# 1
# WHAT IS TAXICAB GEOMETRY?



THE USUAL way to describe a (plane) geometry is to tell what its *points* are, what its *lines* are, how *distance* is measured, and how *angle measure* is determined. When you studied Euclidean coordinate geometry the points were the points of a coordinatized plane. Each of these points could be designated either by a capital letter or by an ordered pair of real numbers (the "coordinates" of the point). For example, in Fig. 1, $P = (-2, -1)$ and $Q = (1, 3)$. The lines were the usual long, straight, skinny sets of points; angles were measured in degrees with a (perfect) protractor; and distances either were measured "as the crow flies" with a (perfect) ruler or were calculated by means of the Pythagorean Theorem.

For example, in Fig. 1 the distance from $P$ to $Q$ could be found by considering a right triangle having $\overline{PQ}$ as its hypotenuse. The dotted segments are the legs of one such triangle. (Are there any other such right triangles?) These legs clearly have lengths 3 and 4. Thus, by the Pythagorean Theorem, the Euclidean distance from $P$ to $Q$ is $\sqrt{3^2 + 4^2} = 5$. We shall use the symbol $d_E$ to represent the Euclidean distance function. Thus, in our example we would write

$$d_E(P, Q) = 5,$$

and read it "The Euclidean distance from $P$ to $Q$ is 5."

Taxicab geometry is very nearly the same as Euclidean coordinate geometry. The points are the same, the lines are the same, and angles are measured in the same way. Only the distance function is different. In Fig. 1 the taxicab distance from $P$ to $Q$, written $d_T(P, Q)$, is determined not as the crow flies, but instead as a taxicab would drive. We count how many blocks it would have to travel horizontally and vertically to get from $P$ to $Q$. The dotted segments suggest one taxi route. Clearly

$$d_T(P, Q) = 7.$$

"The taxi distance from $P$ to $Q$ is 7."

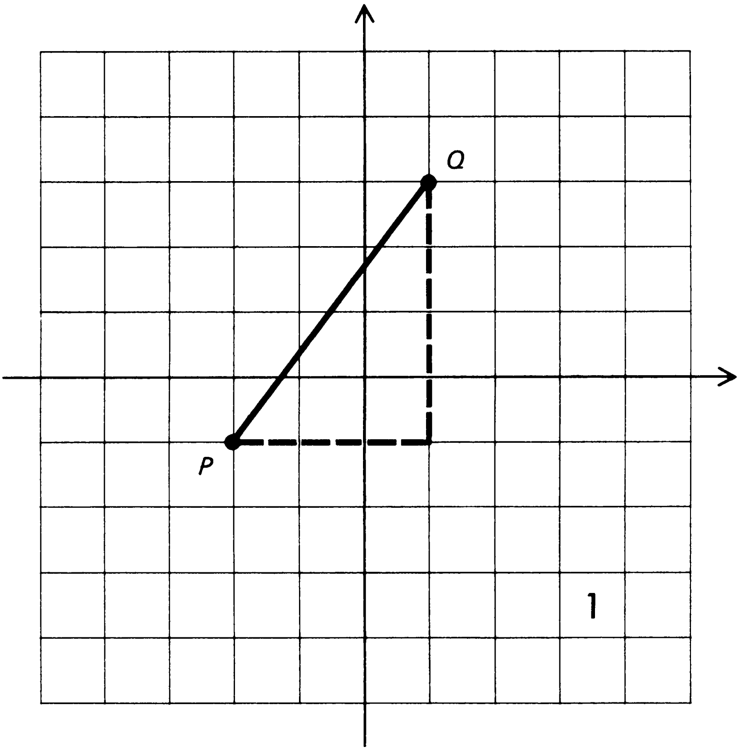



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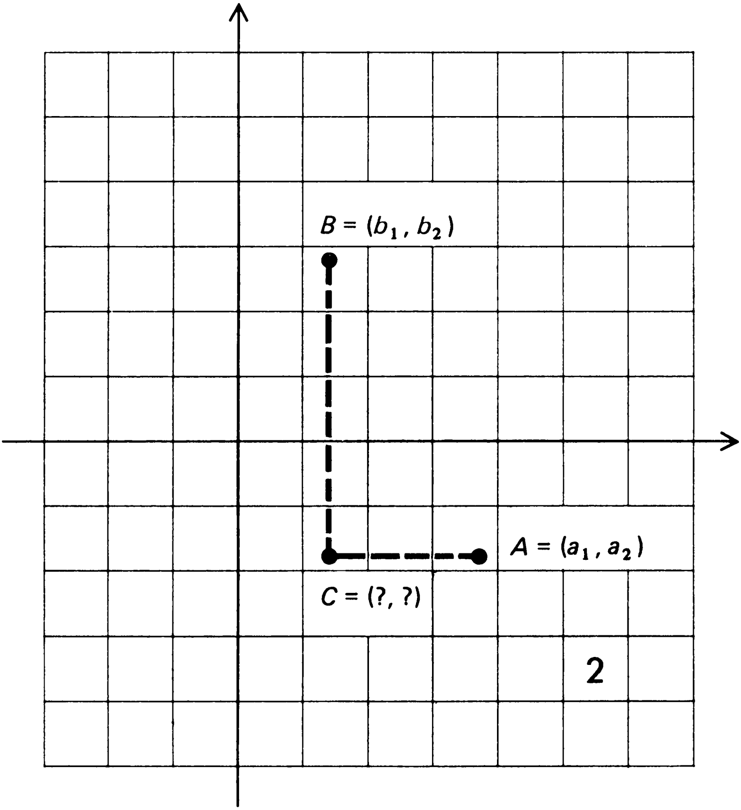
Figure 2 is a reminder that most of the points of the coordinate plane do not have two integer coordinates. In the figure a pair of "arbitrary" points $A = (a_1, a_2)$ and $B = (b_1, b_2)$ is given. What are the coordinates of the point $C$? Write an expression for the length of $\overline{AC}$ in terms of the coordinates of $A$ and $C$. Write an expression for the length of $\overline{BC}$ in terms of the coordinates of $B$ and $C$. The following precise, algebraic definitions of $d_T$ and $d_E$ should now seem reasonable:

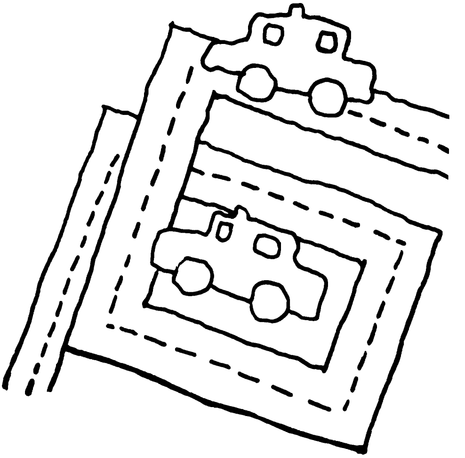
(1) $$d_T(A, B) = |a_1 - b_1| + |a_2 - b_2|;$$

(2) $$d_E(A, B) = \sqrt{(a_1 - b_1)^2 + (a_2 - b_2)^2}.$$

We will make use of these careful definitions only very rarely. The reason for inserting them here is to assure ourselves that (1) there is a mathematically respectable foundation underlying taxicab geometry, and (2) there is a definite taxicab distance between any two points, whether they are located at a "street corner" or not.


B = (b1 , b2 )
A = (a1 , a2 )
C = (?, ?)
2

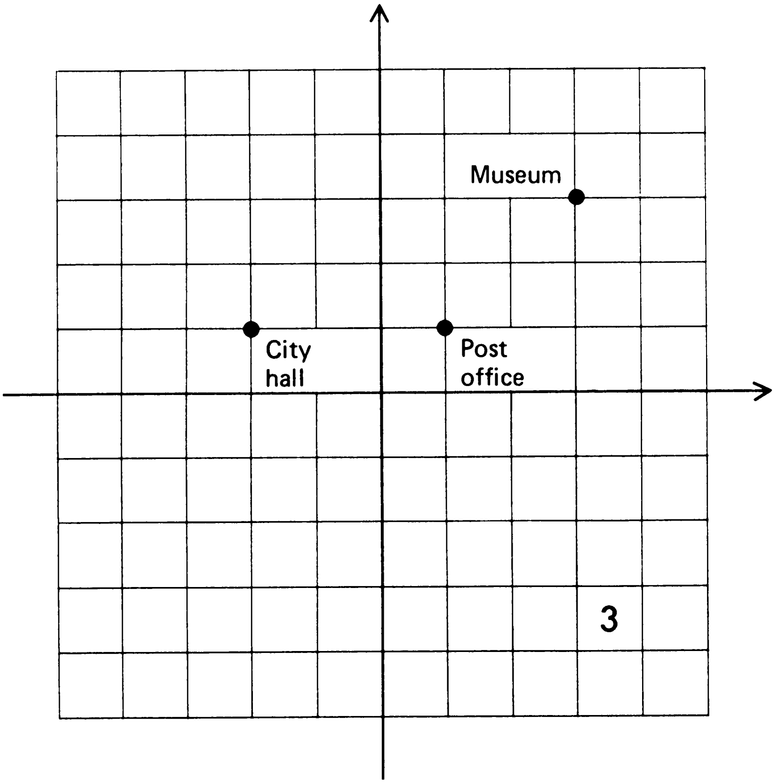
# 2
# SOME APPLICATIONS



TAXICAB GEOMETRY is a more useful model of urban geography than is Euclidean geometry. Only a pigeon would benefit from the knowledge that the Euclidean distance from the Post Office to the Museum (Fig. 3) is $\sqrt{8}$ blocks while the Euclidean distance from the Post Office to the City Hall is $\sqrt{9} = 3$ blocks. This information is worse than useless for a person who is constrained to travel along streets or sidewalks. For people, taxicab distance is the "real" distance. It is *not* true, for people, that the Museum is "closer" to the Post Office than the City Hall is. In fact, just the opposite is true. (What are the two taxicab distances?)

While taxicab geometry is a better mathematical model of urban geography than is Euclidean geometry, it is not perfect. Many simplifying assumptions have been made about the city. All the streets are assumed to run straight north and south or straight east and west; streets are assumed to have no width; buildings are assumed to be of point size . . . You should not be greatly disturbed by these assumptions. True, no city is exactly like the ideal one we have in mind. Still, many parts of many cities are not too different from it. The things we learn about our ideal model will have application in certain real urban situations.

The process of setting up a mathematical model of a real situation nearly always involves making simplifying assumptions. Without them the mathematical problems tend to be too involved and difficult to solve, or even to set up. In Section 6 we shall see some of the mathematical complications that arise when we alter our ideal model to make it more realistic.


Museum
City hall
Post office
3

# exercises

1. List some additional simplifying assumptions that we have made about Ideal City.

2. Alice and Bruno are looking for an apartment in Ideal City. Alice works as an acrobat at amusement park $A = (-3, -1)$. Bruno works as a bread taster in bakery $B = (3, 3)$. (See Fig. 4.) Being ecologically aware, they walk wherever they go. They have decided their apartment should be located so that the distance Alice has to walk to work plus the distance Bruno has to walk to work is as small as possible. Where should they look for an apartment?

3. In a moment of chivalry Bruno decides that the sum of the distances should still be a minimum, but Alice should not have to walk any farther than he does. Now where could they look for an apartment?

4. Alice agrees that the sum of the distances should be a minimum, but she is adamant that they both have exactly the same distance to walk to work. Now where could they live?

5. After a day of fruitless apartment hunting they decide to widen their area of search. The only requirement they keep is that they both be the same distance from their jobs. Now where should they look?

6. After another luckless day they finally agree that all that really matters is that Bruno be closer to his job than Alice is to hers. Now where can they look?

7. The dispatcher for the Ideal City Police Department receives a report of an accident at $X = (-1, 4)$. There are two police cars in the area, car $C$ at $(2, 1)$ and car $D$ at $(-1, -1)$. Which car should she send to the scene of the accident?