Calculating Line Elements in
Cylindrical and Spherical Coordinates

by Corinne Manogue and Katherine Meyer

©1997 & 2004 Corinne A. Manogue

Rectangular Coordinates:

The arbitrary infinitesimal displacement vector in Cartesian coordinates is:

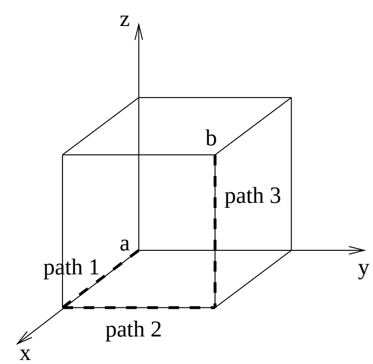
$$d\vec{\boldsymbol{r}} = dx\,\hat{\boldsymbol{\imath}} + dy\,\hat{\boldsymbol{\jmath}} + dz\,\hat{\boldsymbol{k}}$$

Given the cube shown below, find $d\vec{\boldsymbol{r}}$ on each of the three paths, leading from a to b.

Path 1: $d\vec{\boldsymbol{r}} =$

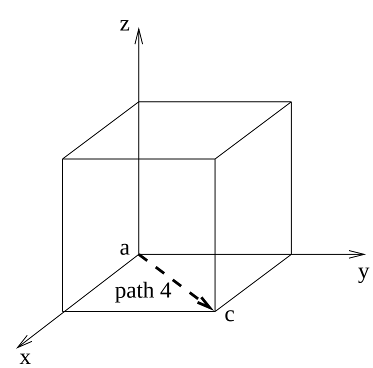
Path 2: $d\vec{\boldsymbol{r}} =$

Path 3: $d\vec{\boldsymbol{r}} =$



The first expression above for $d\vec{\boldsymbol{r}}$ is valid for any path in rectangular coordinates. Find the appropriate expression for $d\vec{\boldsymbol{r}}$ for the path which goes directly from a to c as drawn below.

Path 4: $d\vec{\boldsymbol{r}} =$



However, Cartesian coordinates would be a **poor** choice to describe a path on a cylindrically or spherically shaped surface. Next we will find an appropriate expression in these cases.

Cylindrical Coordinates:

You will now derive the general form for $d\vec{\boldsymbol{r}}$ in cylindrical coordinates by determining $d\vec{\boldsymbol{r}}$ along the specific paths below.

Note that an infinitesmial element of length in the $\hat{\boldsymbol{r}}$ direction is simply dr, just as an infinitesimal element of length in the $\hat{\boldsymbol{\imath}}$ direction is dx. **But**, an infinitesimal element of length in the $\hat{\boldsymbol{\phi}}$ direction is **not** just $d\phi$, since this would be an angle and does not even have the units of length.

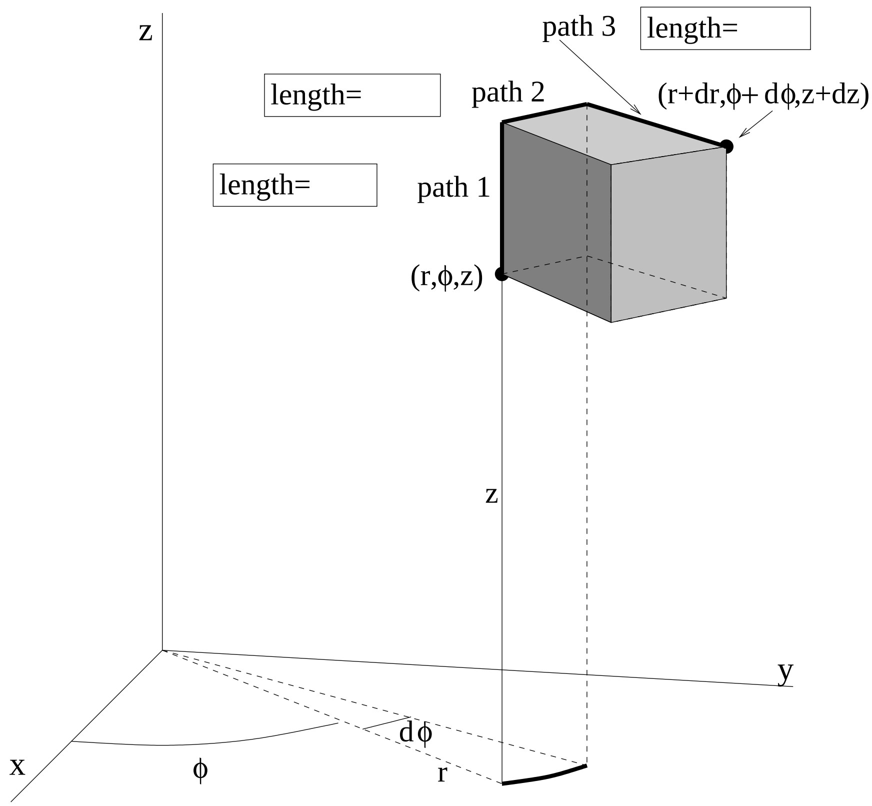
Geometrically determine the length of the three paths leading from a to b and write these lengths in the corresponding boxes on the diagram.

Now, remembering that $d\vec{\boldsymbol{r}}$ has both magnitude and direction, write down below the infintesimal displacement vector $d\vec{\boldsymbol{r}}$ along the three paths from a to b. Notice that, along any of these three paths, only one coordinate r, ϕ, or z is changing at a time. (i.e. along path 1, $dz \neq 0$, but $d\phi = 0$ and $dr = 0$).

Path 1: $d\vec{\boldsymbol{r}} =$

Path 2: $d\vec{\boldsymbol{r}} =$

Path 3: $d\vec{\boldsymbol{r}} =$

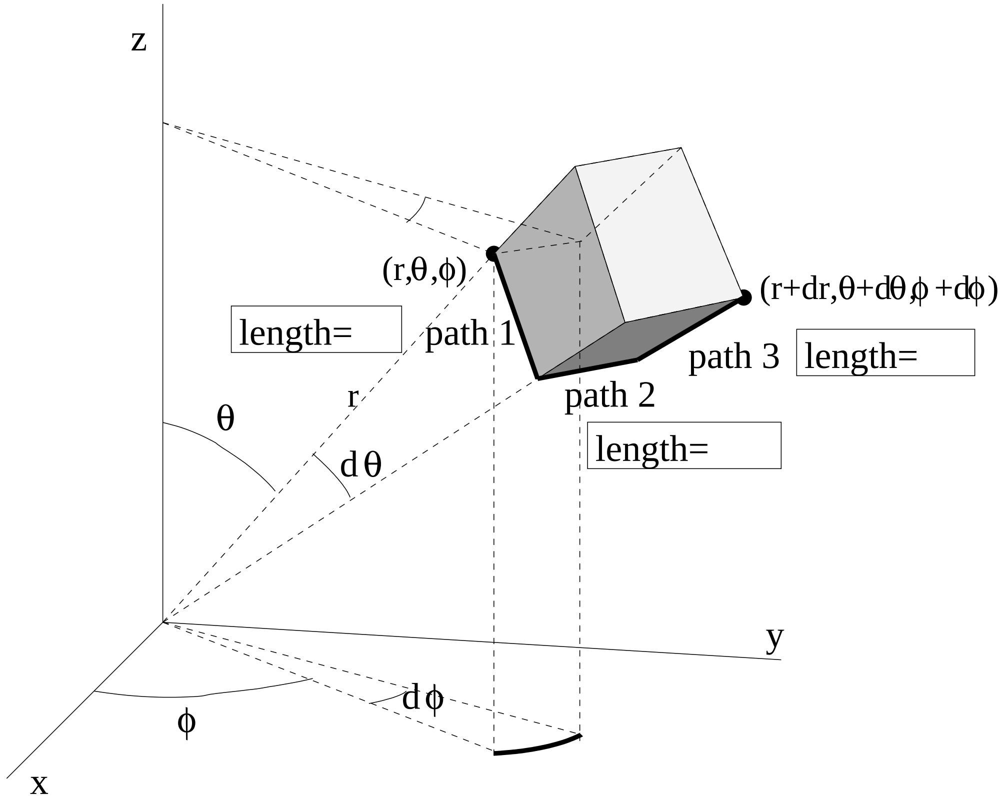


Cylindrical Coordinates

If all three coordinates are allowed to change simultaneously, by an infinitesimal amount, we could write this $d\vec{\boldsymbol{r}}$ for any path as:

$d\vec{\boldsymbol{r}}=$

This is the general line element in cylindrical coordinates.



Spherical Coordinates

Spherical Coordinates:

You will now derive the general form for $d\vec{\boldsymbol{r}}$ in spherical coordinates by determining $d\vec{\boldsymbol{r}}$ along the specific paths below. As in the cylindrical case, note that an infinitesmial element of length in the $\hat{\boldsymbol{\theta}}$ or $\hat{\boldsymbol{\phi}}$ direction is **not** just $d\theta$ or $d\phi$. You will need to be more careful. Geometrically determine the length of the three paths leading from a to b and write these lengths in the corresponding boxes on the diagram. Now, remembering that $d\vec{\boldsymbol{r}}$ has both magnitude and direction, write down below the infintesimal displacement vector $d\vec{\boldsymbol{r}}$ along the three paths from a to b. Notice that, along any of these three paths, only one coordinate r, θ, or ϕ is changing at a time. (i.e. along path 1, $d\theta \neq 0$, but $dr = 0$ and $d\phi = 0$).

Path 1: $d\vec{\boldsymbol{r}} =$

Path 2: $d\vec{\boldsymbol{r}} =$ (Be careful, this is the tricky one.)

Path 3: $d\vec{\boldsymbol{r}} =$

If all 3 coordinates are allowed to change simultaneously, by an infinitesimal amount, we could write this $d\vec{\boldsymbol{r}}$ for any path as:

$d\vec{\boldsymbol{r}}=$

This is the general line element in spherical coordinates.