MTH 255 Final Review

1. Each part of this problem concerns the parametric curve $\vec{r}(t) = (t^2-2)\,\vec{\imath} + (2t-1)\,\vec{\jmath} + \ln(t)\,\vec{k}, \quad t > 0$.

   (a) Find the velocity, $\vec{\mathbf{v}}$, the speed, ν, and the acceleration, $\vec{\mathbf{a}}$, at every $t > 0$.

   (b) Find the length of the parametric curve from $(-1, 1, 0,)$ to $(7, 5, \ln(3))$.

   (c) Find the unit tangent vector $\vec{T}$ and the tangential component of acceleration, a_T, at the point $(-1, 1, 0)$, that is, when $t = 1$.

   (d) Find the curvature κ, normal component of acceleration, a_N, and the principle unit normal vector $\vec{N}$ at $(-1, 1, 0)$, that is when $t = 1$.

2. In this problem the function f is given by $f(x, y, z) = x^3y - 3x^2yz^4$.

   (a) Compute $\nabla f(x, y, z)$ and find the directional derivative of f at the point $(1, 1, 1)$ in the direction towards the origin.

   (b) In what direction does f increase most rapidly at the point $(1, 1, 1)$ and what is the rate of increase of f in that direction?

   (c) Consider the level surface $f(x, y, z) = -2$, where f is as above. Find an equation of the tangent plane and of the normal line to this surface at the point $(1, 1, 1)$.

3. In this problem $f(x, y) = 3xy - x^2y - xy^2$.

   (a) Find all critical points of f and determine whether f has a local maximum, local minimum, or a saddle point at each critical point. (There are 4 distinct critical points.)

   (b) Find the maximum and minimum values of the function $f(x, y)$ on (and inside) the triangle with vertices at $(0, 0)$, $(3, 0)$, and $(3, 6)$.

4. In this problem, the vector field $\vec{F}$ is given by $\vec{F} = (2xz + \sin(y))\,\vec{\imath} + x\cos(y)\,\vec{\jmath} + x^2\,\vec{k}$.

   (a) Show that $\vec{F}$ is conservative.

   (b) Compute the line integral $\int_C \vec{F}\cdot\vec{T}\,ds$ where C is the curve $\vec{r}(t) = \cos(t)\,\vec{\imath} + \sin(t)\,\vec{\jmath} + t\,\vec{k}$, with $0 \le t \le \pi/2$.

5. In this problem $\vec{F}$ is the vector field in the plane given by $\vec{F}(x, y) = \dfrac{x}{x^2+y^2}\,\vec{\imath} + \dfrac{y}{x^2+y^2}\,\vec{\jmath}$.

   (a) Compute $\operatorname{div} F$.

(b) Compute $\int_C \vec{F} \cdot \vec{n} ds$ for each of the three curves: $C_1 = \{(x, y) : \ x^2 + y^2 = 4\}$, $C_2 = \{(x, y) : \ (x - 1)^2 + (y - 1)^2 = 1\}$, and $C_3 = \{(x, y) : \ x^2 + y^2/2 = 1\}$. All curves are oriented counterclockwise.

6. A surface S in the shape of a helicoid is given by the parametric equations

$$\vec{r}(u, v) = u \cos(v)\,\vec{\imath} + u \sin(v)\,\vec{\jmath} + v\,\vec{k}, \quad 0 \le u \le 1, \quad 0 \le v \le 2\pi.$$

Suppose that the density of the surface at the point (x, y, z) is given by $\rho(x, y, z) = [1 + x^2 + y^2]^{1/2}$. Find the total mass of the surface, that is, compute the surface integral $\iint_S \rho(x, y, z)\, dS$.

7. Let $\vec{F} = xy\,\vec{\imath} + yz\,\vec{\jmath} + zx\,\vec{k}$ and let P be the triangular region in space with vertices $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$. Let C be the boundary of P with the positive orientation being counterclockwise as viewed from above. Compute the line integral $\int_C \vec{F} \cdot d\vec{r}$.

8. Let $\vec{F} = x^2\,\vec{\imath} + xy\,\vec{\jmath} + z\,\vec{k}$ and let E be the three dimensional solid region that lies above the paraboloid $z = x^2 + y^2$ and below the plane $z = 4$. Let S be the boundary of E. Compute the total flux of the vector field $\vec{F}$ out of E across the surface S.