Change of Variables

Recall: $x = r\cos\phi$
$y = r\sin\phi$

$$\Rightarrow \quad dx = dr\cos\phi - r\sin\phi\, d\phi$$
$$dy = dr\sin\phi + r\cos\phi\, d\phi$$

$$\Rightarrow \quad \boxed{\begin{array}{ll} \dfrac{\partial x}{\partial r} = \cos\phi & \dfrac{\partial y}{\partial r} = \sin\phi \\[2ex] \dfrac{\partial x}{\partial \phi} = -r\sin\phi & \dfrac{\partial y}{\partial \phi} = r\cos\phi \end{array}}$$

$$r^2 = x^2 + y^2$$
$$\tan\phi = y/x$$
$$\Rightarrow \quad 2r\,dr = 2x\,dx + 2y\,dy$$
$$(1+\tan^2\phi)\,d\phi = \frac{x\,dy - y\,dx}{x^2}$$
$$\Rightarrow \quad dr = \frac{x\,dx + y\,dy}{\sqrt{x^2+y^2}}$$
$$d\phi = \frac{x\,dy - y\,dx}{x^2+y^2}$$

$$\Rightarrow \quad \boxed{\begin{array}{ll} \dfrac{\partial r}{\partial x} = \dfrac{x}{\sqrt{x^2+y^2}} & \dfrac{\partial r}{\partial y} = \dfrac{y}{\sqrt{x^2+y^2}} \\[2ex] \dfrac{\partial \phi}{\partial x} = \dfrac{-y}{x^2+y^2} & \dfrac{\partial \phi}{\partial y} = \dfrac{x}{x^2+y^2} \end{array}}$$

Fact: $$dx\,dy = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} \\ \frac{\partial x}{\partial \phi} & \frac{\partial y}{\partial \phi} \end{vmatrix} dr\,d\phi$$
$$= r\,d\phi$$

Fact: $$dr\,d\phi = \begin{vmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} \\ \frac{\partial \phi}{\partial x} & \frac{\partial \phi}{\partial y} \end{vmatrix} dx\,dy$$
$$= \frac{1}{\sqrt{x^2+y^2}}\,dx\,dy$$

Notation

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

the Jacobian of x, y w.r.t. u, v

determinant

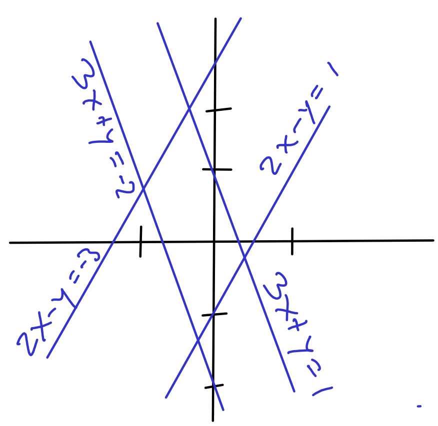
Fact: $$dx\,dy = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$$

absolute value

Example

$u = 2x - y$

$v = 3x + y$



$$\Rightarrow x = \frac{u+v}{5}$$

$$y = \frac{2v-3u}{5}$$

$$\Rightarrow \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{5} & -\frac{3}{5} \\ \frac{1}{5} & \frac{2}{5} \end{vmatrix}$$

$$= \left(\frac{1}{5}\right)\left(\frac{2}{5}\right) - \left(-\frac{3}{5}\right)\left(\frac{1}{5}\right) = \frac{5}{25} = \frac{1}{5}$$

$$\Rightarrow dx\,dy = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv = \frac{1}{5}\,du\,dv$$

$$\Rightarrow A = \int_R dA = \int_{?}^{?}\int_{?}^{?} dx\,dy = \int_{-2}^{1}\int_{-3}^{1} \frac{1}{5}\,du\,dv$$

$$= \frac{1}{5}\, u\Big|_{-3}^{1}\, v\Big|_{-2}^{1} = \boxed{\frac{12}{5}}$$

$$\Rightarrow \iint_R Q\,dx\,dy = \int_{-2}^{1}\int_{-3}^{1} Q\,\frac{1}{5}\,du\,dv$$

But; must express Q in terms of u, v ...