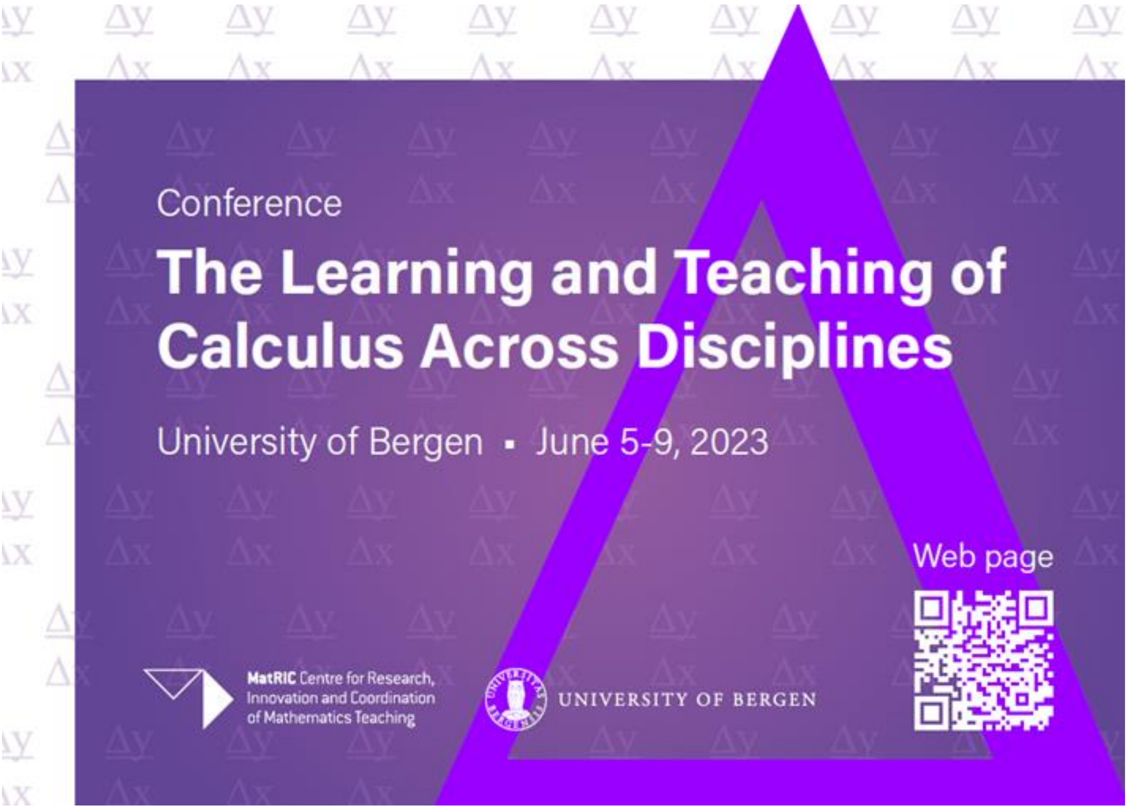

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Instances of confounding when differentiating vector fields

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Introduction

Vector fields are important objects in both mathematics and physics. In vector calculus courses, students are typically introduced to the idea of a vector field and learn about two different types of vector field derivatives, namely divergence and curl. These two vector derivative operators are used frequently in electromagnetism, but students are not usually asked to consider other derivatives of vector fields. There is a gap in the literature on student understanding of derivatives of a vector field other than divergence and curl.

There are two main student approaches when trying to take a more general derivative of a vector field: a component-based approach, where students attempt to differentiate each component individually, and a geometric, vector-valued approach, where students subtract two nearby vectors. Within these broad categories, students can use a variety of methods and strategies to find a derivative, including graphical approaches, numerical approaches, and utilizing what they know about divergence and curl. These techniques may not be available to a student at the same time, depending on the information the student has, and how much experience the student has had with derivatives and vector fields. Due to the unfamiliar nature of a more general vector field derivative than divergence and curl to many undergraduate physics and mathematics students, it is of interest to study the approaches students take when attempting to differentiate a vector field.

In this study, three junior-level physics students were asked to think about taking a derivative of a vector field. Each student displayed evidence of confounding the components of the vector field with either the independent variables (two students) or the basis directions (one student), thus impacting their ability to recognize that a derivative can also be a vector field. We analyze the interview data related to confounding, and discuss implications for instructors teaching calculus.

Literature review

Student understanding of derivatives and vector fields is of interest to both mathematics and physics education, and the current literature shows a wide variety of concepts, misunderstandings, and ideas that students have when thinking about vectors and derivatives separately. This review spotlights the work of a few authors who study how students think about derivatives in a vector calculus setting, and notes the similarities and patterns that show up throughout the literature explored.

There have been several studies that provide insight into students' struggles and understanding of vector fields, particularly in the context of electricity and magnetism. For example, Dray and Manogue (1999) outline the differences between how vector calculus is taught in mathematics and the way vector calculus is used in physics, and the possible impacts this "gap" has on student understanding. They explain that mathematics courses emphasize algebraic understanding and calculations, whereas physics courses typically use graphical understanding and symmetry, with less emphasis placed on algebra. The authors suggest that this disparity may contribute to student difficulties in understanding vector calculus in physics contexts, and suggest more communication between mathematics and physics instructors as a possible solution to the problem. Similarly, Gire and Price (2012) found that students see variable and component as interchangeable when looking at algebraic representations of vector fields, and consequently creating a graph of a vector field from an algebraic function is exceptionally difficult for students They found that students have a particularly difficult time separating variable from component when the *x* component depends on the *y* variable and vice-versa.

The Colorado Upper-Division E&M Instrument (CUE) has been used in many studies to test student understanding of electricity and magnetism. This instrument was developed by physics education researchers at the University of Colorado to measure how students think about a variety of concepts in electricity and magnetism, including vector calculus (Chasteen et al., 2012). Pepper et al. (2012) used interview data in addition to the CUE and found that students tend to focus on one part of vector fields (either direction or magnitude) when doing calculations. Whether the students focused on magnitude or direction differed depending on the problem, but the pattern persisted throughout the exam. The CUE also showed that students had difficulty understanding the physical meaning behind vector field operations, such as gradient, divergence, and curl. The students were able to calculate the gradient, divergence, and curl, but were often unable to explain what the results meant, corroborating the results of similar studies on student understanding of vector calculus.

Methods

Individual interviews had been previously conducted with four students at the end of the Static Fields Paradigms course at Oregon State University. The interviews aimed to determine how students think about partial derivatives of functions. The first phase of the interviews asked students to think about the partial derivative of a scalar-valued function, and the second phase of the interview prompted the students to think about the partial derivative of a vector field. Three of the four students completed both phases of the interview. This paper only focuses on the vector field phase of the protocol, although some students reference their work on the scalar field during the vector field phase. Students were encouraged to say their thoughts and processes out loud, and to write/draw on the provided paper throughout the interview. Each interview lasted approximately 90 minutes, with each phase lasting approximately 45 minutes.

At the time of the interview, the students would have completed the entire calculus sequence, including vector calculus 1 and 2, and the general calculus-based physics sequence. The students also likely had completed a sophomore-level course introducing ideas such as relativity, quantum physics, statistical physics, and other physical ideas from the 20th century.

The interview transcripts and videos were analyzed qualitatively, in the style of Thematic Analysis (Aronson, 1995). Thematic analysis consists of several steps, including the identification of patterns in the data and sorting the data into subthemes. We identified confounding as a commonality among all three students interviewed. Within the confounding pattern, we identified two subthemes, namely confounding variable with component and confounding component with variable, only the first of which will be discussed here.

Results

We define *confounding* as imposing a strong relationship between two unrelated objects or concepts, resulting in a student treating the confounded objects or concepts as though an action on one imposes the same action on the other. There are two different levels of confounding, which we call "strong" confounding and "weak" confounding. A student who fails to recognize that the two objects confounded are distinct would be demonstrating strong confounding, whereas a student who recognizes that the objects are different but nonetheless treats them as indistinct or strongly linked would be demonstrating weak confounding. Alex and Bailey each demonstrated weak confounding of variables and components. Due to space limitations we paraphrase only Bailey's comments here.

The interviewer asked Bailey if there was a way to figure out how the *y* component changes with respect to *x*, which appeared to cause Bailey to doubt his earlier claim that the *y* component changes with respect to *x*. These comments show that Bailey understood the distinction between variable and component, and that *y* was used to represent two things. However, when trying to take a partial derivative with respect to *x*, Bailey did not know whether to hold the *y* component or the *y* variable constant. His previous discussion of divergence may have led him to the conclusion that only the *x* component should be differentiated when taking a partial derivative with respect to *x*, but he struggled when the interviewer asked if the *y* component was changing. This is an example of Bailey's confounding, because although Bailey had previously expressed understanding that both components depend on both variables, he rejected the idea that differentiating with respect to *x* would give information about the *y* component. The interviewer's prompt about finding how the *y* component changes with respect to *x* appeared to cause Bailey to doubt his previous claim, so he began to consider what it would mean to hold *y* constant. Bailey's unconfounding process started when he realized that his previous difficulty was due to *y* representing both the *y* component and *y* variable. Bailey did not know what he was supposed to be holding constant, and this uncertainty combined with the idea of divergence being recently on his mind likely led him to hold both the *y* component and the *y* variable constant, and only differentiate the *x* component with respect to *x*.

Discussion

Bailey started out by discussing divergence and gradient, both of which correlate variable with component, and clearly had difficulty imagining uncorrelated "cross terms", such as those that would show up in curl. Although Bailey understood the difference between variable and component, he was unsure which to hold constant – a clear instance of weak confounding. During the course of the interview, Bailey began to identify the source of his confusion, thus beginning to unconfound variable and component.

Although Alex demonstrated early awareness of the dependence on each vector component on both independent variables, he nonetheless acted as though there was a correlation, thus also weakly confounding variable and component. However, he was then able to use a graphical approach to unconfound these two objects.

When analyzing physics students' use of mathematics terminology, it is important to take into account the different ways that these two disciplines use and refer to derivatives. For instance, physicists predominantly use Leibniz notation for derivatives, assigning physical meaning to the infinitesimals in the numerator and denominator as "small changes", whereas mathematicians predominantly use primes, implicitly emphasizing that differentiation is an operator that acts on functions. Furthermore, physicists use subscripts to denote components, whereas mathematicians use them to denote differentiation. These notational issues reflect different conceptual emphases, which can be especially confusing for physics students when first using physics notation to express gradient and divergence, precisely the context in which these interviews were conducted.

Because the number of students interviewed is small, the conclusions this study draws and the implications thereof may not be indicative of the entire student body. That said, given the strong evidence in this study that students confound variable with component, greater emphasis should be placed on distinguishing between them. Explicit examples could be presented during instruction demonstrating that these two objects are not interchangeable, despite having similar labels. An activity similar to the interview protocol, where students are given a vector field and asked to take its partial derivative, would force students to think about the dependence of the components on the independent variables outside of the context of divergence and curl, thus solidifying their understanding of the underlying concepts. The extent to which notation can and should be chosen so as to reduce such student confusion is worthy of further study. Along these lines, we note recent work of Topdemir et al. (2023) on student understanding of vector field derivatives.

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