THE REGULARIZED LAYERED MEDIUM EQUATION

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Abstract. The *regularized layered medium equation* is proposed as a model of voltage distribution in a medium consisting of alternating thin films of conducting and dielectric materials. This equation is obtained from the *layered medium equation* by the introduction of a regularizing perturbation that takes account of the resistance at the interface between adjacent conducting and dielectric layers. The regularized equation is an implicit evolution equation which is shown to be well-posed, and an explicit measure is derived for the rate of decay of singularities in the initial data. It is shown that, as the regularizing parameter approaches zero, solutions of the initial-boundary value problem for the regularized equation converge to the corresponding solution of the layered medium equation. This gives a method of calculating the exact rate of decay of singularities in the initial data for the layered medium equation.

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1. Introduction.

Multilayered structures offer potential advantages in the design of integrated circuits (Ghausi and Kelly[6]), and a variety of techniques is available for the construction of layered synthetic microstructures, which consist of alternating layers of different semiconductors, different metals, or semiconductors and metals (Doeher[4]; Dresselhaus[5]). One example of a device with such a structure is the

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multilayer ceramic capacitor, a common component of computer memory boards where its functions are to divert spurious signals and to buffer fluctuations in the power supply (Trotter[13]). Such a capacitor consists of around 60 to 120 alternating layers of ceramic (with a very high dielectric constant) and metal. The appropriate setting in which to model such multilayered RC structures is the theory of distributed networks (Ghausi and Kelly[6]). Bosse and Showalter[2] proposed as such a model the *Layered Medium Equation (LME)*

$$-\frac{\partial}{\partial t}\big(\partial_z(C(\vec{x},z)\partial_z u)\big)-\partial_z(G_V(\vec{x},z)\partial_z u)-\vec{\nabla}_{\vec{x}}\cdot(G_H(\vec{x},z)\vec{\nabla}_{\vec{x}}u)=F(\vec{x},z,t). \quad (1.1)$$

This represents a classical continuum approximation to a discretely layered structure. The coefficients G_H, G_V and C measure the distributed horizontal conductivity, vertical conductivity and capacitance of the limiting heterogeneous material obtained by letting the thickness of the layers approach 0. The unknown, u, represents the potential in the structure with respect to a reference level for which it is natural to take the voltage in the substrate. Since the continuum of layers is assumed to be horizontally aligned, the vertical (z) direction is distinguished from the horizontal $\vec{x}=(x_1,x_2)$-plane by capacitance effects. The coefficients (G_H, G_V, C) can be assumed to be independent of z, as the limiting process by which the *LME* is derived is effectively a homogenization in the z-direction.

We propose in Section 2 a modification of the *LME*, which we call the *Regularized Layered Medium Equation (RLME)*. This model is distinguished from the *LME* by the inclusion of resistive losses due to the (usually small) transverse current at the interface between layers, and it thus takes account of the effect of a lossy dielectric on energy dissipation in multilayered structures. Both the *LME* and the *RLME* are examples of implicit evolution equations, as is shown in Section 3 where the abstract variational formulation of these problems is developed. It follows that these equations are well-posed with data from a variety of spaces. In Section 4 it is also shown that a dual form of the *LME* is well-posed.

The two models differ in the regularity of their solutions. The *LME* essentially has the form

$$\partial_t(\partial_z^2 u)+\triangle_{\vec{x},z}u=-F$$

and so we would expect it to be parabolic in $\vec{x}=(x_1,x_2)$ but to preserve regularity in z. The *RLME* is a perturbation of the *LME* of the form

$$\partial_t(\partial_z^2 u+h\triangle_{\vec{x},z}u)+\triangle_{\vec{x},z}u=-F$$

and so should preserve regularity in all variables. Our primary result is to make this intuition precise by demonstrating how singularities in the initial data appear in the solutions of the *LME* and *RLME*, and calculating rates of decay for those singularities as $t \to \infty$. This is achieved for the *RLME* in Section 5 by reformulating it as an integral equation. Then it is shown in Section 6 that solutions of the *RLME* converge to a solution of the (dual form of the) *LME* as the regularizing parameter $h \to 0$. From this is derived a decay rate for singularities in the solution of the *LME*.

2. The Regularized Layered Medium Equation.

We consider the voltage distribution in a cylindrical domain, $\Omega = D \times I$, where $D \subset \mathbb{R}^2$ is bounded and convex and $I = [0, Z] \subset \mathbb{R}^1$. Points of D will be denoted by $\vec{x} = (x_1, x_2)$, while the variable z is reserved to indicate the vertical axis. This domain is filled with thin, alternating, horizontal layers of two conducting and dielectric materials. We model this multilayered structure as a continuum of *microcapacitor* cells: we assume that at each point $(\vec{x}, z) \in \Omega$ there is a cell consisting of a layer of conductive material and a layer of dielectric material, both horizontally aligned, i.e., normal to the z-axis, with a resistive interface between them. Each such cell functions as a capacitor and a resistor in series, as shown in Figure 1, with the horizontal direction represented as one-dimensional.

Let $u(\vec{x}, z, t)$ represent the voltage distribution inside Ω, measured with respect to the base of Ω. Let $\vec{J}_H$ and $\vec{J}_V$ represent respectively the horizontal and vertical components of the current field in Ω, induced by the voltage gradient. Then

$$\vec{\nabla} \cdot (\vec{J}_H + \vec{J}_V) = F(\vec{x}, z, t) \tag{2.1}$$

where F represents any sources of current. By the classical Ohm's Law

$$\vec{J}_H = -G_H(\vec{x})\vec{\nabla}_{\vec{x}}u, \tag{2.2}$$

where G_H is the spatially distributed, horizontal conductance of the conductive layer, and $\vec{\nabla}_{\vec{x}} = (\partial_{x_1}, \partial_{x_2})$ is the gradient in the horizontal direction. In the vertical direction, $\partial_z u$ is the voltage drop across the entire cell, and we introduce $v(\vec{x}, z, t)$ to represent the voltage drop across the purely capacitive component of the cell, as shown in Figure 1. If G_V is the spatially distributed vertical conductance (modeling leakage through the dielectric layers) and g the conductance of the interface between layers, represented as the resistance in Figure 1, then Kirchoff's Laws give us that

$$\vec{J}_V = \big(-G_V(\vec{x})\partial_z u - g(\vec{x})(\partial_z u - v)\big)\vec{e}_3. \tag{2.3}$$

The new variable, v, is related to u_z via the vertical capacitance effect in the cell:

$$C(\vec{x})\partial_t v = g(\vec{x})(\partial_z u - v). \tag{2.4}$$

Combining (2.1)–(2.3) and using (2.4) to eliminate v, we get:

$$\frac{\partial}{\partial t}\big[-C(\vec{x})\partial_z^2 u - h(\vec{x})\vec{\nabla}\cdot G(\vec{x})\vec{\nabla}u\big] - \vec{\nabla}\cdot G(\vec{x})\vec{\nabla}u = (I + h(\vec{x})\partial_t)F \tag{2.5}$$

where I represents the identity operator, $h(\vec{x}) = C(\vec{x})/g(\vec{x})$ and we have written:

$$\vec{\nabla}\cdot G(\vec{x})\vec{\nabla} \equiv \vec{\nabla}_{\vec{x}}\cdot G_H(\vec{x})\vec{\nabla}_{\vec{x}} + G_V(\vec{x})\partial_z^2.$$

The equation (2.5) is what we call the *Regularized Layered Medium Equation (RLME)* with regularizing parameter h. Using the same notation, the *Layered Medium Equation* of Bosse and Showalter[2] can be written

$$\frac{\partial}{\partial t}\big[-C(\vec{x})\partial_z^2 u\big] - \vec{\nabla}\cdot G(\vec{x})\vec{\nabla}u = F, \tag{2.6}$$

and we note that, formally at least, the *LME* is recovered from the *RLME* as $h \to 0$ (i.e., as the interface resistance, $\frac{1}{g} \to 0$).

For boundary conditions, we shall assume that the boundary of Ω is divided into complementary parts, Γ_0 and Γ_1, with $\Gamma_0 = D \times \{0\}$, the base of the cylindrical domain, and shall prescribe the voltage on Γ_0 and the outward normal component of current on Γ_1:

$$\begin{aligned} u(s) &= 0, && s \in \Gamma_0 \\ G(s)\vec{\nabla}u(s)\cdot\vec{n}(s) &= J(s), && s \in \Gamma_1,\ t > 0, \end{aligned} \tag{2.7}$$

where $\vec{n}(s)$ denotes the unit outward normal. The initial charge distribution in the multilayered structure is prescribed by

$$C(\vec{x})u(\vec{x},z,0) = C(\vec{x})u_0(\vec{x},z), \quad (\vec{x},z) \in \Omega. \tag{2.8}$$

3. Abstract Variational Formulation.

Next we develop the variational statement of the *LME* and *RLME* and demonstrate that both are examples of implicit evolution equations. With D, I, $\Omega = D \times I$ and Γ_0 as in Section 2, denote by $H^1(\Omega)$ the Sobolev space of functions in $L^2(\Omega)$ for which each first order distributional derivative belongs to $L^2(\Omega)$. The trace operator, $\gamma : H^1(\Omega) \to H^{\frac{1}{2}}(\partial\Omega)$, extends the notion of boundary values to all functions in $H^1(\Omega)$. For details see Adams[1]. Define the spaces

$$\begin{aligned}
&V_A \equiv \{v \in H^1(\Omega) : \gamma v = 0 \text{ on } \Gamma_0\}\\
&V_B \equiv \{v \in L^2(\Omega) : \partial_z v \in L^2(\Omega) \text{ and } v(\vec{x},0) = 0 \text{ for a.e. } \vec{x} \in D\}\\
&V_C \equiv \{v \in L^2(\Omega) : \partial_{x_j} v \in L^2(\Omega),\ j = 1,2\}.
\end{aligned}$$

These are Hilbert spaces with inner products given respectively by

$$\begin{aligned}
&(u,v)_A \equiv (u,v)_{H^1(\Omega)}\\
&(u,v)_B \equiv (u,v)_{L^2(\Omega)} + (\partial_z u, \partial_z v)_{L^2(\Omega)}\\
&(u,v)_C \equiv (u,v)_{L^2(\Omega)} + (\partial_{x_1} u, \partial_{x_1} v)_{L^2(\Omega)} + (\partial_{x_2} u, \partial_{x_2} v)_{L^2(\Omega)}.
\end{aligned}$$

Finally, we let $H \equiv L^2(\Omega)$ and write the L^2 inner product as $(\ \ ,\ \ )_H$.

Note that $V \equiv \{v \in L^2(\Omega) : \partial_z v \in L^2(\Omega)\} \cong L^2(D, H^1(I))$, the space of Bochner integrable functions $v : D \to H^1(I)$ such that $\int_D \|v(\vec{x})\|^2_{H^1(I)}\, d\vec{x} < \infty$. (For details of the Bochner integral see Wloka[14].) The functions in $H^1(I)$ are uniformly continuous, and so the boundary condition incorporated in the definition of V_B is meaningful.

The above spaces form a hierarchy

$$\begin{tikzcd}
 & V_B \arrow[dr] & \\
V_A \arrow[ur] \arrow[dr] & & H \\
 & V_C \arrow[ur] &
\end{tikzcd}$$

where each of the embeddings is continuous and dense. We define continuous linear operators $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ from the spaces V_A, V_B and V_C into their respective duals V_A', V_B' and V_C' by:

$$\begin{aligned}
\mathcal{A}u(v) &\equiv \int_\Omega \left\{ \left(G_H(\vec{x})\vec{\nabla}_{\vec{x}}u\right)\cdot\vec{\nabla}_{\vec{x}}v + G_V(\vec{x})\partial_z u \partial_z v \right\} d\vec{x}\,dz, && u,v \in V_A\\
\mathcal{B}u(v) &\equiv \int_\Omega C(\vec{x})\partial_z u \partial_z v \, d\vec{x}\,dz, && u,v\in V_B\\
\mathcal{C}u(v) &\equiv \int_\Omega \left(G_H(\vec{x})\vec{\nabla}_{\vec{x}}u\right)\cdot\vec{\nabla}_{\vec{x}}v \, d\vec{x}\,dz, && u,v\in V_C
\end{aligned}$$

We shall assume that $C, G_H, G_V \in C^{0,1}(\bar{D})$ and that each is uniformly positive. The above operators are symmetric and, by Poincaré's Inequality, $\mathcal{A}$ and $\mathcal{B}$ are also *coercive*, i.e.,

$$\begin{aligned}
\mathcal{A}v(v) &\geq k_a\|v\|_A^2, \quad \forall v\in V_A\\
\mathcal{B}v(v) &\geq k_b\|v\|_B^2, \quad \forall v\in V_B
\end{aligned}$$

where k_a, $k_b > 0$, while $\mathcal{C}$ satisfies

$$\mathcal{C}v(v) + \lambda\|v\|_H^2 \geq k_c\|v\|_C^2, \quad \forall v\in V_C$$

for some $\lambda > 0$ and $k_c > 0$. It follows from the Lax-Millgram Theorem (e.g. Showalter[10], p.54) that $\mathcal{A}$ and $\mathcal{B}$ have continuous inverses.

With each of these variational operators we also associate a strong (unbounded) operator in H as follows. Define a subspace $D(A)$ of V_A by

$$D(A) \equiv \{u\in V_A : \exists K_u : |\mathcal{A}u(v)| \leq K_u\|v\|_H, \forall v\in V_A\}.$$

If $u \in D(A)$, then $\mathcal{A}u$ is a linear functional on V_A and is continuous in the H-norm. Since V_A is dense in H, $\mathcal{A}u$ has a unique extension to a continuous linear functional on H, and this functional is represented uniquely by an element $Au \in H$. Thus

$$\mathcal{A}u(v) = (Au, v)_H, \quad \forall u\in D(A),\, v\in V_A.$$

This determines an unbounded linear operator, $A : D(A) \to H$, and we similarly define $B : D(B) \to H$ and $C : D(C) \to H$. A and B are bijections with continuous inverses and so are closed operators in H. C, while not a bijection, is a closed operator in H, and all three operators are self-adjoint with domains dense in H. These conclusions follow from standard results (see Showalter[10], p.77 for example). In each case, the unbounded operator is continuous in the corresponding *graph norm* defined by

$$\|u\|_{D(A)} \equiv \|u\|_A + \|Au\|_H.$$

For the particular operators that we are considering, if we assume that the spatial domain, $D \subset \mathbb{R}^2$, is convex and that the coefficients $C, G_H, G_V \in C^{0,1}(\bar{D})$ are independent of z, then we can use regularity theory (Grisvard[8]) to show that the abstract subspaces and operators defined above can be characterized as follows:

$$\begin{aligned}
&D(A) = \{v \in V_A : v \in H^2(\Omega) \text{ and } (G\vec{\nabla}v) \cdot \vec{n} = 0 \text{ on } \Gamma_1\} \\
&Av = -\vec{\nabla}_{\vec{x}} \cdot (G_H(\vec{x})\vec{\nabla}_{\vec{x}}v) - G_V(\vec{x})\partial_z^2 v \\
&D(B) = \{v \in V_B : v(\vec{x}, \cdot) \in H^2(I) \text{ and } \partial_z v(\vec{x}, Z) = 0 \text{ for a.e. } \vec{x} \in D\} \\
&Bv = -C(\vec{x})\partial_z^2 v \\
&D(C) = \{v \in V_C : v(\cdot, z) \in H^2(D) \text{ and } (G_H\vec{\nabla}_{\vec{x}}v) \cdot \vec{n} = 0 \text{ on } \partial D \text{ for a.e. } z \in I\} \\
&Cv = -\vec{\nabla}_{\vec{x}} \cdot (G_H(\vec{x})\vec{\nabla}_{\vec{x}}v)
\end{aligned}$$

where $\vec{n}$ denotes the unit outward normal. It again follows that these subspaces form a hierarchy of dense, continuous embeddings:

$$\begin{array}{ccccc}
 & & D(B) & & \\
 & \nearrow & & \searrow & \\
D(A) & & & & H \\
 & \searrow & & \nearrow & \\
 & & D(C) & &
\end{array}$$

It is important in the following to note the regularity properties of functions in these spaces. By the Sobolev Embedding Theorem, we have the embedding $H^2(\Omega) \hookrightarrow C_u^0$, the space of uniformly continuous functions on Ω. Thus each $v \in D(A)$ is continuous in $(\vec{x}, z)$. On the other hand, $D(B) \subset L^2(D, H^2(I))$ and hence if $v \in D(B)$ then v is continuous in z but only L^2 with respect to $\vec{x} = (x_1, x_2)$. Similarly, each $v \in D(C)$ is continuous in $\vec{x}$ but only L^2 in z.

We now have the background to formulate the *LME* and *RLME* as implicit evolution equations. With the above notation, the strong form of the *initial boundary value problem* for the *LME* becomes

$$\begin{aligned}
B\frac{du}{dt} + Au(t) = F(t) \qquad \text{in } H \\
u(0) = u_0,
\end{aligned} \tag{3.1}$$

while the strong form of the corresponding problem for the *RLME* is

$$\begin{aligned}(B+hA)\frac{du^h}{dt}+Au^h(t)&=F^h(t) \qquad \text{in } H\\ u^h(0)&=u_0,\end{aligned} \tag{3.2}$$

where $h=C(\vec{x})/g(\vec{x})>0$ and $F^h(t)=(I+h\frac{d}{dt})F(t)$. Moreover, the operator A has the special structure, $A=bB+C$ where $b=G_V(\vec{x})/C(\vec{x})$. This structure is crucial for our analysis of the asymptotic behavior of solutions of these problems. For these equations to be appropriately well-posed, it is necessary that the operator A be B-positive, i.e. that

$$(A\varphi,B\varphi)_H\geq 0, \qquad \forall\varphi\in D(A),$$

and we shall also need to assume that A is B-symmetric, i.e. that

$$(A\varphi,B\psi)_H=(B\varphi,A\psi)_H, \qquad \forall\varphi,\psi\in D(A).$$

We now show that both of these conditions are satisfied by the operators defined above, provided that the distributed capacitance, $C(\vec{x})$, and the leakage conductance, $G_V(\vec{x})$, are constant, which we assume in the sequel.

Firstly, a straightforward integration by parts proves

Lemma 3.1. *For any $\varphi\in D(A)\cap C^\infty(\bar{\Omega})$ and $v\in D(B)$,*

$$(A\varphi,Bv)_{L^2(\Omega)}=\int_\Omega(G\vec{\nabla}\partial_z\varphi)\cdot(C\vec{\nabla}\partial_z v)\,d\vec{x}dz.$$

If D has a C^∞ boundary or is rectangular, then we also have

Lemma 3.2. $A\left(C^\infty(\bar{\Omega})\cap D(A)\right)$ *is dense in $H=L^2(\Omega)$.*

Proof. Ω has the cone property: there is a finite cone, K, such that each point in Ω is the vertex of a finite cone congruent to K and contained in Ω. By the Rellich-Kondrachev Theorem (see Adams[1], p.144), since $D\subset\mathbb{R}^2$, $I\subset\mathbb{R}^1$ and both have the cone property, the embeddings of $H^1(I)$ in $L^2(I)$ and $H^1(D)$ in $L^2(D)$ are compact. We may treat B, C as operators in $H^1(I)$, $H^1(D)$ respectively. It is immediate that B is coercive over $\{v\in H^1(I): v(0)=0\}$, while $C+\lambda I$ is coercive over $H^1(D)$ for some $\lambda>0$. Since they are also symmetric, they both have complete sets of eigenfunctions:

$$Cu_i=\lambda_i u_i, \qquad Bv_j=\mu_j v_j$$

with $u_i\in C^\infty(\bar{D})$ and $v_j\in C^\infty(\bar{I})$. Then $\{u_iv_j\}$ constitutes a complete set of eigenfunctions for $A=bB+C$ (b constant). Thus, every function in $L^2(\Omega)$ can be approximated by a linear combination of the u_iv_j's, and hence of the Au_iv_j's. $\square$

Proposition 3.3. *If the coefficients C and G_V are constant, then A is B-positive and B-symmetric.*

Both results follow easily by approximating Au, where $u \in D(A) \hookrightarrow D(B)$, by $A\varphi_n$, where $\varphi_n \in D(A) \cap C^\infty(\bar{\Omega})$, and then invoking Lemma 3.1.

4. The *LME* and the *RLME* are Well-posed.

With the *LME* and *RLME* both expressed as implicit evolution equations, we can apply abstract existence and uniqueness results from Showalter[10,11,12] or Carroll and Showalter[3] to show that both are well-posed in a number of different spaces.

Theorem. *Let V, W and H be Hilbert spaces with $V \hookrightarrow W \hookrightarrow H$, where we identify H with H', and let $\mathcal{L} : V \to V'$, $\mathcal{M} : W \to W'$ be continuous linear operators. Define $D(L) \equiv \{v \in V : \mathcal{L}v \in H\}$ and $D(M)$ similarly, with $L = \mathcal{L}|D(L)$, $M = \mathcal{M}|D(M)$. Assume that $\mathcal{L}$ and $\mathcal{M}$ are coercive on V and W respectively, that $D(L) \subset D(M)$ and that*

$$(L\varphi, M\varphi)_H \geq 0, \qquad \forall \varphi \in D(L).$$

Then, for each $u_0 \in D(M)$ and $f \in C^\lambda(0,T;H)$ with $0 < \lambda < 1$, there is a unique solution, u, of

$$\begin{aligned} M\frac{du}{dt} + Lu(t) &= f(t) \qquad \text{in } H \\ u(0) &= u_0 \end{aligned} \tag{4.1}$$

such that $u \in C([0,T]; D(M)) \cap C^1((0,T]; D(M))$ and $u(t) \in D(L)$, $\quad \forall t > 0$.

With $L \equiv A$ and $M \equiv B$, this result shows immediately that the strong form of the *LME* is well-posed. The same theorem, with $L \equiv A$ and $M \equiv B + hA$, also shows that the *RLME* is well-posed. These results can be summarized as:

	Initial Data	Solution
LME	$u_0 \in D(B)$	$u(t) \in D(A)$
RLME	$u_0 \in D(B + hA) \subset D(A)$	$u(t) \in D(A)$

Recalling the characterizations of the spaces and operators from Section 3, we see that for the *LME* if our initial data, u_0, is in $L^2(\Omega)$ with $\partial_z^2 u_0 \in L^2(\Omega)$, then we get a solution $u(t) \in H^2(\Omega)$ for all $t > 0$. Thus the *LME* is regularizing (parabolic) in $\vec{x}$, while preserving regularity in z. In contrast to the *LME*, the *RLME* exactly preserves regularity in all variables: with data u_0 in $H^2(\Omega)$, the solution also takes values in $H^2(\Omega)$. This is intuitively to be expected, since the perturbed operator $B + hA$ and the operator A both contain exactly the same derivatives.

Our aim is to characterize more precisely the regularity-preserving properties of these two equations by calculating an exact rate of decay of those singularities in the data which are preserved in the solutions of the *LME* and the *RLME*. However, the interpretation of the result of this calculation for the *LME* requires a more general notion of solution than the strong solutions provided by the above theorem. We now prove a generalization of this theorem which shows that the dual of (4.1):

$$\begin{aligned} B' \frac{du}{dt} + A' u(t) &= f(t) \qquad \text{in } D(B)' \\ u(0) &= u_0. \end{aligned} \tag{4.2}$$

is well-posed. As usual, $D(B)'$ denotes the space of continuous linear functionals on $D(B)$, and the dual operators $A' : H \to D(A)'$, $B' : H \to D(B)'$ are defined by, e.g.,

$$\langle A'v, w \rangle_{D(A)', D(A)} = (v, Aw)_H$$

for $v \in H$, $w \in D(A)$, where $\langle \quad , \quad \rangle_{D(A)', D(A)}$ denotes the $D(A)'$–$D(A)$ duality pairing. Since $D(A)$ is dense and continuously embedded in $D(B)$, the same is true of $D(B)'$ in $D(A)'$. Recall that A and B are continuous (in the graph norm) bijections, and hence so are A' and B'.

Lemma 4.1. *For A, B as given above, if A is B-symmetric then* $(A')^{-1} (D(B)') \subset B (D(A))$.

Proof. Since A' is a bijection, this is equivalent to $D(B)' \subset A' (B (D(A)))$. Now

$$\begin{aligned} & u \in A' (B (D(A))) \\ \iff & \exists w \in D(A) : \, u = A'Bw \text{ in } D(A)' \\ \iff & \exists w \in D(A) : \, \forall v \in D(A), \langle u, v \rangle_{D(A)', D(A)} = (Bw, Av)_H. \end{aligned}$$

Since $A : D(A) \to H$ and $B' : H \to D(B)'$ are bijections, given any $u \in D(B)'$, there is $w \in D(A) : \, u = B'Aw$. Thus, for any $v \in D(A)$,

$$\langle u, v \rangle_{D(A)', D(A)} = \langle B'Aw, v \rangle_{D(A)', D(A)} = (Aw, Bv)_H = (Bw, Av)_H$$

since A is B-symmetric. Thus $u \in A'(B(D(A)))$. $\square$

For brevity write $D \equiv (A')^{-1}(D(B)')$. The preceding lemma shows that $D \subset B(D(A))$. We now prove the promised existence result.

Theorem 4.2. *Let $V \hookrightarrow W \hookrightarrow H$ be Hilbert spaces, where '$\hookrightarrow$' denotes a dense and continuous embedding. We identify H with H', and let $\mathcal{A} : V \to V'$, $\mathcal{B} : W \to W'$ be continuous linear operators. Define $A = \mathcal{A}|D(A)$, $B = \mathcal{B}|D(B)$ where $D(A) = \{u \in V : \mathcal{A}u \in H\}$ and $D(B)$ is defined similarly. Assume that $\mathcal{A}$ and $\mathcal{B}$ are coercive, that $D(A)$ is a dense subset of $D(B)$, and that A is B-positive and B-symmetric, i.e.,*

$$(A\varphi, B\varphi)_H \geq 0 \text{ and } (A\varphi, B\psi)_H = (B\varphi, A\psi)_H, \quad \forall \varphi, \psi \in D(A).$$

Then, for each $u_0 \in D$ and $f \in C^1(0, T; D(B)')$, there is a unique solution, $u \in C^1(0, T; H)$, of the Cauchy problem (4.2). Moreover, $u(t) \in D, \forall t \geq 0$.

Proof. Recall that the graph norm on $D(A)$ is $\|u\|_{D(A)} = \|u\|_V + \|Au\|_H$. Since $\mathcal{A}$ is coercive, A is closed and $D(A)$ is complete in the graph norm and dense in H. Similar remarks hold for B and $D(B)$. Suppose that $u_n \to u$ in $D(A)$ and $u_n \to v$ in $D(B)$. Then $u_n \to u$ in V while $u_n \to v$ in W. Since $V \hookrightarrow W$, $u = v$ and the identity map $i : D(A) \to D(B)$ is closed, therefore continuous (by the Closed Graph Theorem), and $D(A) \hookrightarrow D(B)$. It follows that $D(B)' \hookrightarrow D(A)'$.

We solve the Cauchy problem, (4.2), in the equivalent form

$$\frac{du}{dt} + (B')^{-1}A'u(t) = (B')^{-1}f(t).$$

Let $M' = (B')^{-1}A' : D \to H$. We show that M' is m-accretive, from which the result follows by standard theorems on semigroups (see Goldstein[7], for example).

M' is accretive on D. Indeed, if $v \in D$ then $v = Bw$ for some $w \in D(A)$ by Lemma (4.1). Thus $(M'v, v)_H = \big((B')^{-1}A'v, v\big)_H = \langle A'v, B^{-1}v\rangle_{D(B)',D(B)}$ $= \langle A'v, w\rangle_{D(A)',D(A)} = (v, Aw)_H = (Bw, Aw)_H \geq 0$, since A is B-positive.

Moreover M' is closed, since it is the inverse of the continuous map

$$(M')^{-1} = (A')^{-1}B' = (A')^{-1} \circ i' \circ B'$$

where each of the maps

$$H \xrightarrow{B'} D(B)' \xrightarrow{i'} D(A)' \xrightarrow{(A')^{-1}} H$$

is continuous. Since M' is accretive, closed and surjective (as A', B' are bijections), it is m-accretive, and the theorem is proved. $\square$

The notion of solution in this case is

$$(\frac{du}{dt}, B\varphi)_H + (u(t), A\varphi)_H = \langle f(t), \varphi\rangle_{D(B)',D(B)}$$

for $t \geq 0$ and $\varphi \in D(A) \subset D(B)$. For the operators A and B of the *RLME*, for which we have the special structure, $A = bB + C$, we can simplify the condition $u_0 \in D = (A')^{-1}(D(B)')$ to $u_0 \in D(C)$, as shown by the following lemma.

Lemma 4.3. *A' maps $D(C)$ into $D(B)'$.*

Proof. For $u \in D(C)$ and $v \in D(A)$ we have

$$\begin{aligned}
|\langle A'u, v\rangle| &= |(u, Av)_H| \\
&\leq |(u, bBv)_H| + |(Cu, v)_H| \\
&\leq b\|u\|_H\|v\|_{D(B)} + k\|Cu\|_H\|v||_{D(B)} \\
&\leq k\|u\|_{D(C)}\|v\|_{D(B)}.
\end{aligned}$$

Since $D(A)$ is dense in $D(B)$, $A'u$ has a unique extension to an element of $D(B)'$. $\square$

5. Regularity of Solutions of the *RLME*.

We now examine the evolution of singularities in the form of jump discontinuities in the initial data. From the results of Section 4, these will be preserved in the solution of the *RLME*, and we shall exhibit explicitly how they decay as $t \to \infty$. This will be achieved by recasting the problem as an integral equation, a form which is more revealing of the structure of the solutions. We work with the *RLME* in the strong form

$$\begin{gathered}
(B + hA)\frac{du^h}{dt} + Au^h(t) = F^h(t) \qquad \text{in } H \\
u^h(0) = u_0,
\end{gathered} \tag{5.1}$$

and, for the moment, we shall suppress the superscript, h. Since $B + hA$ is an isomorphism, this is equivalent to

$$\frac{du}{dt} + (B + hA)^{-1}Au(t) = (B + hA)^{-1}F(t) \tag{5.2}$$

and rewriting $(B+hA)^{-1}A$ as $\dfrac{1}{h}\left(I-(B+hA)^{-1}B\right)$ and rearranging terms, we can express (5.2) as:

$$\frac{du}{dt}+\frac{1}{h}u(t)=(B+hA)^{-1}\left\{\frac{1}{h}Bu(t)+F(t)\right\}, \tag{5.3}$$

an ordinary differential equation which is equivalent to the integral equation in Hilbert space

$$u(t)=e^{-\frac{t}{h}}u_0+\int_0^t e^{-\frac{1}{h}(t-\tau)}(B+hA)^{-1}\left\{\frac{1}{h}Bu(\tau)+F(\tau)\right\}d\tau \tag{5.4}$$

in which $F \in C(0,T;H)$. This is of the form $u(t)=Su(t)$, where S is an operator on either $C(0,T;D(B))$ or $C(0,T;D(A))$ according as $u_0 \in D(B)$ or $D(A)$ respectively. An inductive argument shows that for n sufficiently large, S^n is a contraction, and so S has a unique fixed point. Thus with $u_0 \in D(B)$ and $F \in C(0,T;H)$, the Cauchy problem for (5.3) has a unique solution $u \in C^1(0,T;D(B))$ of the form

$$u(t)=e^{-\frac{t}{h}}u_0+u_2(t)$$

where

$$u_2(t)=\int_0^t e^{-\frac{1}{h}(t-\tau)}(B+hA)^{-1}\left\{\frac{1}{h}Bu(\tau)+F(\tau)\right\}d\tau.$$

Thus $u_2 \in C^1(0,T;D(A))$, since $B: D(B)\to H$ while $(B+hA)^{-1}: H \to D(A)$. Since $D(A)\subset H^2(\Omega)$, $u_2(t)$ is continuous in $(\vec{x},z)$. On the other hand, $D(B)\subset L^2(D,H^2(I))$, and so the initial data, u_0, and hence also the solution, $u(t)$, while continuous in z, are only L^2 with respect to $\vec{x}=(x_1,x_2)$. Denoting the jump in, for example, the x_1-direction by σ_{x_1}, we conclude that

$$\sigma_{x_1}(u(t))=e^{-\frac{t}{h}}\sigma_{x_1}(u_0),$$

giving an explicit decay rate for singularities in u_0 with respect to x_1 (or x_2) as $t\to\infty$.

We would similarly like to analyze the decay of singularities with respect to z. This requires that we examine solutions of the *RLME* with initial data $u_0 \in D(C)\subset L^2(I,H^2(D))$. Taking account of the structure of A, namely $A=bB+C$, $b>0$, we can make the substitution

$$A=\frac{b}{1+hb}(B+hA)+\frac{1}{1+hb}C$$

in (5.2), rewriting it as

$$\frac{du}{dt} + \frac{1}{1+hb}\left(bI + (B+hA)^{-1}C\right)u(t) = (B+hA)^{-1}F(t) \tag{5.5}$$

which can be recast as the equivalent integral equation

$$u(t) = e^{-\frac{bt}{1+hb}}u_o + \int_0^t e^{-\frac{b}{1+hb}(t-\tau)}(B+hA)^{-1}\left\{F(\tau) - \frac{1}{1+hb}Cu(\tau)\right\}d\tau. \tag{5.6}$$

As before, with $u_0 \in D(C)$ and $F \in C(0,T;H)$, a fixed point argument yields a unique solution $u \in C^1(0,T;D(C))$ of (5.6) and hence of the Cauchy problem for (5.5). This solution has the form

$$u(t) = u_1(t) + u_2(t)$$

where

$$u_1(t) = e^{-\frac{bt}{1+hb}}u_0 \tag{5.7}$$

and

$$u_2(t) = \int_0^t e^{-\frac{b}{1+hb}(t-\tau)}(B+hA)^{-1}\left\{F(\tau) - \frac{1}{1+hb}Cu(\tau)\right\}d\tau \tag{5.8}$$

so that $u_1 \in C^\infty(0,T;D(C))$, preserving singularities in the z-direction, while $u_2 \in C^1(0,T;D(A))$ and is uniformly continuous in $(\vec{x},z)$. We now have in

$$\sigma_z\left(u(t)\right) = e^{-\left(\frac{bt}{1+hb}\right)}\sigma_z(u_0)$$

an explicit decay rate for z-singularities in the initial data.

6. Regularity of Solutions of the *LME*.

We previously observed that the *LME* is regularizing in $\vec{x}$ but preserves regularity in z. In order to establish for the *LME* a measure of the decay rate of z-discontinuities in the initial data similar to that derived in Section 5 for the *RLME*, we shall work with $u_0 \in D(C) \subset L^2(I,H^2(D))$. It is not possible to proceed directly, as in Section 5, for the corresponding integral formulation of the *LME* is

$$u(t) = e^{-bt}u_0 + \int_0^t e^{-b(t-\tau)}B^{-1}\left\{F(\tau) - Cu(\tau)\right\}d\tau,$$

and this does not define an operator on $C(0,T;D(C))$, as required for the fixed-point argument that we gave above. The obstacle is the term $B^{-1}C$, which maps $D(C) \to D(B) \not\subset D(C)$. Replacing $B^{-1}C$ by the regularized term $(B+hA)^{-1}C : D(C) \to D(A) \subset D(C)$ in the *RLME* avoids this difficulty. For the *LME* we shall calculate a decay rate for singularities by obtaining a solution of the *LME* as a limit of solutions of the *RLME* as $h \to 0$.

We start with the solution

$$u^h(t) = u_1^h(t) + u_2^h(t)$$

of the *RLME* with data $u_0 \in D(C)$ and $F^h \in C(0,T;H)$, where u_1^h and u_2^h are given by (5.7) and (5.8). We note that $u_1^h(t) = e^{-\frac{bt}{1+hb}}\, u_0$ solves:

$$\begin{aligned} \frac{du_1^h}{dt} + \frac{b}{1+hb} u_1^h(t) &= 0 \\ u_1^h(0) &= u_0 \end{aligned} \tag{6.1}$$

and clearly $u_1^h \to u_1$ in $L^2(0,T;H)$, where $u_1(t) = e^{-bt}u_0$. On the other hand, by reversing the manipulations which led to the integral equation form of the *RLME*, we see that $u_2^h(t)$ solves:

$$\begin{aligned} (B+hA)\frac{du_2^h}{dt} + Au_2^h(t) &= G^h(t) \qquad \text{in } H \\ u_2^h(0) &= 0 \end{aligned} \tag{6.2}$$

where

$$G^h(t) = F^h(t) - \frac{1}{1+hb} C u_1^h(t)$$

and

$$F^h(t) = F(t) + h\frac{dF}{dt}.$$

We shall assume that $F \in C^1(0,T;H)$. It follows that $G^h \to G$ in $C(0,T;H)$ and $L^2(0,T;H)$ where $G(t) = F(t) - e^{-bt}Cu_0$.

We need several estimates on $\{u_2^h\}$. To simplify the notation, we shall suppress subscripts and superscripts, except on G^h. Let $0 \le s \le T$. Since A is self-adjoint and coercive,

$$\int_0^s \left(Au(t), \frac{du}{dt}\right)_H dt = \frac{1}{2}\left(Au(s), u(s)\right)_H \ge \frac{1}{2} k_a \|u(s)\|_A^2.$$

Thus, if we take the inner product of (6.2) with $\frac{du}{dt}$ and integrate $\int_0^s$, then using the coercivity of A and B and an application of Young's Inequality we get

$$k_b \left\| \frac{du}{dt} \right\|^2_{L^2(0,s;V_B)} + \frac{1}{2} k_a \|u(s)\|_A^2 \le \frac{1}{2} k_b^{-1} \left\| G^h \right\|^2_{L^2(0,T;H)} + \frac{1}{2} k_b \left\| \frac{du}{dt} \right\|^2_{L^2(0,s;V_B)}$$

where we have also exploited the embedding of V_B in H. This establishes the following:

(B1) $\left\{ \frac{du_2^h}{dt} \right\}$ is bounded in $L^2(0,T;V_B)$;

(B2) $\{u_2^h\}$ is bounded in $L^\infty(0,T;V_A)$ and hence in $L^2(0,T;V_A)$.

Since A is B-symmetric, we note that

$$\frac{d}{dt} \left(Bu(t), Au(t)\right)_H = 2\big(B\frac{du}{dt}, Au(t)\big)_H$$

and so:

$$\int_0^s \left((B + hA)\frac{du}{dt}, Au(t) \right)_H dt = \frac{1}{2} \left(Bu(s), Au(s)\right)_H + \frac{1}{2} h \|Au(s)\|_H^2 \ge 0,$$

as A is also B-positive. Thus, taking the inner product of (6.2) with $Au(t)$ and integrating $\int_0^T$ yields the following:

(B3) $\{Au_2^h\}$ is bounded in $L^2(0,T;H)$,

and hence from **(B2)**:

(B4) $\{u_h^2\}$ is bounded in $L^2(0,T;D(A))$.

Finally, if we take the inner product of (6.2) with $B\frac{du}{dt}$ and again use the B-positivity and B-symmetry of A, then we find that

(B5) $\left\{ B\frac{du_2^h}{dt} \right\}$ is bounded in $L^2(0,T;H)$,

which combines with **(B1)** to give

(B6) $\left\{ \frac{du_2^h}{dt} \right\}$ is bounded in $L^2(0,T;D(B))$.

We conclude from **(B4)** and **(B6)** that we may choose a subsequence, which we still denote by $\{u_2^h\}$, such that

$$u_2^h \rightharpoonup u_2 \text{ in } L^2(0,T;D(A)), \tag{6.3}$$

$$\frac{du_2^h}{dt} \rightharpoonup \frac{du_2}{dt} \text{ in } L^2(0,T;D(B)), \tag{6.4}$$

and

$$Au_2^h \rightharpoonup Au_2 \quad \text{and} \quad B\frac{du_2^h}{dt} \rightharpoonup B\frac{du_2}{dt} \tag{6.5}$$

in $L^2(0,T;H)$ (and hence in $L^2(0,s;H)$ for $s \in [0,T]$). To complete the convergence argument we also need to verify that the initial condition, $u_2^h(0) = 0$, is preserved in the limit.

Lemma. *Suppose that H is a Hilbert space, $\{v_\lambda\} \subset C^1(0,T;H)$, and that $v_\lambda \rightharpoonup v$, $\dfrac{dv_\lambda}{dt} \rightharpoonup \dfrac{dv}{dt}$ in $L^2(0,T;H)$ as $\lambda \to 0$. If $v_\lambda(0) = 0$, $\forall \lambda > 0$, then $v(0) = 0$.*

Proof. It follows from the representation

$$(v_\lambda(s), \xi)_H = \int_0^s \left(\frac{dv_\lambda}{dt}(t), \xi \right)_H dt$$

for $\xi \in H, s \in [0,T]$, that

$$v_\lambda(s) \rightharpoonup v(s) - v(0) \text{ in } H \tag{6.6}$$

for $s \in [0,T]$. With $\xi \in H$ we also have, using an integration by parts, that

$$\begin{aligned} \int_0^T \left(\frac{dv_\lambda}{dt}, t\xi \right)_H dt &= \left(T v_\lambda(T) - \int_0^T v_\lambda(t)\, dt, \xi \right)_H \\ &\to (v(T) - v(0), T\xi)_H - \int_0^T (v(t), \xi)_H \, dt. \end{aligned}$$

On the other hand, since $t\xi \in L^2(0,T;H)$, if we first take limits in $\int_0^T \left(\frac{dv_\lambda}{dt}, t\xi \right)_H dt$ and then integrate by parts we see that

$$\int_0^T \left(\frac{dv_\lambda}{dt}, t\xi \right)_H dt \to (v(T), T\xi)_H - \int_0^T (v(t), \xi)_H \, dt$$

whence $v(0) = 0$. $\square$

Recalling that $u_2^h \in C^1(0,T;D(A))$ and u_2 is AC, we conclude that $u_2(0) = 0$.

We can now conclude the convergence argument to obtain the following.

Theorem. *Suppose that $\{u_2^h\} \subset C^1(0,T;D(A))$ are solutions of the RLME*

$$\begin{aligned} \frac{d}{dt}(B + hA)u_2^h(t) + Au_2^h(t) = G^h(t) \text{ in } H \\ u_2^h(0) = 0 \end{aligned} \tag{6.7}$$

where A, B are as above and $G^h \to G$ in $L^2(0,T;H)$. Then a subsequence of $\{u_2^h\}$ converges weakly in $L^2(0,T;H)$ to an absolutely continuous $u_2 \in L^2(0,T;D(A))$ with $\dfrac{du_2}{dt} \in L^2(0,T;D(B))$, which is a solution of the LME in H

$$\begin{aligned} \frac{d}{dt}Bu_2(t) + Au_2(t) = G(t), \ \text{a.e. } t \in [0,T] \\ u_2(0) = 0 \end{aligned} \tag{6.8}$$

with $u_2(t) \in D(B)$ *for every* $t \in [0,T]$.

Proof. Let $\xi \in D(A)$. Taking the inner product with ξ in (6.7) and integrating, we get, for any $s \in [0,T]$:

$$\int_0^s \left((B+hA)\frac{du_2^h}{dt},\xi\right)_H dt + \int_0^s \left(Au_2^h(t),\xi\right)_H \, dt = \int_0^s \left(G^h(t),\xi\right)_H \, dt. \tag{6.9}$$

Then

$$\begin{aligned}\int_0^s \left((B+hA)\frac{du_2^h}{dt},\xi\right)_H dt &= \left(Bu_2^h(s),\xi\right)_H + h\left(Au_2^h(s),\xi\right)_H \\ &\to \left(Bu_2(s),\xi\right)_H\end{aligned}$$

by (6.6), using that A is self-adjoint and that $\{u_2^h\}$ is bounded in $L^2(0,T;H)$, from **(B2)**. Thus, taking limits in (6.9) gives

$$\left(Bu_2(s),\xi\right)_H + \int_0^s \left(Au_2(t),\xi\right)_H \, dt = \int_0^s \left(G(t),\xi\right)_H \, dt$$

for any $s \in [0,T]$ and $\xi \in D(A)$, which is dense in H, so (6.8) follows. That $u_2(t) \in D(B)$ follows from the representation, $u_2(t) = \int_0^t \frac{du_2}{ds}\,ds$. $\square$

To complete the asymptotic analysis we must reassemble $u_1(t) = e^{-bt}u_0 \in C^\infty(0,T;D(C))$ and $u_2(t)$ as given by the theorem. Let

$$u(t) = u_1(t) + u_2(t) \in L^2(0,T;D(C)).$$

Note that, since $u_2(t) \in D(A)$ and $\dfrac{du_2}{dt}(t) \in D(B)$, for a.e. t,

$$\begin{aligned}B'\frac{du}{dt} + A'u(t) &= B\frac{du_2}{dt} + Au_2(t) - be^{-bt}B'u_0 + e^{-bt}A'u_0 \\ &= F(t),\end{aligned}$$

recalling that $G(t) = F(t) - e^{-bt}Cu_0$. Since $u(t) \in D(C)$ for a.e. t, and $A' : D(C) \to D(B)'$ (Lemma 4.3), the equation holds in $D(B)'$ for a.e. t. Moreover, u is AC, $u_0 \in D(C) \subset (A')^{-1}(D(B)')$, and $F \in C^1(0,T;H) \subset C^1(0,T;D(B)')$. Thus u is the unique solution of the abstract Cauchy problem for the dual version of the *LME* (see Theorem 4.2). It follows that u, and hence also u_2, is in $C^1(0,T;H)$.

Thus the structure of the solution of the (dual of the) *LME* is

$$u(t) = e^{-bt}u_0 + u_2(t)$$

where $u_2(t) \in D(B) \subset \{v \in L^2(D \times I) : v(\vec{x}, \cdot) \in H^2(I)$ for a.e. $\vec{x} \in D\}$, and so $u_2(t, \vec{x}, \cdot)$ is continuous for a.e. $\vec{x} \in D$. On the other hand, the data $u_0 \in D(C) \subset \{v \in L^2(D \times I) : v(\cdot, z) \in H^2(D)$ for a.e. $z \in I\}$. Hence $u_0(\cdot, z)$ is continuous for a.e. $z \in I$, but may have jump discontinuities in z, which are preserved in $u(t)$. The above representation now tells us that such singularities decay at the rate e^{-bt}, where $b = G_V/C > 0$ is the ratio of the (average) conductance to capacitance of the dielectric layers.

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