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CAUCHY PROBLEM FOR HYPER-PARABOLIC PARTIAL DIFFERENTIAL EQUATIONS

R.E. Showalter

Department of Mathematics, RLM 8.100
University of Texas at Austin
Austin, Texas
U.S.A.

INTRODUCTION

We shall consider initial-boundary-value problems for partial differential equations of the form

(1) $$u_t = u_{xx} - u_{yy}.$$

Such equations arise in a discussion of classification. Specifically, every linear second-order constant-coefficient partial differential equation in n variables can be reduced to the form

$$\sum_{i=1}^{k} u_{x_i x_i} - \sum_{i=1}^{j} u_{y_i y_i} + au_t + bu = f(x,y,t,\ldots).$$

The general parabolic case is $k+j+1 = n$; we are interested in the non-normal case of $kj \neq 0$ which we call <u>hyper-parabolic</u>.

Equations of this type have arisen in diverse non-standard applications [1,4,7-11, 15], most of which require a solution subject to classical initial and boundary conditions on a space-time cylinder. However such a problem is not well posed but is "hyper-sensitive" to variations in the data [12]. It is clear that neither the initial-value nor the final-value problem is well posed for (1) and, moreover, the boundary-value problem for stationary solutions is ill posed.

Our plan is to develop some elementary notions of generalized solution of an abstract model of (1) as an evolution equation in Hilbert space. The Cauchy problem can be approximated by a <u>quasi-reversibility method</u> [5]. Then we present some well posed problems for this equation, and these results suggest a more natural method of approximating solutions of the ill posed Cauchy problem. This new approximation scheme we call the <u>quasi-boundary-value method</u>.

INITIAL-VALUE PROBLEM

Hereafter A and B denote self-adjoint non-negative operators on a Hilbert space H, and we assume their resolvents commute. Thus $-A$ generates a (holomorphic) semigroup of contractions $\{\exp(-At) : t \geq 0\}$ on H; their inverses are unbounded operators $\exp(At)$ which could also be obtained by the spectral theorem. If $u \in C^1$ is a solution of the evolution equation

$$u'(t) + Au(t) = 0, \quad \tau < t$$

then $\frac{d}{ds}\exp(-A(t-s))u(s) = 0$, hence, $\xi(t) \equiv \exp(-A(t-s))u(s)$, $\tau \leq s \leq t$, is independent of s. Thus one obtains the semi-group representation $u(t) = \exp(-A(t-s))u(s)$, $s \leq t$; the operators $\exp(-A(t-s))$ are the <u>propagators</u> for the evolution equation and their continuity implies the initial-value problem is well posed.

Proceeding similarly for the hyper-parabolic equation

$$u'(t) + Au(t) - Bu(t) = 0, \tag{2}$$

we see that if $u \in C^1$ is a solution on the interval $[\tau, t]$, then $\frac{d}{ds}\exp(-A(t-s))\exp(-B(s-\tau))u(s) = 0$, $\tau < s < t$, so

$$\xi(t,\tau) = \exp(-A(t-s))\exp(-B(s-\tau))u(s), \quad \tau \le s \le t \tag{3}$$

is independent of s. Thus, we are led to define a weak solution of (2) on $[\tau, t]$ as a continuous H-valued function for which (3) holds, i.e., the right side of (3) is independent of $s \in [\tau, t]$.

Lemma. If u is a weak solution on $[\tau, t]$ then it is a weak solution on each $[\tau_1, t_1] \subset [\tau, t]$ and then $\xi(t,\tau) = \exp(-A(t-t_1))\exp(-B(\tau_1-\tau))\xi(t_1,\tau_1)$, and $u(t) = \xi(t, t^-)$.

If u is a continuous H-valued function on $[0,1]$, then u is a weak solution iff $\exp(-Bt)u(t) = \exp(-At)u(0)$, $0 \le t \le 1$. Thus the initial-value problem of finding a weak solution of (2) on $[0,1]$ with $u(0) = f$ given in H is equivalent to

$$u \in C^0([0,1], H) \text{ with } \exp(-Bt)u(t) = \exp(-At)f, \quad 0 \le t \le 1.$$

Since each $\exp(-Bt)$ is one-to-one, there is at most one solution of the initial-value problem. Also, the representation via unbounded operators as $u(t) = \exp(Bt)\exp(-At)f$ shows the initial-value problem is not well posed. Considering existence, we see that if $f \in \mathrm{Rg}\{\exp(-B)\} = \mathrm{dom}\{\exp(B)\}$, then

$$u(t) \equiv \exp(-At) \cdot \exp(-B(1-t)) \cdot \exp(B)f$$

defines a strong solution (C^∞) of the initial-value problem. More generally we have the following

Proposition 1. There exists a weak solution of the initial-value problem if and only if $\exp(-A)f = \exp(-B)g$ for some $g \in H$, i.e., $f \in \mathrm{dom}\{\exp(B-A)\}$.

QUASI-REVERSIBILITY METHOD

Since the lack of well-posedness of the initial-value problem for (2) is due to the unboundedness of B, we use a Q-R method [3,5,13,14] to obtain an approximate solution. First replace B by its bounded Yosida approximation $B_\varepsilon \equiv B(I+\varepsilon B)^{-1}$, $\varepsilon > 0$, and solve the equation (2) for $\exp((B_\varepsilon - A)t)f$, $0 \le t \le 1$. The final-value $\exp(B_\varepsilon - A)f$ belongs to $\mathrm{Rg}\{\exp(-A)\}$ so we obtain a (strong) solution of (2) backward from here,

$$\begin{aligned} u_\varepsilon(t) &= \exp((A-B)(1-t))\exp(B_\varepsilon - A)f \\ &= \exp(-At)\exp(-B(1-t))\exp(B_\varepsilon)f, \quad 0 \le t \le 1, \end{aligned}$$

and it satisfies $u_\varepsilon(0) = \exp(B_\varepsilon - B)f$. Using results from [14] we obtain the following

Theorem 1. For any $f \in H$, $\lim_{\varepsilon\to 0} u_\varepsilon(0) = f$ and $\|u_\varepsilon(0)\| \le \|f\|$. There exists a (weak) solution u of the initial-value problem for (2) on $[0,1]$ if and only if

$\lim_{\varepsilon \to 0} \{u_\varepsilon(t)\}$ exists in H for all $t \in [0,1]$, and then $\lim_{\varepsilon \to 0} u_\varepsilon(t) = u(t)$.

This QR-method is theoretically incisive: there exists a solution if and only if it converges. It is slightly more subtle than the usual case $(A = 0)$ in [14] since the method must give a $u_\varepsilon(1) \in \mathrm{Rg}\{\exp(-A)\}$ in order that the backward problem have a strong solution. The method is always stable and convergent at $t = 0$, but only at $t = 0$. For example, $\|u_\varepsilon(1)\|$ may grow like $\mathcal{O}(\exp(1/\varepsilon))$, so as a numerical method it is essentially worthless. Even if one uses log-convexity estimates to stabilize the method [2,6], the use of initial-value or final-value problems in the procedure is not natural for (2).

BOUNDARY-VALUE PROBLEM

Suppose there is a weak solution of (2) on $[0,1]$. From the Lemma above it follows that $\xi(0,1)$ and hence the solution u will depend on both $u(0)$ and $u(1)$. One need only determine the domain of influence of u on ξ through the formulae of the Lemma. This suggests that a boundary-value problem on the interval $[0,1]$ is more appropriate than an initial-value problem.

We can substantiate this observation as follows. First let C be a self-adjoint operator whose spectrum is unbounded in both positive and negative real numbers, thus $C = A - B$ as above where A and B are the positive and negative parts of C, respectively. We seek a representation of a solution of

$$u'(t) + Cu(t) = 0, \quad 0 < t < 1, \tag{4}$$

in the form $u(t) = \int_\Gamma \exp(Cz)U(t,z)dz$. In order to choose the contour Γ in $\mathbb{C}$ so $\{\exp(Cz) : z \in \Gamma\}$ is bounded, we take $z = i\tau$, $\tau \in \mathbb{R}$, so we have

$$u(t) = \int_{\mathbb{R}} \exp(i\tau C)U(t,\tau)d\tau . \tag{5}$$

Substitution of (5) into (4) yields

$$\int_{\mathbb{R}} \exp(i\tau C)(U_t + iU_\tau)d\tau + (1/i)\exp(i\tau C)U(t,\tau)\Big|_{\tau=-\infty}^{\tau=\infty},$$

so we need require the kernel $U(t,\tau)$ to satisfy the Cauchy-Riemann equation in the slab $0 < t < 1$ and to vanish at $\tau \to \pm\infty$. Thus U will be determined by its remaining boundary-values, $U(0,\tau)$, $U(1,\tau)$, $-\infty < \tau < +\infty$. These in turn are determined by $u(0)$ and $u(1)$ through (5). These formal calculations can (and will) be made precise elsewhere but they already suggest that the equation (1) is elliptic and that the following problem is well posed. The boundary-value problem is to find a weak solution u of (2) on $[0,1]$ for which $au(0) + bu(1) = f$. Here $f \in H$ and $a,b \in \mathbb{R}$ are given.

Proposition 2. If u is a solution of the boundary-value problem then

$$(a\exp(-B) + b\exp(-A))u(t) = \exp(-At)\exp(-B(1-t))f, \quad 0 \le t \le 1.$$

If also $a,b \ge 0$ and not both are zero, then there is at most one solution. If both a and b are strictly positive then there exists a solution u for each $f \in H$ and it satisfies

$$u(t) \le \|f\|/a^{1-t}b^t, \quad 0 \le t \le 1.$$

QUASI-BOUNDARY-VALUE METHOD

Consider again the initial-value problem for (2) on the interval $[0,1]$ with

$u(0) = f$. The quasi-reversibility method of approximation was to regularize the problem by perturbing the equation, i.e., replace B by B_ε. The method suggested by Proposition 2 is to regularize the problem by perturbing the initial condition, i.e., replace it by the boundary condition

$$u(0) + \varepsilon u(1) = f . \tag{6}$$

Thus for each $\varepsilon > 0$ we let u_ε be the solution of the boundary-value problem (2), (6).

Theorem 2. For any $f \in H$, $\lim_{\varepsilon \to 0} u_\varepsilon(0) = f$. There exists a solution u of the initial-value problem for (2) on $[0,1]$ if and only if $\lim_{\varepsilon \to 0} \{u_\varepsilon(t)\}$ exists in H for all $t \in [0,1]$, and then $\lim_{\varepsilon \to 0} u_\varepsilon(t) = u(t)$. The solutions u_ε of (2), (6) satisfy the estimates

$$\|u_\varepsilon(t)\| \leq \|f\|/\varepsilon^t , \quad 0 \leq t \leq 1 , \quad \varepsilon > 0 . \tag{7}$$

The regularization procedure of Theorem 2, the QB-method, and the QR-method of Theorem 1 both approximate with a well posed problem for each $\varepsilon > 0$. Moreover, the estimate (7) is $\mathcal{O}(1/\varepsilon)$ at $t = 1$ in contrast to $\mathcal{O}(\exp(1/\varepsilon))$ in the QR-method, so the QB-method is reasonable for numerical implementation. However the regularized problems in the QB-method are global in t, so marching methods and their resultant sparse matrices and reduced storage requirements are not directly available in numerical work. Our preceding remarks on the "elliptic" nature of these equations suggest that such difficulties may be implicit in the problem, not just this method.

There is a fundamental deficiency in the use of Theorem 2 to actually find a solution u from data f; namely, the data is never measured exactly. This measurement error can be handled if we stabilize the problem by considering only those solutions which satisfy a prescribed global bound. Whereas Theorem 2 merely guarantees a good approximation at the initial time $t = 0$, we shall get a global approximation on $t \in [0,1]$.

Theorem 3. Let u be a weak solution of (2) on $[0,1]$. Let $M \geq 1$, $\delta > 0$, and $f \in H$ be given such that $\|u(0) - f\| < \delta$ and $\|u(1)\| \leq M$. Choose $\varepsilon \equiv M/\delta$ and let u_ε be the solution of the boundary-value problem (2), (6). Then we have the estimate

$$\|u(t) - u_\varepsilon(t)\| \leq 2\delta^{1-t} M^t , \quad 0 \leq t \leq 1 ,$$

for the error.

The procedure above is the stabilized quasi-boundary-value method. It is appropriate in applied problems where one knows from physical considerations there is a solution with a bound but the data $u(0)$ is not known exactly.

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