A Transport Model with Adsorption Hysteresis

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Abstract

The linear transport equation is supplemented with a hysteresis operator to obtain a local model of adsorption–desorption. The resulting nonlinear initial-boundary-value problem is shown to have a unique differentiable global solution when the hysteresis functional has a symmetric convex graph. An explicit solution is given for an elementary example, and a formula is given for the representation of smooth cases of such functionals.

1 Introduction

We study the partial differential equation

$$\frac{\partial}{\partial t}(u+v)+\frac{\partial}{\partial x}u=0, \quad v=\mathcal{M}(u), \tag{1}$$

as a generic model for the transport and adsorption of a chemical of concentration $u(x,t)$ carried in a solution with constant (unit) velocity in a tube $x \in (0,L)$ for $t>0$. Here $\mathcal{M}(\cdot)$ is a rather general functional describing adsorption and desorption of the chemical on the particles of solid filling up the tube. In the classical case $\mathcal{M}(\cdot)$ is given by an *isotherm*, a real-valued function corresponding to an experimentally determined relation between the concentration $v(x,t)$ of the adsorbed species on the surface of the particles and the concentration $u(x,t)$ in solution. In the general situation considered here, the adsorption–desorption functional $\mathcal{M}(\cdot)$ exhibits *hysteresis*, i.e., the relations between u and v for the cases when u is increasing (adsorption) and decreasing (desorption) follow *different* curves. A typical adsorption-desorption graph is presented in Figure 1 (compare with the graphs in [16]). The motivation for our study comes from applications in chemical and geological engineering. For example, the chromatographic elution process is modeled by (1); see [21, 20, 13] for a general study. Here one studies the relationship between the concentration input, $u(0,t)=\varphi(t)$, and the *break–through curves*, $u(L,t),\ t>0$. Hysteresis coupled to transport phenomena occurs also in the modeling of oil–water interaction [27] and of waste treatment in subsurface reservoirs [10]. The phenomenon of hysteresis in adsorption has been observed and studied for many years (see [16, 21, 11, 7]), frequently in parallel with capillary condensation hysteresis (see [9]). See also [3] for more rheological models of adsorption–desorption hysteresis. To our knowledge there has been no detailed mathematical study beyond the general theoretical formulation of the transport problem subject to adsorption–desorption hysteresis, and this is what we initiate here. A more general problem, including kinetics, diffusion, and multiscale phenomena will be discussed elsewhere.

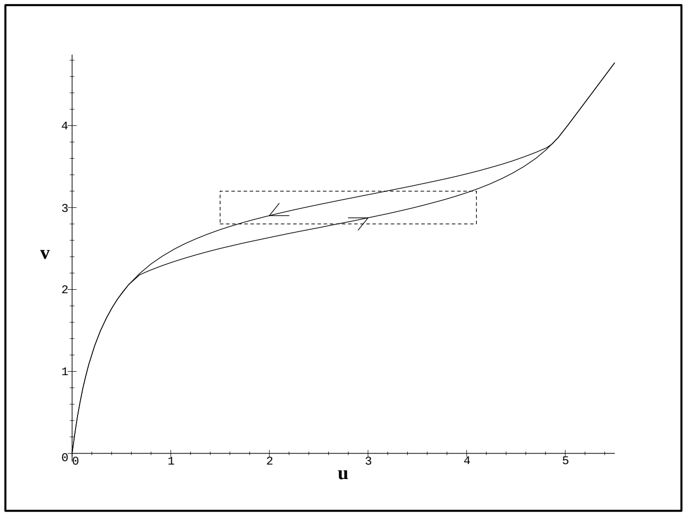

Figure 1: Typical adsorption/desorption hysteresis graph.

The corresponding classical initial-boundary-value problem

$$(u + m(u))_t + u_x = 0, \quad u(0,t) = \varphi(t), \quad u(x,0) = 0 \tag{2}$$

in which $m(\cdot) : R^+ \mapsto R^+$ is a smooth function has been thoroughly investigated by numerous authors, see [20, 8] for analysis with applications to adsorption problems (also multicomponent). Assume the inlet concentration $\varphi(t)$ is a Gaussian impulse. For the case of a *Langmuir isotherm*, $m(u) = \frac{cu}{a+u}$, the resultant concentration profiles consist of a shock discontinuity followed by a rarefaction wave developing in finite time and travelling with speed $\alpha = \frac{1}{1+m'(u)}$. When $m(\cdot)$ is not convex, e.g., as with a BET isotherm, more complicated profiles arise. Most cases are well understood, numerically treatable and used by engineers (see e.g. [14]). The theory of such conservation laws as (2) is rather complete, even in the case when $m(\cdot)$ is not a single-valued function but rather a general *monotone* graph (see below). These have been studied in [6, 2, 12, 5, 24] where one can find general theorems on existence and uniqueness of solutions in an appropriately weak sense. One obtains estimates only in the space $L^1(0, L)$, so it is necessary to use the theory of m-accretive operators in a *general* Banach space, and from this we obtain only a very weak notion of an *integral solution* $u \in C(0, T; L^1(0, L))$. Such solutions are not necessarily differentiable or even within the domain of the operator at any time $t > 0$.

Our goal in this work is to study the case where the functional $\mathcal{M}$ is multivalued because its values $v(t) = \mathcal{M}(u)(t)$ depend not only on the current value of $u(\cdot, t)$ at $t > 0$ but on the past *history* $u(\cdot, s)$, $0 < s < t$. The equation (1) with a general hysteresis functional $\mathcal{M}$ has previously been presented, e.g., in [26], where integral solutions were obtained from L^1 techniques. Here we obtain *differentiable* solutions by using the L^2-theory of evolution equations. In addition to the usual formulation of the existence–uniqueness results, we obtain a second one by switching the variables x and t to represent the solution by a semigroup of break–through curves. This second formulation could not be achieved if there were an additional viscosity term in (1), whereas the first formulation extends easily as indicated below.

In order to illustrate the typical features of such a history–dependent process, we study here the case that arises when $\mathcal{M}$ is taken to be a representative *local* part of the graph in Figure 1. We compute in Section 2 an explicit solution, and in Section 3 we prove the existence and uniqueness of strong (differentiable) solutions for the elementary case of those hysteresis functionals whose sides are bounded by parallel lines, a result that does not follow from the aforementioned references. It will be obtained from the theory of maximal monotone or m-accretive operators in Hilbert space, and as such it is not comparable to the L^1 results above. In Section 4 we discuss a somewhat more general case of convex adsorption-desorption hysteresis functionals and indicate the corresponding extension of the results of Section 3. The construction of the functional $\mathcal{M}$ in the convex case is obtained by taking a weighted sum of graphs of the elementary type of Section 3, with different switching points. This is known as a Prandtl-Ishlinskii construction [26]. The geometric representation of $\mathcal{M}$ in terms of the elementary functionals means that the increasing curves are convex and the decreasing ones are concave, so that the distance between them is necessarily nonzero. These complementary types of the bounding "isotherms" prevent the formation of shocks. It is for this reason that such regular solutions can be obtained for (1) with these nonlinear hysteresis functionals.

2 An explicit analytical solution

Here we consider a simple example in the form

$$(u+v)_t + u_x = 0, \;\; v = \mathcal{M}(u), \quad x \in (0,L), t > 0 \tag{3}$$

with the boundary condition specified at the left end of the tube by

$$u(0,t) = \varphi(t), \quad t > 0,$$

and the initial condition

$$u(x,0) = v(x,0) = 0, \quad x \in (0,L).$$

The functional $\mathcal{M}(\cdot)$ constructed here is called a *simple play.* This example arises from the general situation in Figure 1 if we consider only small variations of u which remain *inside* the hysteretic loop. For the sake of exposition the sides of the loop have unit slope, and the graph is shifted down to the origin. This simplification allows us to concentrate on the special character of the elementary history–dependent process modeled by $\mathcal{M}$.

We shall implement a very useful realization of the hysteresis functional, $\mathcal{M}(\cdot)$, in (3). It will be characterized by the evolution equation

$$v_t + sgn^{-1}(v-u) \ni 0, \quad t > 0. \tag{4}$$

The second term in this equation, sgn^{-1}, is the inverse of the signum graph, and it is the *maximal monotone graph* defined by $sgn^{-1}(-1) = (-\infty, 0]$, $sgn^{-1}(1) = [0,\infty)$, $sgn^{-1}(u) = \{0\}$, $u \in (-1,1)$. Thus (4) implies the *constraint*

$$-1 \le v - u \le 1.$$

The values of v_t are then allowed to take any positive value when $v - u = -1$ (but this means $v = u - 1$ and so $v_t = u_t$), any negative value when $v - u = 1$ (here $v = u + 1$ and again $v_t = u_t$), and $v_t = 0$ if the strict inequality holds in the constraint. In summary this means $v_t = u_t$ or 0, and inserting those values into the first equation gives

$$u_t + u_x + \left\{ \begin{array}{ll} 0 & u-1 < v < u+1 \\ u_t & v = u+1 \textit{ decreasing} \\ u_t & v = u-1 \textit{ increasing} \end{array} \right\} = 0\,.$$

$region$	$descr$	$u(x,t)$	$v(x,t)$
A	$t < x$	0	0
B	$x < t < x+1$	$t-x$	0
C	$x+1 < t < 2x+1, x \leq 4$	1	0
v	$x+1 < t < x+5, x > 4$		
D	$2x+1 < t < \frac{3}{2}x+3, x \leq 4$	$t-2x$	$t-2x-1$
E	$\frac{3}{2}x+3 < t < x+5, x \leq 4$	$6-(t-x)$	$2-\frac{1}{2}x$
F	$x+5 < t < \frac{5}{2}x+5, x \leq 2$	$\frac{1}{3}(8+x-t)$	$2-\frac{1}{2}x, x \leq 4$
	$x+5 < t < x+8, x > 2$		$0, x > 4$
G	$\frac{5}{2}x+5 < t < 2x+6, x \leq 2$	$6-t+2x$	$7-t+2x$
H	$2x+6 < t, x \leq 2$	0	1
	$x+8 < t, x > 2$		

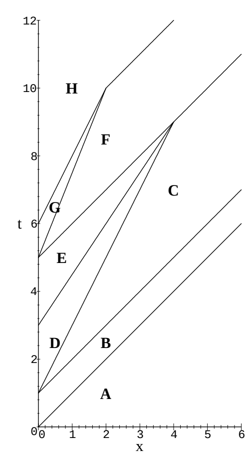

Figure 2: Characteristics in (x,t) plane.

This means that the *speed* of the wave front in the first case is $\alpha = 1$, since only u values are modified and v remains locally constant, and in the next two cases it is $\alpha = 1/2$ since the modification of values of u across the wave must be accompanied by the same modification of v. That is, v is "dragged" behind u by the constraint. It is also necessary for the constraint to hold at the initial time, so we set $v(x,0) = 0$.

For an explicit example, let us take the boundary condition

$$u(0,t) = \varphi(t) \equiv \begin{cases} t & 0 \le t \le 3 \\ 6-t & 3 < t < 6 \\ 0 & 6 \le t. \end{cases}$$

This choice of the boundary input data will drive the solution $(u(t), v(t))$ once around the loop (see Figure 1).

In order to compute the exact solution, we use the method of characteristics. If our original equation was $u_t + \alpha u_x = 0$, then the solution subject to the above boundary condition would preserve its shape and travel with speed α and $u(x,t) = \varphi(t - \frac{1}{\alpha}x)$, with characteristics in the form $t - \frac{1}{\alpha}x = \xi$. In our case, with different values of α, the characteristics must cross and this leads to discontinuities in the derivatives of the solution. The solution itself remains continuous. The analysis of this example presented in Section 3 leads to the same results.

The computations of the solution along with the sketch of the characteristics are given in Figure 2.

3 A Linear-Sided Play Model

Our objective in this section is to establish general results on the existence and uniqueness of solutions to the initial-boundary-value problem for (3). We begin by recalling some relevant definitions. A (possibly multi-valued) *operator* A in a real Hilbert space H is a collection of related pairs $[x,y] \in H \times H$ denoted by $y \in A(x)$; the *domain* $Dom(A)$ is the set of all such x and the *range* $Rg(A)$ consists of all such y. The operator A is called *accretive* if for all $y_1 \in A(x_1)$, $y_2 \in A(x_2)$, and $\varepsilon > 0$, we have

$$\|x_1 - x_2\| \le \|x_1 - x_2 + \varepsilon(y_1 - y_2)\| \ .$$

This is equivalent to requiring that $(I + \varepsilon A)^{-1}$ be a contraction on $Rg(I + \varepsilon A)$ for every $\varepsilon > 0$. This in turn is equivalent to requiring

$$\langle y_1 - y_2, x_1 - x_2 \rangle_H \ge 0 \quad \forall\, x_1, x_2 \in Dom(A),\ \forall\, y_1 \in A(x_1),\ \forall\, y_2 \in A(x_2)\ .$$

If additionally $Rg(I + \varepsilon A) = H$ for some (equivalently, for all) $\varepsilon > 0$, then A is *maximal*, and we say A is *m-accretive*. For such an operator, the *Cauchy problem*

$$\begin{cases} u'(t) + A(u(t)) & \ni f(t), t > 0 \\ \quad u(0) & = u_0 \end{cases} \tag{5}$$

is known to be well-posed. The general result for this abstract Cauchy problem is the following. See [15, 1, 4, 22].

Theorem 3.1 *(Kato–Komuro–Dorroh) Let A be m-accretive in the Hilbert space H. If $T > 0$, $u_0 \in Dom(A)$ and $f \in W^{1,1}(0,T;H)$, then there exists a unique solution $u \in W^{1,\infty}(0,T;H)$ of the Cauchy problem (5) with $u(t) \in Dom(A)$ for all $0 \le t \le T$, hence, $A(u(\cdot)) \in L^\infty(0,T;H)$.*

We shall realize our initial-boundary-value problem as such a problem in an appropriate function space. Moreover, we shall do this in two different ways and obtain thereby two notions of solution which differ in their regularity. For perspective, we note that Theorem 3.1 applies directly to the linear initial-boundary-value problem

$$u_t + u_x = 0, \quad u(x,0) = \psi(x), \quad x \in (0,L), \quad u(0,t) = 0$$

for which the solution is given by $u(x,t) = \psi(x-t)$ for $0 < t < x$ and $u(x,t) = 0$ for $0 < x < t$, $x < L$. Here the operator $A \equiv \partial_x$ is m-accretive on the domain $D(A) = \{u \in H^1(0,L) | u(0) = 0\ \}$, and the initial condition is chosen with $\psi \in D(A)$. We see that the corresponding semigroup operators are just rightward translation, so the solution at each time $t > 0$, $u(\cdot,t)$, has the same smoothness as the initial data, ψ, just as is specified by Theorem 3.1. When the initial condition $u(x,0) = \psi(x)$ is replaced by the boundary condition $u(0,t) = \varphi(t)$, then the setup of the above problem follows after the substitution $w(x,t) = u(x,t) - \varphi(t)$. That is, we exchange the problem with nonhomogeneous boundary condition for the corresponding initial-boundary-value problem,

$$w_t + w_x = -\varphi'(t), \quad w(0,t) = 0, \quad w(x,0) = -\varphi(0).$$

We shall assume that $\varphi(0) = 0$ and that $\varphi \in W^{2,1}(0,L)$, so Theorem 3.1 applies. Note, however, that this construction does *not* give the optimal regularity, since the solution is exactly as smooth as the boundary data: Theorem 3.1 requires here one more derivative on $\varphi(\cdot)$ than is necessary.

For our first formulation of the initial–boundary–problem as an abstract Cauchy problem (5), we let $c(\cdot)$ be a maximal monotone graph in $\mathcal{R}$. In our application to (3) and (4) it will be the signum inverse graph, $c(\cdot) = sgn^{-1}(\cdot)$, and our system becomes

$$\begin{aligned} u_t + \partial_x u - c(v-u) &\ni 0 \\ v_t + c(v-u) &\ni 0 \\ u(0,t) &= \varphi(t) \\ u(x,0) = 0 &= v(x,0). \end{aligned}$$

Here and below, it is understood that one takes the *same* choice from the multivalued $c(\cdot)$ in each component of such a system. As before, we replace u by $u + \varphi$ and also v by $v + \varphi$. This gives the problem with homogeneous boundary conditions

$$u_t + Lu - c(v-u) \ni -\varphi'(t) \tag{6}$$
$$v_t + c(v-u) \ni -\varphi'(t) \tag{7}$$
$$u(t) \in Dom(L) \tag{8}$$
$$u(x,0) = 0 = v(x,0) \tag{9}$$

where we have denoted by L the m-accretive *linear* operator on $H \equiv L^2(0,L)$ which is a realization of ∂_x with the domain $Dom(L) = \{u \in H^1(0,L) : \ u(0) = 0\}$. This system is in the form of an evolution equation $\vec{u}' + A(\vec{u}) \ni \vec{f}$. The corresponding *resolvent equation*, $\vec{u} + A(\vec{u}) \ni \vec{f}$, takes the form

$$u + Lu - c(v-u) \ni f \tag{10}$$
$$v + c(v-u) \ni g, \tag{11}$$

in which $\vec{u} = [u,v]$ and $\vec{f} = [f,g]$. This suggests that we set

$$Dom(A) = \{[u,v] \in Dom(L) \times L^2(0,L) : \ v - u \in Dom(c)\}$$

and define $[f,g] \in A([u,v])$ if $f,\ g \in L^2(0,L),\ g \in c(v-u)$, and $Lu = f + g$.

Let's show that this operator A is accretive in $H \times H$, $H = L^2(0,L)$. If we have $[f_1, g_1] \in A([u_1, v_1])$ and $[f_2, g_2] \in A([u_2, v_2])$, then

$$(f_1 - f_2, u_1 - u_2)_{L^2} + (g_1 - g_2, v_1 - v_2)_{L^2} =$$
$$(L(u_1 - u_2), u_1 - u_2)_{L^2} + (g_1 - g_2, v_1 - v_2 - u_1 + u_2)_{L^2}.$$

Since L is accretive, the first term on the right side is nonnegative, and the second is nonnegative similarly, since $c(\cdot)$ is monotone. Next we verify that $I + A$ is onto. By eliminating v from the resolvent system, we obtain the equivalent equation

$$u + Lu + u + (I + c)^{-1}(g - u) = f + g \tag{12}$$

for which the second component is given by

$$v = u + (I + c)^{-1}(g - u).$$

Since the left side of (12) is the sum of an m-accretive operator and an accretive Lipschitz operator, it is always solvable, so the resolvent system has a solution. That is, A is m-accretive, and we obtain the following result.

Theorem 3.2 *If $\varphi \in W^{2,1}(0,T)$ and $\varphi(0) = 0$, then there is a unique solution of (6-9) with $u,\ v \in W^{1,\infty}(0,T; L^2(0,L))$.*

Note that for this solution we have $u_t,\ u_x, v_t \in L^\infty(0,T; L^2(0,L))$. Furthermore, all of the above carries through for the case of a general m-accretive operator L on $L^2(0,L)$. In particular, it permits the example of diffusive transport, $Lu = u_x - \mu u_{xx}$ with $\mu \geq 0$ and appropriate boundary conditions on $Dom(L)$.

For our second formulation of the initial–boundary–problem as an abstract Cauchy problem, we treat the spatial variable x as another time variable, and therefore we rename it: $x = \tau$. Then the system is given by

$$u_\tau + u_t - c(v - u) \ni 0 \tag{13}$$
$$v_t + c(v - u) \ni 0 \tag{14}$$
$$u(0,t) = \varphi(t) \tag{15}$$
$$u(\tau, 0) = 0 = v(\tau, 0). \tag{16}$$

We shall regard this as an evolution *in the first component* on $H \equiv L^2(0,T)$. That is, this system is of the form $u' + A(u) \ni 0$ for an appropriate definition of A, and then the corresponding resolvent equation, $u + A(u) \ni f$, takes the form

$$u + u_t - c(v - u) \ni f \tag{17}$$
$$v_t + c(v - u) \ni 0, \quad 0 < t < T, \tag{18}$$
$$u(0) = 0 = v(0). \tag{19}$$

In order to define A, we set $Dom(A) = \{u \in H^1(0,T) :\ u(0) = 0\}$. Note that for each such $u \in Dom(A)$, the Cauchy problem

$$(v - u)_t + c(v - u) \ni -u_t, \quad (v - u)(0) = 0$$

has a unique solution $v \in H^1(0,T)$. Then we define $A(u) = u_t + v_t$ with $u \in Dom(A)$ and v as given.

We shall verify that A is m-accretive on $L^2(0,T)$. If $u_1,\ u_2 \in Dom(A)$ and if $v_1,\ v_2$ are defined as above, then we compute

$$((u_1 + v_1)_t - (u_2 + v_2)_t, u_1 - u_2)_{L^2(0,T)}.$$

By rearranging terms and integrating, we obtain

$$\frac{1}{2}|u_1(T) - u_2(T)|^2 + ((v_1)_t - (v_2)_t, u_1 - u_2)_{L^2(0,T)},$$

and then by adding and subtracting $(v_1 - v_2)$ to the right side of the second term and again integrating, we have

$$\frac{1}{2}|u_1(T) - u_2(T)|^2 + \frac{1}{2}|v_1(T) - v_2(T)|^2 + ((v_1)_t - (v_2)_t, u_1 - v_1 - u_2 + v_2)_{L^2(0,T)}.$$

Since $c(\cdot)$ is monotone, it follows from the definition of v_1 and v_2 that the last term is nonnegative, hence, we have shown that

$$(A(u_1) - A(u_2), u_1 - u_2)_{L^2(0,T)} \geq 0,$$

so A is accretive. In order to verify the range condition, we consider the resolvent equation above with $f \in L^2(0,T)$. But this system is an evolution equation on the product space $L^2(0,T) \times L^2(0,T)$, and the corresponding operator is a *subgradient*, so the system has a unique solution $[u,v] \in H^1(0,T) \times H^1(0,T)$ with $u(0) = v(0) = 0$.

Since A is m-accretive, we obtain the following.

Theorem 3.3 *If $\varphi \in W^{1,2}(0,T) = H^1(0,T)$ there is a unique solution of (6-9) with $u_x \in L^\infty(0,L;L^2(0,T))$, and for all x, $u(x,\cdot) \in H^1(0,T)$, $u(x,0) = 0$.*

Remark 1 *This second formulation represents the solution as the family $u(x,\cdot)$ of break–through curves. These comprise the data that is normally observed and recorded in experiment. Finally, the regularity assumption on $\varphi(\cdot)$ is optimal.*

4 A Convex-Sided Play Model

Here we shall extend the preceding existence and uniqueness results to the case of convex-sided play. As above, this will be done for each of two formulations as an abstract Cauchy problem. Let $\{\mathcal{A}, \mu\}$ be a positive finite measure space and $\{c_\alpha(\cdot) : \alpha \in \mathcal{A}\}$ be a family of maximal monotone graphs in $\mathcal{R}$. We assume that the resolvents $\{(I + c_\alpha)^{-1}$ are *strongly measurable*, i.e., that $\alpha \mapsto (I + c_\alpha)^{-1}(x)$ is measurable on $\mathcal{A}$ for each $x \in R$. For our first formulation with this notation the system that we consider is given by

$$\frac{\partial}{\partial t}(u + \int_{\mathcal{A}} v^\alpha d\mu_\alpha) + \partial_x u = 0 \tag{20}$$

$$v_t^\alpha + c_\alpha(v^a - u) \ni 0, \quad a \in \mathcal{A} \tag{21}$$

$$u(0,t) = \varphi(t) \tag{22}$$

$$u(x,0) = 0 = v^\alpha(x,0). \tag{23}$$

As before, we replace u by $u + \varphi$ and each v^α by $v^\alpha + \varphi$ to obtain the corresponding problem with homogeneous boundary conditions

$$u_t + Lu - \int_{\mathcal{A}} c_\alpha(v^\alpha - u) d\mu_\alpha \ni -\varphi'(t)$$

$$v_t^\alpha + c_\alpha(v^\alpha - u) \ni -\varphi'(t), \quad a \in \mathcal{A}$$

$$u(t) \in Dom(L)$$

$$u(x,0) = 0 = v^\alpha(x,0)\,.$$

We have denoted by L an m-accretive *linear* operator on $H \equiv L^2(0,L)$. This system is in the form of an evolution equation; we define

$$Dom(A) = \{[u,v] \in Dom(L) \times L^2((0,L) \times \mathcal{A}) : \ v^\alpha - u \in Dom(c_\alpha), \ \alpha \in \mathcal{A}\}$$

and the operator A by $[f,g] \in A([u,v])$ if $f \in L^2(0,L)$, $g \in L^2((0,L)\times\mathcal{A})$, $g^\alpha \in c_\alpha(v^\alpha - u)$, and $Lu = f + \int_{\mathcal{A}} g^\alpha d\mu_\alpha$. Then we show as before that this operator A is accretive, but here in the space $H \times \mathcal{H}$, $H = L^2(0,L)$, $\mathcal{H} = L^2((0,L)\times\mathcal{A})$.

Theorem 4.1 *If $\varphi \in W^{2,1}(0,T)$ and $\varphi(0) = 0$, then there is a unique solution of (20) with $u,\ v^\alpha \in W^{1,\infty}(0,T;L^2(0,L))$ for almost every $\alpha \in \mathcal{A}$.*

This solution satisfies $u_t,\ u_x, v_t^\alpha \in L^\infty(0,T;L^2(0,L))$.

For the second formulation we denote the spatial variable by $x = \tau$, and then the system is

$$\begin{aligned} u_\tau + u_t - \int_{\mathcal{A}} c_\alpha(v^\alpha - u)d\mu_\alpha &\ni 0 \\ v_t^\alpha + c_\alpha(v^\alpha - u) &\ni 0 \\ u(0,t) &= \varphi(t) \\ u(\tau,0) = 0 &= v^\alpha(\tau,0)\,. \end{aligned}$$

We define A on the same domain $Dom(A) = \{u \in H^1(0,T):\ u(0) = 0\}$ as before. For each such $u \in Dom(A)$, the Cauchy problem

$$(v^\alpha - u)_t + c_\alpha(v^\alpha - u) \ni -u_t, \quad (v^\alpha - u)(0) = 0$$

has a unique solution $v^\alpha \in H^1(0,T)$, and then we define

$$A(u) = u_t + \int_{\mathcal{A}} v_t^\alpha d\mu_\alpha, \quad u \in Dom(A),$$

with v^α given as above. Finally, we check that A is m-accretive on $L^2(0,T)$, so we have the following.

Theorem 4.2 *If $\varphi \in W^{1,2}(0,T) = H^1(0,T)$ there is a unique solution of (20) with $u_x \in L^\infty(0,L;L^2(0,T))$, and for all x, $u(x,\cdot) \in H^1(0,T)$, $u(x,0) = 0$.*

In our applications we shall employ the collection of elementary maximal monotone graphs $c_\alpha(\cdot)$ which are constructed by rescaling the signum inverse: $c_\alpha(s) = sgn^{-1}(\frac{2}{\alpha}s+1)$ for $\alpha \in \mathcal{A}$. For an instructive example, we choose $\mathcal{A} = \{1,2,3\}$ with discrete measure μ given by

$$\mu(1) = \frac{1}{4}, \quad \mu(2) = \frac{1}{2}, \quad \mu(3) = 1.$$

Then the hysteresis functional $v = \mathcal{M}(u)$ is given by the system

$$\begin{aligned} \frac{d}{dt}v^1 + c_0(v^1 - u) &\ni 0, \\ \frac{d}{dt}v^2 + c_1(v^2 - u) &\ni 0, \\ \frac{d}{dt}v^3 + c_2(v^3 - u) &\ni 0, \\ v(t) = \frac{1}{4}v^1(t) + \frac{1}{2}&v^2(t) + v^3(t). \end{aligned}$$

Note that the first equation is equivalent to $v^1 = u$. Suppose $u(t)$ starts at $u(0) = 0$ and increases until $u = 3$. Then

$$v^\alpha(t) = (u(t) - \alpha)^+, \quad \alpha \in \mathcal{A},$$

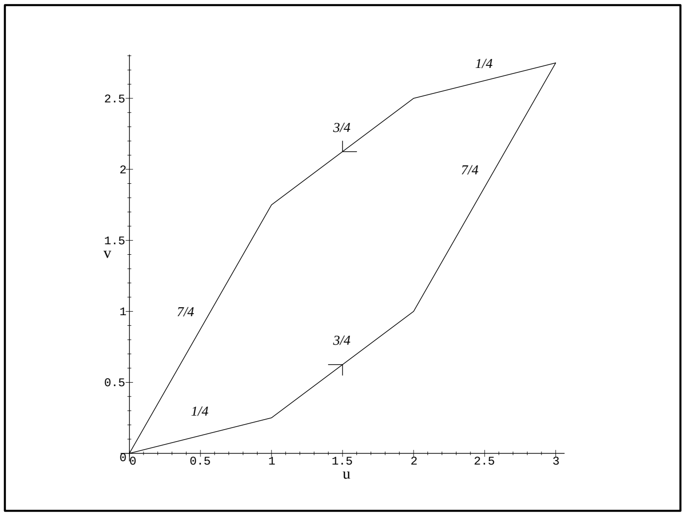

Figure 3: Convex-sided hysteresis graph, slopes indicated, u is the input, $v = \frac{1}{4}v_1 + \frac{1}{2}v_2 + v_3$ is the combined output.

where x^+ denotes the *positive part* of x. Thus $v(\cdot)$ follows the convex piecewise linear curve with slope $\frac{1}{4}$ on $[0,1]$, slope $\frac{3}{4}$ on $[1,2]$, and slope $\frac{7}{4}$ on $[2,3]$. If $u(t)$ then decreases from 3 down to 0, then $v(\cdot)$ follows the symmetric piecewise linear concave curve with slopes $\frac{1}{4}$ on $[2,3]$, slope $\frac{3}{4}$ on $[1,2]$, and slope $\frac{7}{4}$ on $[0,1]$ back down to $v = 0$. This is indicated in Figure 3. Similarly, we can recover any such bounding curve on the right consisting of piecewise linear sides with increasing slopes followed by the corresponding symmetric curve on the left.

By taking a large number of such components, we can approximate any such center–symmetric convex region which is bounded on the right by an increasing curve and on the left by the corresponding symmetric decreasing curve. Moreover, in order to construct another useful class of such functionals, let the family of hysteresis functionals $\{v^\alpha(\cdot)\}$ be given as above for $\alpha \in \mathcal{A} \equiv [0,a]$ for some $a > 0$, that is,

$$\frac{d}{dt}v^\alpha(t) + c_\alpha(v^\alpha(t) - u(t)) \ni 0.$$

For any convex function $\psi(\cdot) \in W^{2,1}(0,a)$ with $\psi(0) = \psi'(0) = 0$, we let μ be the positive measure on $[0,a]$ with $d\mu_\alpha = \psi''(\alpha)d\alpha$ so that our hysteresis functional $v = \mathcal{M}(u)$ is given by

$$v(t) = \int_0^a v^\alpha(t)\psi''(\alpha)\,d\alpha\,.$$

If $u(t)$ is increasing from 0 to a, we have

$$v(t) = \int_0^a (u(t) - \alpha)^+\psi''(\alpha)\,d\alpha\,,$$

and an integration-by-parts shows that

$$v(t) = \psi(u(t)).$$

Similarly, if $u(t)$ is decreasing from a to 0, $v(t)$ follows the center-symmetric curve according to

$$v(t) = \psi(a) - \psi(a - u(t)).$$

This provides an explicit formula for the representation of any such smoothly–bounded, convex and center–symmetric hysteresis functional as a continuous weighted sum or integral of the elementary functionals $\{v^\alpha(\cdot)\}$, and then it can be used in the transport system (1).

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