DIFFUSION IN PARTIALLY FISSURED MEDIA AND IMPLICIT EVOLUTION SYSTEMS

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Abstract. We study the Cauchy problem for systems of the form

$$\frac{d}{dt}\mathcal{A}(u(t)) + \mathcal{B}(u(t)) \ni f(t) \ ,$$

where $\mathcal{A}$ is a subgradient and $\mathcal{B}$ is maximal monotone on a product of three Hilbert spaces. The operator $\mathcal{A}$ is compact in the first component and linear in the third. The second component is treated as an ordinary differential equation. The results here on existence and uniqueness extend those of *DiBenedetto and Showalter* to systems of equations in which $\mathcal{A}$ is not necessarily compact in all its components. The technique used to prove existence is also used to prove continuous dependence of solutions on the subgradient $\mathcal{A}$. A large class of models for flow in partially fissured porous media are of the type considered here, and they provide the motivation for our study.

Keywords: doubly nonlinear, secondary flux, partially fissured, Nunzioto's equation, degenerate evolution equations, existence, uniqueness, continuous dependence.

1. Introduction.

We present in Section 2 a model for flow in fractured or fissured media which contains many classical and contemporary models. This model is the motivation for the hypotheses made in the theorems proven in Sections 3, 4, and 5. We have made an effort to make the discussion of the physical models independent of the discussion of the abstract results. Readers interested only in the theory of abstract evolution equations may skip Sections 2

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and 6, while those who are interested only in the consequences of the results to the model problems may skip Sections 3–5. We feel, however, that the elegance of the abstract results enhances the insight into the model, and the model lends justification and independent interest to the formulation of the abstract results.

We study the initial-value problem

$$
\begin{cases} \dfrac{d}{dt}\mathcal{A}(u(t)) + \mathcal{B}(u(t)) \ni f(t) , \\ \mathcal{A}(u(0)) \ni v_0 \end{cases} \tag{1.1}
$$

where $\mathcal{A}$ is the subgradient of a convex, lower semicontinuous function $\varphi : \mathcal{V} \to \mathbb{R}^1$ and $\mathcal{B}$ is maximal monotone from the Hilbert space $\mathcal{V}$ to its dual $\mathcal{V}^*$. The data $f \in L^2(0, T; \mathcal{V}^*)$ and $v_0 \in \mathcal{V}^*$ are given. Many models currently used for flow in fractured porous media yield relations of the type (1.1). This problem is well understood when $\mathcal{A}$ is compact; we shall briefly review the results of [10] in Section 3. The approach used in [10] to solve (1.1) is to pick $\epsilon > 0$ and solve

$$
\begin{cases} \dfrac{d}{dt}(\mathcal{A}(u_\epsilon(t)) + \epsilon\mathcal{R}(u_\epsilon(t))) + \mathcal{B}(u_\epsilon(t)) \ni f(t) , \\ \mathcal{A}(u_\epsilon(0)) + \epsilon\mathcal{R}(u_\epsilon(0)) \ni v_0 + \epsilon\mathcal{R}(u_0) \end{cases} \tag{1.2}
$$

where $\mathcal{R} : \mathcal{V} \to \mathcal{V}^*$ is the Riesz mapping and $[u_0, v_0] \in \mathcal{A}$. If *a priori* bounds independent of ϵ exist in $\mathcal{V}$ for the solutions u_ϵ, then one expects that u_ϵ converges to a solution of (1.1) in some sense. Our objective here is to relax the compactness assumption and thereby extend these results to a larger class of problems arising in the study of fluid flow in fractured media. Two alternative assumptions on the structure of the problem which have been utilized are that $\mathcal{A}$ is linear and that the composition of $\mathcal{B}$ and $\mathcal{A}^{-1}$ is Lipschitz (cf. [19], [5]). Each of these three assumptions will play a role below.

We study these problems in Section 4 for systems with the additional structure $\mathcal{V} = V_0 \times H_1 \times H_2$ (all Hilbert spaces), and bounded $\mathcal{A} = \mathcal{A}_0 \times \mathcal{A}_1 \times \mathcal{A}_2$, where the first component $\mathcal{A}_0 : V_0 \to \mathcal{V}^*$ is compact and the third component $\mathcal{A}_2 : H_2 \to \mathcal{V}^*$ is linear. We also require that the projection of the mapping $u_1 \mapsto \mathcal{B}(u_0, u_1, u_2) : H_1 \to \mathcal{V}^* = V_0^* \times H_1^* \times H_2^*$ into H_1^* is uniformly Lipschitz continuous for (u_0, u_2) in a bounded subset of $V_0 \times H_2$. In this way, each of the three hypotheses mentioned above plays a role in the system.

Once existence of solutions to (1.2) is established, we give sufficient conditions for existence of solutions to (1.1). The estimates are wholly analogous to those of [10]. In Section 5 we consider the dependence of the solutions on the subgradient $\mathcal{A}$. Sufficient conditions for continuous dependence of the solutions are given, even if $\mathcal{A}$ degenerates to, say, the zero operator. In the final Section 6, we discuss the continuous dependence of the solutions obtained from each model on the operator $\mathcal{A}$ in terms of the physical assumptions used to derive the model.

2. Models.

A fissured medium consists of a matrix of porous and permeable blocks of pores or *cells* which are partially isolated from each other by a highly developed system of *fissures* through which the bulk of the fluid transport occurs. Due to this tendency for the cells to be separated by the fissure system, there is a comparatively smaller amount of transport directly from cell to cell. Another feature of fissured media is that the volume occupied by the fissures is frequently considerably smaller than that occupied by the matrix of cells. Thus, most transport occurs in the fissures and substantial storage takes place in the cells. The system is by nature very much unsymmetric in the structure and function of its components. Specifically, the fissure system provides the primary or global transport component and any interaction with the matrix of cells is regarded as contributing a localized perturbation. Thus, any transport between cells or storage in fissures can be regarded as "secondary" effects, and the study of these phenomena is our objective.

The double porosity model is a classical description of diffusion in heterogeneous media. The idea is to introduce at each point in space a density, pressure or concentration for each component, each being obtained by averaging in the respective medium over a generic neighborhood sufficiently large to contain a representative sample of each component. The rate of exchange between the components must be expressed in terms of these quantities, and the resulting expressions become distributed source and sink terms for the diffusion equations in the individual components. Thus, one obtains a system of diffusion-type equations, one for each component. We shall construct and compare models of this type as they apply to our situation.

The classical double porosity model of [2] for the flow of fluid in a general heterogeneous medium consisting of two components is

$$\frac{d}{dt}\, a_0 u_0 + \mathcal{B}_0 u_0 + \frac{1}{\alpha}(u_0 - u_1) = f \ , \tag{2.1.a}$$

$$\frac{d}{dt}\, a_1 u_1 + \mathcal{B}_1 u_1 + \frac{1}{\alpha}(u_1 - u_0) = 0 \ , \tag{2.1.b}$$

where $\mathcal{B}_i = -\nabla \cdot B_i \nabla$ for some positive semidefinite matrices B_i, $i = 0, 1$. The top line quantifies the rate of flow in fissures — regions of small relative volume but large permeability. The second line quantifies the rate of flow in the cell system — regions of large porosity or volume, but largely isolated from one another. Both of these equations are to be understood macroscopically; that is, these quantifications are to be idealized by averaging over neighborhoods containing a large number of cells (called "blocks" of pores) and fissures. These ideas are described in detail in [2]. Although the components of (2.1) are structured symmetrically, fissured media characteristics are modeled by very small coefficients a_0 and B_1. In the extreme model, $a_0 = 0$ because the relative volume of the fissures is zero, and $B_1 = 0$ (the zero matrix) because there is no direct flow from one block of pores to another; only indirect flow through the fissures occurs. Thus, the condition $B_1 = 0$ corresponds to a *completely fissured* medium in which each cell is isolated from adjacent cells by the fissure system. The last term on the left of each equation represents the exchange of fluid between the cells and the fissures. The parameter α represents the resistance of the medium to this exchange. When $\alpha = \infty$, no exchange flow is possible. An alternative interpretation is that $1/\alpha$ represents the degree of fissuring in the medium. When the degree of fissuring is infinite, the exchange flow encounters no resistance and $u_0 = u_1$. The external sources of fluid represented by f are located in the fissures in this model, although there is no technical difficulty in assuming that fluid may be injected or extracted in the cells as well.

Our objective is to modify this model to allow flow between the cells, resulting in a *secondary flux* from this bridging between cells. This is the sense in which the medium is *partially fissured*. The model given here does not have the inappropriate symmetry of (2.1); it arises in a natural way from both mathematical and physical considerations, and we discuss both. First, we consider the formal operators

$$\mathcal{P} = a_0 \frac{d}{dt} \ + \mathcal{B}_0, \qquad \mathcal{Q} = a_1 \frac{d}{dt} \ + \mathcal{B}_1 \ .$$

Then (2.1.b) implies that

$$u_1 = (I + \alpha\mathcal{Q})^{-1} u_0 \ .$$

Substituting into (2.1.a) gives

$$\mathcal{P}u_0 + \frac{1}{\alpha}\left(I - (I+\alpha\mathcal{Q})^{-1}\right) u_0 = \mathcal{P}u_0 + \mathcal{Q}_\alpha u_0 = f \ , \tag{2.2}$$

where

$$\mathcal{Q}_\alpha = \frac{1}{\alpha}\left(I - (I+\alpha\mathcal{Q})^{-1}\right) = \mathcal{Q}\,(I+\alpha\mathcal{Q})^{-1}$$

is the Yosida approximation of the maximal monotone operator $\mathcal{Q}$ (cf. [7]). Since $1/\alpha$ is a measure of the fissuring of the medium, $\mathcal{Q}_0 = \mathcal{Q}$ corresponds to a homogeneous medium in which flow is modeled by the classical diffusion equation.

Let us consider first the *completely fissured* case $\mathcal{B}_1 = 0$, in which no direct flow between cells takes place. Then

$$\mathcal{Q}_\alpha = a_1\frac{d}{dt}\ \left(I + \alpha a_1\frac{d}{dt}\ \right)^{-1} \ ,$$

and (2.2) becomes

$$a_0\frac{d}{dt}\,u_0 + \mathcal{B}_0 u_0 + \frac{d}{dt}\ \left(I + \alpha a_1\frac{d}{dt}\ \right)^{-1} u_0 = f \ , \tag{2.3}$$

the *kinetic* or *first-order rate model* (cf. [14], [22], [13]). If, in addition, the fissure volume is zero, then $a_0 = 0$ and the classical *fissured-medium equation* [4] is recovered. [See (2.8).]

Let us investigate the operators in (2.3) more closely. The first property we consider is the separation of the temporal and spatial operators. The temporal operator is an ordinary differential operator and is studied using ordinary differential equation techniques (cf. [4] and [18] for examples), while the spatial operator is an elliptic differential operator. Note that the operators are not separated in (2.2) as they are in (2.3). It is natural to ask whether some corresponding (approximate) separation into temporal and spatial operators exists for (2.2).

To achieve the separation of temporal and spatial operators, we consider the approximation

$$\left(I + \alpha\frac{d}{dt}\ \right)^{-1} u_0(t) \approx u_0(t-\alpha) \ ;$$

i.e., the resolvent of d/dt is approximately a translation. This observation suggests that we may think of the cells as providing a *delayed* storage or capacitance. The parameter α in the Yosida approximation of $\mathcal{Q}$ now plays the role of a response time. These observations lead us to models of the form

$$\mathcal{P}u_0 + \frac{d}{dt} \quad (\text{ delayed storage }) + \vec{\nabla}(\text{ delayed flux }) = f \ .$$

We wish to model the delayed storage and flux terms so that when $\mathcal{B}_1 = 0$, the kinetic model is recovered, and so that the model can be derived from physical considerations.

In order to quantify these considerations, we first ask what approximation to $\mathcal{Q}_\alpha$ of the type indicated above is good for "small $\mathcal{B}_1$". Let us formally expand

$$\left(a_1 \frac{d}{dt} + \mathcal{B}_1\right)_\alpha = \frac{1}{\alpha}\left(I - \left(I + \alpha\left(a_1 \frac{d}{dt} + \mathcal{B}_1\right)\right)^{-1}\right)$$

in powers of $\mathcal{B}_1$. The formal power series of the right side suggests that

$$\left(a_1 \frac{d}{dt} + \mathcal{B}_1\right)_\alpha = \left(a_1 \frac{d}{dt}\right)_\alpha + \left(I + \alpha a_1 \frac{d}{dt}\right)^{-2} \mathcal{B}_1 + \mathcal{O}(\mathcal{B}_1^2) \ . \tag{2.4}$$

If $\mathcal{B}_1$ is the L^2 realization of a linear maximal monotone operator and the correct interpretation is given to the terms on the right, then this formal expansion is actually correct.

PROPOSITION 1. *Let $-C$ be the linear generator of a C_0 contraction semigroup $S(t)$ on a Banach space X. Suppose $u_0 \in D(C^2)$ and $f \in C(0,T; D(C^2))$. Define the functions u_1, u_2, u_3 and v by*

$$\begin{cases} \left(I + \alpha a \dfrac{d}{dt} + \alpha C\right) u_1 = f \ , & u_1(0) = u_0 \ , \\ \left(I + \alpha a \dfrac{d}{dt}\right) u_2 = f \ , & u_2(0) = u_0 \ , \\ \left(I + \alpha a \dfrac{d}{dt}\right) v = Cf \ , & v(0) = Cu_0 \ , \\ \left(I + \alpha a \dfrac{d}{dt}\right) u_3 = v \ , & u_3(0) = 0 \ . \end{cases}$$

Then

$$\left\| \frac{1}{\alpha}(f-u_1) - \frac{1}{\alpha}(f-u_2) - u_3 \right\|_{L^\infty(0,T;X)}$$

$$\leq \alpha \left\{ \frac{1}{2}\left(\frac{t}{\alpha a}\right)^2 e^{-t/\alpha a} \left\|C^2 u_0\right\|_X + \left\|C^2 f\right\|_{L^2(0,T;X)} \right\} .$$

PROOF. It is a simple matter to verify the following representations:

$$u_1(t) = e^{-t/a\alpha} S\left(\frac{t}{a}\right) u_0 + \frac{1}{a\alpha}\int_0^t e^{-(t-s)/a\alpha} S\left(\frac{t-s}{a}\right) f(s)\, ds$$

$$u_2(t) = e^{-t/a\alpha} u_0 + \frac{1}{a\alpha}\int_0^t e^{-(t-s)/a\alpha} f(s)\, ds$$

$$v(t) = e^{-t/a\alpha} C u_0 + \frac{1}{a\alpha}\int_0^t e^{-(t-s)/a\alpha} C f(s)\, ds$$

$$u_3(t) = \frac{1}{a\alpha}\int_0^t e^{-(t-s)/a\alpha}\left\{ e^{-s/a\alpha} C u_0 + \frac{1}{a\alpha}\int_0^s e^{-(s-r)/a\alpha} C f(r)\, dr \right\} ds$$

$$= \frac{t}{a\alpha} e^{-t/a\alpha} C u_0 + \frac{1}{(a\alpha)^2}\int_0^t (t-r) e^{-(t-r)/a\alpha} C f(r)\, dr .$$

Armed with these representations, it is a simple matter to compute

$$\mathcal{Q}_\alpha f = \frac{f-u_1}{\alpha}, \quad \left(a\frac{d}{dt}\ \right)_\alpha f = \frac{f-u_2}{\alpha},$$

and

$$\mathcal{Q}_\alpha f - \left(a\frac{d}{dt}\ \right)_\alpha f - u_3 = -\frac{1}{\alpha} e^{-t/a\alpha}\left[S\left(\frac{t}{a}\right) - I + \frac{t}{a}C\right] u_0$$

$$- \frac{1}{a\alpha^2}\int_0^t e^{-(t-s)/a\alpha}\left[S\left(\frac{t-s}{a}\right) - I + \frac{t-s}{a}C\right] f(s)\, ds .$$

Since

$$\left[S\left(\frac{t}{a}\right) - I + \frac{t}{a}C\right] x = \int_0^t \frac{t-s}{a^2} S\left(\frac{s}{a}\right) C^2 x\, ds$$

for all $x \in D(C^2)$,

$$\left\| \left[S\left(\frac{t}{a}\right) - I + \frac{t}{a}C\right] x \right\|_X \leq \frac{t^2}{2a^2}\left\|C^2 x\right\|_X .$$

The proof is concluded by using this estimate in the representation above:

$$\left\| \left(\mathcal{Q}_\alpha f - \left(a\frac{d}{dt}\ \right)_\alpha f - u_3\right)(t) \right\|_X \leq \frac{\alpha}{2}\left(\frac{t}{a\alpha}\right)^2 e^{-t/a\alpha}\left\|C^2 u_0\right\|_X + \alpha\left\|C^2 f\right\|_{L^\infty(0,T)} .$$

REMARKS:

1) The complicated expressions above are merely the rigorous expressions for the formal computation (2.4). The only technical detail is the assignment of the values of $v(0)$ and $u_3(0)$, which are chosen to make the approximation correct when $t = 0$. Other theorems of the same flavor may be proven under various hypotheses on f and on $S(t)$.
2) If we interpret $(I + ad/dt)^{-1}$ as a delay by a, then $\mathcal{Q}_\alpha$ decomposes approximately into

$$\mathcal{Q}_\alpha \approx a_1 \frac{d}{dt} \ \text{(delay by } a_1\alpha) + \mathcal{B}_1(\text{delay by } 2a_1\alpha) \ .$$

The flux is delayed twice as long as the capacitance. Further remarks on this interpretation follow the discussion below.

We consider now a physical derivation for our model for flow in a partially fissured medium, motivated in part by the decomposition above. The double porosity models above are based on the assumption that the exchange flow between the components has a spatially distributed density proportional to the pressure differences. That is, the fluid stored in the cell system at a point in space is determined solely by the history of the *values* of the pressures of the components at that point. In order to induce a flow within the cell matrix as a response to local coupling, however, it is necessary to apply a pressure *gradient* from the fissure system. Thus, we shall model the flux exchange as a response to both the value and gradient of the pressure. Equivalently, we assume that the local cell structure at a point is responsive to the best linear approximation of the pressure in a neighborhood of that point. Furthermore, if the geometry of the cell matrix is symmetric with respect to the coordinate system, then the response of the cell to the value and to the gradient of the pressure is additive, the terms representing even and odd responses, respectively, to even and odd input. Thus, we are led to model the resulting storage and transport responses within the cell matrix as two independent processes whose effects are additive.

Let u_0 denote the pressure in the fissures. Then the conservation of fluid in the fissures requires that

$$\frac{d}{dt}(a_0 u_0) - \vec{\nabla} \cdot B_0 \vec{\nabla} u_0 + q = f \tag{2.5.a}$$

where a_0 is relative porosity, B_0 is the permeability matrix for the fissure system, f is the distributed source density, and q is the distributed mass flow rate into the surrounding cell matrix. This exchange flow will be computed below. The primary effect of this exchange is the storage of fluid in the cells, and we model this response as

$$\frac{d}{dt}(a_1 u_1) + \frac{1}{\alpha}(u_1 - u_0) = 0 \tag{2.5.b}$$

where u_1 is the pressure in the cells. This is just (2.1.b) with $B_1 = 0$.

In order to appropriately model the flux in the cell matrix, we recall that the cells are substantially isolated from each other by the highly developed fissure system. Thus, there will be a limited global cell-to-cell secondary flow and it is driven not by the pressure gradient in the cells but by the history of the pressure gradient $\vec{\nabla} u_0$ in the surrounding fissure system. Fluid from the fissure system enters the cell matrix at a point of higher pressure, it flows through the matrix to a cell at a nearby point of lower fissure pressure, and there it exits the cell back into the fissure system. This results in a *secondary flux* $\vec{u}_2$ which we assume follows the fissure pressure gradient according to

$$\frac{d}{dt}(a_2 \vec{u}_2) + \frac{1}{\beta}(\vec{u}_2 + B_2 \vec{\nabla} u_0) = \vec{0} \; . \tag{2.5.c}$$

Thus, the rate of change of this secondary flux is proportional to its difference with $-B_2 \vec{\nabla} u_0$, and is thereby determined by the fissure pressure gradient. The matrix B_2 arises from "bridging" between the nearly isolated cells, and this distinguishes "partially" from "fully" fissured media. The resistance β may be independent of α, but we suggest the following heuristic model: since the behavior of $\vec{u}_2$ is connected to the fissure pressure gradient, it is reasonable to set $\beta = 2\alpha$. This assignment represents a resistance α to flow from the fissure to the cell, and additional resistance α to flow from the cell back to the fissure. In the interpretation of delays, the delay of information exchange between the change in fissure pressure and the flux is the sum of the delays of the fissure to cell interaction and the cell to fissure interaction. If these latter quantities are equal, then $\beta = 2\alpha$. In light of the approximation scheme (2.4), these interpretations are no surprise. There is, however, no need to assume a particular relation between α and β in what follows. We note that (2.5.c) follows from (2.1.b) if $B_1 = 0$, $\vec{u}_2 = -B_2 \vec{\nabla} u_1$, $a_1 = a_2$ and $\alpha = \beta$.

Conservation of fluid in the fissures and the pores requires that the flux q equal the exchange rate between the fissures and pores,$-d(a_1u_1)/dt$, plus the rate due to the "secondary flux" $\vec{u}_2$. Using (2.5.b) for the exchange rate and Fick's law for the secondary flux rate, we write

$$q = \frac{1}{\alpha}(u_0 - u_1) + \vec{\nabla}\cdot B_2^T\vec{u}_2. \tag{2.5.d}$$

Notice that the cell flux as seen from the fissures is $B_2^T\vec{u}_2$. This model arises naturally when $a_2 = 0$, for then $\vec{u}_2 = -B_2\vec{\nabla}u_0$ and this flux becomes $-B_2^TB_2\vec{\nabla}u_0$. Since Fick's law can be written in this form whenever the diffusivity matrix is positive semidefinite and symmetric, (2.5) reduces to the usual diffusion equation for a binary system when $a_2 = 0$. In the general case we can eliminate u_1 and $\vec{u}_2$ from the system (2.5) to obtain the functional differential equation

$$\begin{aligned}\frac{d}{dt}(a_0u_0) - \vec{\nabla}\cdot B_0\vec{\nabla}u_0 + \frac{d}{dt}a_1\left(I + \alpha\frac{d}{dt}a_1\right)^{-1} u_0 \\ - \vec{\nabla}\cdot B_2^T\left(I + \beta\frac{d}{dt}a_2\right)^{-1} B_2\vec{\nabla}u_0 = f\ .\end{aligned} \tag{2.6}$$

The resolvent operators above can be expressed as *convolutions*, and then (2.6) is recognized as a special case of *Nunzioto's equation* [16]. See [17] for the mathematical and numerical analysis of this approach with *microstructure* in the completely-fissured case, $B_2 = 0$.

It is interesting to note the special limiting cases of the system (2.5) that result by letting certain coefficients vanish. For example, if $\beta \to 0$ we obtain formally the system

$$\frac{d}{dt}(a_0u_0) - \vec{\nabla}\cdot(B_0 + B_2^TB_2)\vec{\nabla}u_0 + \frac{1}{\alpha}(u_0 - u_1) = f \tag{2.7.a}$$

$$\frac{d}{dt}(a_1u_1) + \frac{1}{\alpha}(u_1 - u_0) = 0 \tag{2.7.b}$$

which is the *first-order rate* model (2.3). (The same form also results by letting $B_2 \to 0$.) Further, letting $a_0 \to 0$ in (2.7) leads to the *fissured medium equation*

$$\frac{d}{dt}a_1\big(u_0 - \alpha\vec{\nabla}\cdot(B_0 + B_2^TB_2)\vec{\nabla}u_0\big) - \vec{\nabla}\cdot(B_0 + B_2^TB_2)\vec{\nabla}u_0 = \left(I + \frac{d}{dt}a_1\right)f\ . \tag{2.8}$$

On the other hand, if we let $a_0 \to 0$ directly in (2.5) (or (2.6)) we obtain formally the *Sobolev-type* equation [8]

$$
\begin{aligned}
&-\left(I+\alpha\frac{d}{dt}a_1\right)\left(I+\beta\frac{d}{dt}a_2\right)\vec{\nabla}\cdot B_0\vec{\nabla}u_0+\frac{d}{dt}a_1\left(I+\beta\frac{d}{dt}a_2\right)u_0\\
&\quad-\left(I+\alpha\frac{d}{dt}a_1\right)\vec{\nabla}\cdot B_2^TB_2\vec{\nabla}u_0=\left(I+\alpha\frac{d}{dt}a_1\right)\left(I+\beta\frac{d}{dt}a_2\right)f\ .
\end{aligned}
\tag{2.9}
$$

This in turn becomes a fissured medium equation if either $\beta \to 0$, $B_2 \to 0$, or if $\alpha a_1 = \beta a_2$. The continuous dependence of the solutions of (2.5) on coefficients will be the subject of Section 5.

3. Definitions, notation, and review of previous results.

We recall some well-known facts on maximal monotone operators and some existence results on doubly-nonlinear evolution equations that are immediate consequences of [10]. We shall also develop the semi-linear case by the method of [19]. Let $\mathcal{V}$ be a Hilbert space with inner product $(\cdot,\cdot)$. If $\mathcal{V}^*$ denotes the dual of $\mathcal{V}$, the Riesz representation theorem guarantees the existence of a map $\mathcal{R} : \mathcal{V} \to \mathcal{V}^*$, called the Riesz isomorphism, which satisfies

$$(u, v) = \langle \mathcal{R}u, v\rangle \ \ \forall\ u, v \in \mathcal{V}\ ,$$

where $\langle\cdot,\cdot\rangle$ is the duality pairing between $\mathcal{V}^*$ and $\mathcal{V}$. A subset $\mathcal{B} \subset \mathcal{V} \times \mathcal{V}^*$, is called *monotone* if

$$\langle v_2 - v_1, u_2 - u_1\rangle \geq 0$$

whenever $[u_i, v_i] \in \mathcal{B}$, $i = 1, 2$. In this case, $\mathcal{B}$ is thought of as a (possibly) multivalued operator from $\mathcal{V}$ to $\mathcal{V}^*$. To say $v \in \mathcal{B}(u)$ means $[u, v] \in \mathcal{B}$. $\mathcal{B}$ is *maximal* monotone if, in addition to being monotone, $\mathcal{B}$ has no monotone proper extension in $\mathcal{V} \times \mathcal{V}^*$. This is equivalent to the condition that $(\mathcal{R} + \lambda\mathcal{B})^{-1} = J_\lambda$, the resolvent of $\mathcal{B}$, is a contraction defined on all of $\mathcal{V}^*$ for any $\lambda > 0$. The *Yosida approximation* of $\mathcal{B}$ is $\mathcal{B}_\lambda = \mathcal{R}(I - J_\lambda \circ \mathcal{R})/\lambda$; it is Lipschitz continuous and monotone : $\mathcal{V} \to \mathcal{V}^*$. If $u \in \mathcal{V}$, $\mathcal{B}_\lambda(u) \in \mathcal{B}(J_\lambda(u))$. If $\mathcal{B}$ is maximal monotone, $[u_n, v_n] \in \mathcal{B}$, $u_n \rightharpoonup u$ (*i.e.*, u_n converges weakly to u), $v_n \rightharpoonup v$, and $\liminf \langle u_n, v_n\rangle \leq \langle u, v\rangle$, then $[u, v] \in \mathcal{B}$. If also $\limsup \langle u_n, v_n\rangle \leq 0$, then we have the additional information $\lim \langle u_n, v_n\rangle = \langle u, v\rangle$. A maximal monotone operator $\mathcal{B}$ on $\mathcal{V}$

induces a maximal monotone operator (still denoted by $\mathcal{B}$) on $L^2(0,T;\mathcal{V})$ by $v \in \mathcal{B}(u)$ iff $v(t) \in \mathcal{B}(u(t))$ *a.e.* on $[0,T]$. It is often convenient to interpret maximal monotone operators as maps from $\mathcal{V}$ to $2^{\mathcal{V}}$ *via* the Riesz isomorphism $\mathcal{R}^{-1} : \mathcal{V}^* \to \mathcal{V}$. We shall use these two notions interchangeably.

A special class of maximal monotone operators is the class of subgradients. If $\varphi : \mathcal{V} \to (-\infty, \infty]$ is a lower semicontinuous, proper, convex function, then the *subgradient* $\partial\varphi \subset \mathcal{V} \times \mathcal{V}^*$ is defined by

$$\partial\varphi(u) = \{v \in \mathcal{V} : \varphi(\hat{u}) - \varphi(u) \geq \langle v, \hat{u} - u\rangle \ \ \forall\, \hat{u} \in \mathcal{V}\} \ .$$

In this case, $\partial\varphi$ is maximal monotone. The *conjugate* of φ is the convex function $\varphi^* : \mathcal{V}^* \to \mathbb{R}^1$ defined by

$$\varphi^*(v) = \sup_{u \in \mathcal{V}} \left(\langle v, u\rangle - \varphi(u)\right) \ .$$

This function is chosen so that $\partial\varphi^{-1} = \partial\varphi^*$; thus $v \in \partial\varphi(u)$ iff $u \in \partial\varphi^*(v)$ iff $\varphi(u) + \varphi^*(v) = \langle u, v\rangle$. We assume throughout that $\varphi(0) \leq 0$ so that $\varphi^*(v) \geq 0$ for all $v \in \mathcal{V}^*$. If $v \in H^1(0,T;\mathcal{V}^*)$ and $[u,v]$ belongs to the $L^2(0,T;\mathcal{V})$ realization of $\partial\varphi$, then

$$\frac{d}{dt}\varphi^*(v(t)) = \left(\frac{d}{dt}v(t), u(t)\right) \ a.e. \text{ on } \ [0,T] \ .$$

We now recall the existence results from [10]. Let $\mathcal{A} : \mathcal{V} \to \mathcal{V}^*$ be the subgradient of a proper, convex, and lower semicontinuous function $\varphi : \mathcal{V} \to \mathbb{R}^1$, and suppose $\mathcal{A}$ is bounded. Let $\mathcal{B} : \mathcal{V} \to \mathcal{V}^*$ be maximal monotone and bounded. Let $\mathcal{R}$ denote the Riesz map $: \mathcal{V} \to \mathcal{V}^*$. Fix $f \in L^2(0,T;\mathcal{V}^*)$ and $[u_0, v_0] \in \mathcal{A}$. Then for each $\lambda > 0$, there is a pair $u_\lambda \in H^1(0,T;\mathcal{V})$, $v_\lambda \in H^1(0,T;\mathcal{V}^*)$ such that

$$\text{(3.1)} \qquad \begin{cases} v_\lambda(t) \in \mathcal{A}(u(t)) \text{ for all } t \in [0,T] \ , \\ \dfrac{d}{dt}\left(\mathcal{R}u_\lambda(t) + v_\lambda(t)\right) + \mathcal{B}_\lambda(u_\lambda(t)) = f(t), \\ \mathcal{R}u_\lambda(0) + v_\lambda(0) = \mathcal{R}u_0 + v_0 \ . \end{cases}$$

These functions satisfy the *a priori* estimates

$$\|u_\lambda\|_{L^\infty(0,T;\mathcal{V})} \quad , \quad \|v_\lambda\|_{L^\infty(0,T;\mathcal{V}^*)} \quad , \quad \|J_\lambda(\mathcal{R}u_\lambda)\|_{L^\infty(0,T;\mathcal{V})} \ ,$$
$$\|\mathcal{B}_\lambda(u_\lambda)\|_{L^\infty(0,T;\mathcal{V}^*)} \quad , \quad \|u'_\lambda\|_{L^2(0,T;\mathcal{V})} \quad , \quad \|\mathcal{R}u'_\lambda\|_{L^2(0,T;\mathcal{V}^*)}$$

are bounded independent of $\lambda > 0$. Choose a subsequence (still denoted by subscript λ) for which

$$u_\lambda \rightharpoonup u,\ u'_\lambda \rightharpoonup u' \text{ in } L^2(0,T;\mathcal{V})\ ,$$

$$v_\lambda \rightharpoonup v,\ v'_\lambda \rightharpoonup v' \text{ in } L^2(0,T;\mathcal{V}^*)\ ,\ \text{and}$$

$$\mathcal{B}_\lambda(u_\lambda) \rightharpoonup w \text{ in } L^2(0,T;\mathcal{V}^*)\ .$$

Note that, since $\{v_\lambda\}$ and $\{u_\lambda\}$ are uniformly equicontinuous functions, if follows that

$$u_\lambda(t) \to u(t) \text{ and } v_\lambda(t) \to v(t) \text{ for all } t \in [0,T]\ . \tag{3.2}$$

The differential equation

$$\frac{d}{dt}\left(\mathcal{R}u(t) + v(t)\right) + w(t) = f(t)\ \ a.e. \text{ on } [0,T]$$

follows from the convergence hypotheses. The difficulty lies in checking that $v \in \mathcal{A}(u)$ and $w \in \mathcal{B}(u)$ *a.e.* on $[0,T]$. The first result halves the work needed to verify these two conditions.

THEOREM 1. *If it can be shown that $v(t) \in \mathcal{A}(u(t))$ a.e. on $[0,T]$, then the triple $[u,v,w]$ satisfies*

$$\left\{\begin{array}{l} u \in H^1(0,T;\mathcal{V})\ ,\ \ v \in H^1(0,T;\mathcal{V}^*)\ ,\ \ w \in L^2(0,T;\mathcal{V}^*)\ , \\ v \in \mathcal{A}(u)\ ,\ \ w \in \mathcal{B}(u)\ \ a.e.\ \ on\ [0,T]\ , \\ \dfrac{d}{dt}(\mathcal{R}u(t) + v(t)) + w(t) = f(t)\ \ a.e.\ \ on\ [0,T]\ ,\ \ and \\ \mathcal{R}(u(0)) + v(0) = \mathcal{R}(u_0) + v_0\ . \end{array}\right. \tag{3.3}$$

We remark that in [10], the condition $v \in \mathcal{A}u$ *a.e.* follows from a compactness assumption on the operator $\mathcal{A}$. For the systems which follow, this assumption is inappropriate for at least some of the components of $\mathcal{A}$. We shall overcome this difficulty by requiring further structure on these components.

The second existence result of [10] concerns the (possibly) degenerate Cauchy problem

$$
(3.4) \qquad \begin{cases} u \in L^2(0,T;\mathcal{V}) \ , \ \ v \in H^1(0,T;\mathcal{V}^*) \ , \ \ w \in L^2(0,T;\mathcal{V}^*) \ , \\[2ex] v \in \mathcal{A}(u) \ , \ \ w \in \mathcal{B}(u) \ \ a.e. \text{ on } [0,T] \ , \\[2ex] \dfrac{d}{dt}v(t) + w(t) = f(t) \ \ a.e. \text{ on } [0,T] \ , \text{ and} \\[2ex] v(0) = v_0 . \end{cases}
$$

The additional hypotheses for Theorem 2 are that the realizations $\mathcal{A} : L^2(0,T;\mathcal{V}) \to L^2(0,T;\mathcal{V}^*)$ and $\mathcal{B} : L^2(0,T;\mathcal{V}) \to L^2(0,T;\mathcal{V}^*)$ are bounded, and that the solutions to

$$
(3.5) \qquad \begin{cases} v_\lambda \in \mathcal{A}(u_\lambda), w_\lambda \in \mathcal{B}(u_\lambda) \ \ a.e. \text{ on } [0,T] \ , \\ \dfrac{d}{dt}\big(\lambda \mathcal{R} u_\lambda(t) + v_\lambda(t)\big) + w_\lambda(t) = f(t) \ \ a.e. \text{ on } [0,T], \text{ and} \\ \lambda \mathcal{R}(u_\lambda(0)) + v_\lambda(0) = \lambda \mathcal{R}(u_0) + v_0 \ . \end{cases}
$$

satisfy $\|u_\lambda\|_{L^2(0,T;\mathcal{V})} \le M$ for some M independent of λ. Then additional *a priori* bounds are derived, from which it follows that some subsequence (still denoted by subscript λ) satisfies

$$
\begin{aligned}
& u_\lambda \rightharpoonup u \text{ in } L^2(0,T;\mathcal{V}) \\
& v_\lambda \rightharpoonup v \text{ and } v'_\lambda \rightharpoonup v' \text{ in } L^2(0,T;\mathcal{V}^*) \\
& w_\lambda \rightharpoonup w \text{ in } L^2(0,T;\mathcal{V}^*)
\end{aligned}
$$

Again, the difficulty lies in the verification that $v \in \mathcal{A}(u)$ and $w \in \mathcal{B}(u)$, and once again, the gist of the theorem is that only one of these conditions need be verified.

THEOREM 2. *If it can be shown that $v \in \mathcal{A}(u)$ a.e., then the triple $[u,v,w]$ is a solution to (3.4).*

The proof that $v \in \mathcal{A}(u)$ *a.e.* follows in [10] from a compactness argument on $\mathcal{A}$; we have already discussed the modifications we shall make to this hypothesis.

In [10], the *a priori* bound on $\|u_\lambda\|_{L^2(0,T;\mathcal{V})}$ is obtained by requiring $\mathcal{B}$ to be $L^2(0,T;\mathcal{V})$-coercive. This hypothesis is not valid for at least some models in which we are interested. If, in these cases, we require that $\mathcal{A}$ be linear, a rather complete theory is available.

THEOREM 3. *Let $\mathcal{V}$ be a topological vector space with dual $\mathcal{V}^*$. Let $\mathcal{A} : \mathcal{V} \to \mathcal{V}^*$ be linear, symmetric, monotone and continuous, and denote by $\mathcal{V}_a$ the semi-norm-space obtained from the semi-scalar-product $\mathcal{A}u(v)$ on $\mathcal{V}$. (Thus $\mathcal{V}_a^*$ is a Hilbert space.) Let $\mathcal{B} \subset \mathcal{V} \times \mathcal{V}^*$ be a monotone relation and assume $Rg(\mathcal{A} + \mathcal{B}) \supset \mathcal{V}_a^*$. Then for each $v_0 \in Rg(\mathcal{B}) \cap \mathcal{V}_a^*$ and absolutely continuous $f : [0,T] \to \mathcal{V}_a^*$ there exist functions $u, w : [0,T] \to \mathcal{V}$ for which $\mathcal{A}u : [0,T] \to \mathcal{V}_a^*$ is absolutely continuous, $\mathcal{A}u(0) = v_0$, and*

$$\frac{d}{dt}\mathcal{A}u(t) + w(t) = f(t) \ , \quad a.e. \quad t \in [0,T] \ , \tag{3.6.a}$$

$$w \in L^\infty(0,T;\mathcal{V}_a^*) \quad , \quad w(t) \in \mathcal{B}\big(u(t)\big) \ , \quad t \in [0,T] \ . \tag{3.6.b}$$

The functions $\mathcal{A}u(\cdot)$ and w are uniquely determined. If $\mathcal{A} + \mathcal{B}$ is strictly monotone, then u is unique.

PROOF. Denote the kernel of $\mathcal{A}$ by K and form the quotient space $\mathcal{V}/K$ and its completion W. Thus W is a Hilbert space with scalar product given by

$$(\tilde{u}, \tilde{v})_W = \mathcal{A}u(v) \ , \qquad \tilde{u} = q(u) \ , \ \tilde{v} = q(v) \ ,$$

where the quotient map $q : \mathcal{V}_a \to W$ is a strict homomorphism with dense range. Thus the dual $q^* : W^* \to \mathcal{V}_a^*$ is an isomorphism, and the Riesz isomorphism $\mathcal{A}_0 : W \to W^*$ is determined by $\mathcal{A} = q^*\mathcal{A}_0 q$. Define $D = \{u \in \mathcal{V} : \mathcal{B}(u) \cap \mathcal{V}_a^* \neq \emptyset\}$ and restrict $\mathcal{B}$ to $\mathcal{V}_a^*$. That is, we replace $\mathcal{B}$ by $\mathcal{B} \cap (D \times \mathcal{V}_a^*)$. Then define $\mathcal{B}_0 : q[D] \to \mathcal{V}_a^*$ by $\mathcal{B}_0 \equiv (q^*)^{-1} \cdot \mathcal{B} \cdot q^{-1}$. Finally, we define $\mathcal{C} : q[D] \to W$ by $\mathcal{C} = \mathcal{A}_0^{-1} \cdot \mathcal{B}$. It follows that

$$\mathcal{B}u(v) = \mathcal{A}_0\mathcal{C}q(u)\big(q(v)\big) = (\mathcal{C}\tilde{u}, \tilde{v})_W \qquad u, v \in D \ ,$$

so $\mathcal{B}$ is monotone implies $\mathcal{C}$ is monotone in the Hilbert space W. Also $q^*\mathcal{A}_0$ is an isomorphism and

$$\mathcal{A} + \mathcal{B} = q^*\mathcal{A}_0(I + \mathcal{C})q$$

so $Rg(\mathcal{A} + \mathcal{B}) = \mathcal{V}_a^*$ implies $Rg(I + \mathcal{C}) = W$. Thus $\mathcal{C}$ is maximal monotone in W. Finally note that (3.6) is equivalent to the evolution equation

$$\frac{d}{dt}\tilde{u}(t) + \mathcal{C}\big(\tilde{u}(t)\big) = (q^*\mathcal{A}_0)^{-1}f(t) \ , \quad \text{a.e.} \quad t \in (0,T)$$

for $\tilde{u}(t) = q(u(t))$ in W, so the desired results follow from [7]. ∎

The semigroup theory gives the continuous dependence on $\mathcal{B}$:

COROLLARY. *Let $\mathcal{A}, \mathcal{V}, \mathcal{V}_a$ be given as above. For each $n = 0, 1, 2, \ldots$, let $\mathcal{B}_n$ be monotone, $Rg(\mathcal{A} + \mathcal{B}_n) \supset \mathcal{V}_a^*$, $v_n \in Rg(\mathcal{B}_n) \cap \mathcal{V}_a^*$, and $f_n \in L^1(0, T; \mathcal{V}_a^*)$ and let u_n and u_0 be solutions of*

$$\frac{d}{dt}(\mathcal{A}u_n) + \mathcal{B}_n(u_n) \ni f_n \quad , \quad \mathcal{A}u_n(0) = v_n \ .$$

If $v_n \to v_0$ in $\mathcal{V}_a^*$, $f_n \to f_0$ in $L^1(0, T; \mathcal{V}_a^*)$ and if for every $g \in \mathcal{V}_a^*$ and solutions w_n of $(\mathcal{A} + \mathcal{B}_n)w_n \ni g$ we have $w_n \to w_0$ in $\mathcal{V}_a$, where $(\mathcal{A} + \mathcal{B}_0)w_0 \ni g$, then $\mathcal{A}u_n \to \mathcal{A}u_0$ in $\mathcal{C}(0, T; \mathcal{V}_a^*)$.

See [8] for a review of results on various classes of equations of the form (3.6) and an extensive collection of examples of initial-boundary-value problems which are thereby obtained. For related recent results see [11], [12], [20], [1], [3].

4. Systems.

We shall apply Theorems 1 and 2 in the following setting for which the additional structure is motivated by the model problems given in Section 2. Let V, H_1, and H_2 be Hilbert spaces, and denote by $\mathcal{V}$ the (Hilbert space) product $V \times H_1 \times H_2$. Identify H_1 and H_2 with their respective dual spaces, so the Riesz mappings are the identity mappings on these two spaces. We suppose that there is a (pivot) Hilbert space H_0 with V dense and compactly embedded in H_0. Let $i : V \to H_0$ be the embedding. Let $\varphi_k : H_k \to \mathbb{R}^1$, $k = 0, 1, 2$ be convex, lower semicontinuous, proper functions. Define $\varphi : \mathcal{V} \to \mathbb{R}^1$ by

$$\varphi(u) = \varphi_0(iu_0) + \varphi_1(u_1) + \varphi_2(u_2) \ , \qquad u = [u_0, u_1, u_2] \in \mathcal{V} \ .$$

Then φ is convex, lower semicontinuous and proper. Let $\mathcal{A} = \partial\varphi$. Clearly this subgradient is computed componentwise as

$$\mathcal{A}u(v) = \partial\varphi_0(iu_0)(iv_0) + \partial\varphi_1(u_1)(v_1) + \partial\varphi_2(u_2)(v_2) \ ,$$
$$u = [u_0, u_1, u_2] \ , \quad v = [v_0, v_1, v_2] \in \mathcal{V} \ .$$

Let $\mathcal{B} : \mathcal{V} \to \mathcal{V}^*$ be given, and denote the components of $\mathcal{B}u$ by

$$\mathcal{B}u = [\mathcal{B}_0 u, \mathcal{B}_1 u, \mathcal{B}_2 u] \in 2^{V^*} \times H_1^* \times 2^{H_2^*} \ , \quad u = [u_0, u_1, u_2] \in \mathcal{V} \ .$$

In order to apply Theorem 1, we make the following assumptions:

[A$_1$] φ_0 is continuous at some point of V and $\partial\varphi_0 \circ i : V \to H_0^*$ is bounded; $\partial\varphi_1 = \mathcal{A}_1 : H_1 \to H_1^*$ is bounded; and $\partial\varphi_2 = \mathcal{A}_2 : H_2 \to H_2^*$ is bounded and linear.

[B$_1$] $\mathcal{B} : \mathcal{V} \to \mathcal{V}^*$ is maximal monotone and bounded; $\mathcal{B}_1$ is Lipschitz continuous, and if $u_{\lambda 0} \rightharpoonup u_0$ in V and $u_{\lambda 2} \rightharpoonup u_2$ in H_2, then $\mathcal{B}_1(u_{\lambda 0}, \cdot, u_{\lambda 2})$ converges uniformly on bounded subsets of H_1^* to $\mathcal{B}_1(u_0, \cdot, u_2)$.

THEOREM 1$'$. *Under hypotheses* [A$_1$] *and* [B$_1$], *there exists a solution* (u, v, w) *of* (3.3).

PROOF. Suppose $[u_\lambda, v_\lambda]$ are solutions to (3.1), and that $u_\lambda \rightharpoonup u$ in $\mathcal{V}$ and $v_\lambda \rightharpoonup v$ in $\mathcal{V}^*$. Our goal is to show that $v \in \partial\varphi(u)$. We consider each of the components separately.

The last component is easiest to understand. Since $\mathcal{A}_2$ is a bounded linear operator, it is continuous from the weak topology on H_2 to the weak topology on H_2^*. Since $(v_\lambda)_2 = \mathcal{A}_2(u_{\lambda 2})$ for each $\lambda > 0$, it follows that $v_2 = \mathcal{A}_2(u_2)$.

The first component is handled as in [10]. We know that $\{v_{\lambda 0}\} = \{\partial\varphi_0 \circ i(u_{\lambda 0})\}$ is bounded in $L^2(0, T; H_0^*)$, and that $\{dv_{\lambda 0}/dt\}$ is bounded in $L^2(0, T; \mathcal{V}^*)$. We conclude from Aubin's Theorem (cf. p.58 of [15]) that $\{v_{\lambda 0}\}$ is relatively compact in the strong topology in $L^2(0, T; V^*)$. It then follows (using the uniform equicontinuity of $\{v_{\lambda 0}\}$) that $v_{\lambda 0}(t) \to v_0(t)$ in V^* for all $t \in [0, T]$; thus $v_0(t) \in \mathcal{A}_0(u_0(t))$ for all $t \in [0, T]$.

The second component is handled using ordinary differential equation techniques. Let

$$z_\lambda = u_{\lambda 1} + v_{\lambda 1}, \text{ so } u_{\lambda 1} = (I + \mathcal{A}_1)^{-1} z_\lambda \; .$$

Then

$$\begin{cases} \dfrac{d}{dt} z_\lambda + F_\lambda(z_\lambda, t) = f_1(t) \; , \\ z_\lambda(0) = u_{01} + v_{01} \; , \end{cases} \tag{4.1}$$

where $F_\lambda(z, t) = \mathcal{B}_{\lambda 1}(u_{\lambda 0}(t), (I + \mathcal{A}_1)^{-1} z, u_{\lambda 2}(t))$. Integrating (4.1) for parameters λ and μ yields

$$0 = z_\lambda(t) - z_\mu(t) + \int_0^t F_\lambda\big(z_\lambda(s), s\big) - F_\mu\big(z_\mu(s), s\big)\, ds \; .$$

We use the triangle inequality to get

$$\begin{aligned} \|z_\lambda(t) - z_\mu(t)\|_{H_1} \le \bigg(\int_0^t & \big\|F_\lambda\big(z_\lambda(s), s\big) - F_\lambda\big(z_\mu(s), s\big)\big\|_{H_1} \\ & + \big\|F_\lambda\big(z_\mu(s), s\big) - F_\mu\big(z_\mu(s), s\big)\big\|_{H_1}\bigg)\, ds \; . \end{aligned}$$

Since $\mathcal{B}_1$ is Lipschitz continuous and $\mathcal{B}_\lambda(x) \in \mathcal{B}J_\lambda(x)$ where $J_\lambda(x) = (I + \lambda\mathcal{B})^{-1}$,

$$\begin{aligned}
&\left\|F_\lambda\big(z_\lambda(t), t\big) - F_\lambda\big(z_\mu(t), t\big)\right\|_{H_1} = \\
&\Big\|\mathcal{B}_1\big(J_\lambda(u_{\lambda 0}(t), (I+\mathcal{A}_1)^{-1} z_\lambda(t), u_{\lambda 2}(t))\big) \\
&\qquad\qquad -\mathcal{B}_1\big(J_\lambda(u_{\lambda 0}(t), (I+\mathcal{A}_1)^{-1} z_\mu(t), u_{\lambda 2}(t))\big)\Big\|_{H_1} \\
&\quad \leq M \left\|z_\lambda(t) - z_\mu(t)\right\|_{H_1} \ ,
\end{aligned}$$

where M is a Lipschitz constant for $\mathcal{B}_1$. Here we have used the facts that J_λ and $(I+\mathcal{A}_1)^{-1}$ are contractions.

We now apply Gronwall's inequality to estimate

$$\left\|z_\lambda(t) - z_\mu(t)\right\|_{H_1} \leq \int_0^t \mathrm{e}^{M(t-s)} \left\|F_\lambda\big(z_\mu(s), s\big) - F_\mu\big(z_\mu(s), s\big)\right\|_{H_1} \, ds \ .$$

Since $\{z_\mu\}$ is bounded in $L^\infty(0, T; H_1)$, it suffices to show that F_λ converges uniformly on bounded subsets of H_1. It will then follow that $u_{\lambda 1} = (I + \mathcal{A}_1)^{-1}(z_\lambda)$ is strongly convergent in H_1, and thus $v_1 \in \mathcal{A}_1(u_1)$. As in the previous argument, we have

$$\begin{aligned}
&\left\|F_\lambda(z,t) - F_\mu(z,t)\right\|_{H_1} \leq \\
&\Big\|\mathcal{B}_1(J_\lambda(u_{\lambda 0}(t), (I+\mathcal{A}_1)^{-1} z, u_{\lambda 2}(t))) - \mathcal{B}_1(u_{\lambda 0}(t), (I+\mathcal{A}_1)^{-1} z, u_{\lambda 2}(t))\Big\|_{H_1} \\
&\quad + \Big\|\mathcal{B}_1(u_{\lambda 0}(t), (I+\mathcal{A}_1)^{-1} z, u_{\lambda 2}(t)) - \mathcal{B}_1(u_{\mu 0}(t), (I+\mathcal{A}_1)^{-1} z, u_{\mu 2}(t))\Big\|_{H_1} \\
&\quad + \Big\|\mathcal{B}_1(u_{\mu 0}(t), (I+\mathcal{A}_1)^{-1} z, u_{\mu 2}(t)) - \mathcal{B}_1(J_\mu(u_{\mu 0}(t), (I+\mathcal{A}_1)^{-1} z, u_{\mu 2}(t)))\Big\|_{H_1} \\
&\leq M \Big[\lambda \Big\|\mathcal{B}_1(J_\lambda(u_{\lambda 0}(t), (I+\mathcal{A})^{-1} z, u_{\lambda 2}(t)))\Big\|_{H_1} \\
&\quad + \mu \Big\|\mathcal{B}_1(J_\mu(u_{\mu 0}(t), (I+\mathcal{A})^{-1} z, u_{\mu 2}(t)))\Big\|_{H_1}\Big] \\
&\quad + \Big\|\mathcal{B}_1(u_{\lambda 0}(t), (I+\mathcal{A}_1)^{-1} z, u_{\lambda 2}(t)) - \mathcal{B}_1(u_{\mu 0}(t), (I+\mathcal{A}_1)^{-1} z, u_{\mu 2}(t))\Big\|_{H_1} .
\end{aligned}$$

It follows from the hypotheses on $\mathcal{B}_1$ that $\{F_\lambda(\cdot, t)\}$ is uniformly Cauchy on bounded subsets of H_1. ∎

We may apply the above arguments in the situation of Theorem 2 as well, provided

[A$_2$] $\mathcal{A} : L^2(0, T; \mathcal{V}) \to L^2(0, T; \mathcal{V}^*)$ is bounded,

[B$_2$] $\mathcal{B} : L^2(0, T; \mathcal{V}) \to L^2(0, T; \mathcal{V}^*)$ is bounded and coercive.

THEOREM 2′. *Assume* [A$_2$] *and* [B$_2$] *in addition to the above. Then there exists a solution* (u, v, w) *of* (3.4).

PROOF. The arguments concerning $[u_{\lambda 0}, v_{\lambda 0}]$ and $[u_{\lambda 2}, v_{\lambda 2}]$ are unchanged; we need only modify the ordinary differential equation argument for the second component.

Let $[u_\lambda, v_\lambda, w_\lambda]$ satisfy (3.5), and define

$$z_\lambda = \lambda u_{\lambda 1} + v_{\lambda 1}, \text{ so } u_{\lambda 1} = (\lambda I + \mathcal{A}_1)^{-1} z_\lambda \ .$$

Then

$$\begin{aligned} &\frac{d}{dt} z_\lambda + F_\lambda(z_\lambda, t) = f_1(t) \ , \\ &z_\lambda(0) = \lambda u_{01} + v_{01} \ , \end{aligned} \tag{4.2}$$

where $F_\lambda(z,t) = \mathcal{B}_{\lambda 2}(u_{\lambda 0}(t), (\lambda I + \mathcal{A}_1)^{-1} z, u_{\lambda 2}(t))$. Integrating (4.2) for parameters λ and μ yields

$$z_\lambda(t) - z_\mu(t) + \int_0^t F_\lambda(z_\lambda(s), s) - F_\mu(z_\mu(s), s)\, ds = (\lambda - \mu) u_{02} \ .$$

We use the triangle inequality and the fact that $F_\lambda(\cdot, t)$ is Lipschitz continuous as in the previous argument to get

$$\begin{aligned} \|z_\lambda(t) - z_\mu(t)\|_{H_1} \le |\lambda - \mu|\, \|u_{02}\|_{H_1} + \int_0^t \Big(& M\, \|z_\lambda(s) - z_\mu(s)\|_{H_1} \\ & + M \left\| (\lambda I + \mathcal{A}_1)^{-1} (z_\mu(s), s) - (\mu I + \mathcal{A}_1)^{-1} (z_\mu(s), s) \right\|_{H_1} \Big)\, ds \ . \end{aligned}$$

We now apply Gronwall's inequality and use the fact that $(\lambda I + \mathcal{A}_1)^{-1} z_\mu(s) - z_\mu(s) = -\lambda \mathcal{A}_{1\lambda} z_\mu(s)$ to estimate

$$\|z_\lambda(t) - z_\mu(t)\|_{H_1} \le \mathrm{e}^{Mt} |\lambda - \mu|\, \|u_{01}\|_{H_1} + \int_0^t M \mathrm{e}^{M(t-s)} (\lambda + \mu)\, \|\mathcal{A}_1(z_\mu(s))\|_{H_1}\, ds \ .$$

Since $\{z_\mu\}$ and $\mathcal{A}_1$ are bounded in $L^\infty(0, T; H_1)$, we conclude that $z_\lambda(t)$ converges strongly in H_1, uniformly on $[0, T]$. From the definition of z_λ,

$$v_{\lambda 2} - v_{\mu 2} = (z_\lambda - z_\mu) - (\lambda u_{\lambda 2} - \mu u_{\mu 2}) \ .$$

We know from the *a priori* estimates that $\lambda u_\lambda \to 0$, so $v_{\lambda 2} \to v_2$; hence $v_2 \in \mathcal{A}_2(u_2)$. Theorem 2 is, therefore, applicable. ∎

The requirement that $\mathcal{B}$ be coercive is too restrictive for some applications. In these cases, an existence theory exists when $\mathcal{A}$ is linear (cf. Theorem 3). Examples are given in Section 6.

5. Continuous dependence on $\mathcal{A}$.

We study now the dependence of the solutions to (3.4) upon the subgradient $\mathcal{A} = \partial\varphi$ and the data f and v_0. In particular, we are interested in the dependence of the solutions to the model problems in Section 2 upon the choice of model. The results in this section are independent of the system structure from Section 4. Let $\{\mathcal{A}_n\}$, $\{f_n\}$, and $\{v_{0n}\}$ be sequences to which correspond (not necessarily unique) solutions $[u_n, v_n, w_n]$ to the problems

$$(5.1)\qquad \begin{cases} v_n \in \mathcal{A}_n(u_n), w_n \in \mathcal{B}(u_n) \ , \\ \dfrac{d}{dt} v_n + w_n = f_n \ , \\ v_n(0) = v_{0n} \ . \end{cases}$$

We assume that $\varphi_n^* \to \varphi^*$ uniformly on bounded subsets of $\mathcal{V}^*$, that $v_{0n} \to v_0$ for some $v_0 \in \mathcal{V}^*$, and that $f_n \to f$ in $L^2(0,T;\mathcal{V}^*)$. Assume that for some subsequence (still denoted by subscript n),

$$(5.2)\qquad \begin{aligned} &u_n \rightharpoonup u \text{ in } L^2(0,T;\mathcal{V}) \ , \\ &\frac{d}{dt} v_n \rightharpoonup \frac{d}{dt} v \text{ in } L^2(0,T;\mathcal{V}^*) \ , \\ &w_n \rightharpoonup w \text{ in } L^2(0,T;\mathcal{V}^*) \ , \\ &\text{and that } v \in \mathcal{A}(u), w \in \mathcal{B}(u) \ . \end{aligned}$$

If $[u,v,w]$ is a solution to (3.4), then the weak convergence above is actually strong convergence in many cases.

Proposition 2. *If the hypotheses discussed above hold, then*

$$(5.3)\qquad \begin{aligned} &\varphi_n^*(v_n) \to \varphi^*(v) \ \textit{in } L^\infty(0,T;\mathbb{R}^1) \ \textit{and} \\ &\lim_{n\to\infty} \int_0^T \langle w_n(t) - w(t), u_n(t) - u(t)\rangle \, dt = 0. \end{aligned}$$

The following remarks are in order:

1) The solution $[u, v, w]$ to (3.4) need not be unique.
2) The point of Proposition 2 is that we can deduce *strong* convergence of u_n or v_n in certain cases. Specific examples follow.
3) It's possible to have $\mathcal{B}$ depend on n, too, but the technical details are more distracting than enlightening.
4) In practice, the weak convergences of (5.2) are guaranteed from *a priori* bounds arising from $\mathcal{V}$-coercivity of $\mathcal{B}$ and hypotheses on the convergence of $\mathcal{A}_n$. See the last example of Section 6.

PROOF. Compute the $L^2(0,T;\mathcal{V}^*) - L^2(0,T;\mathcal{V})$ duality pairing of the equations in (5.1) and (3.4) with $u_n - u$ to get

$$\int_0^t \langle v_n'(s) - v'(s), u_n(s) - u(s)\rangle + \langle w_n(s) - w(s), u_n(s) - u(s)\rangle\, ds$$
$$= \int_0^t \langle f_n(s) - f(s), u_n(s) - u(s)\rangle\, ds\ .$$

Since $d\varphi_n^*(v_n(t))/dt = \langle v_n'(t), u_n(t)\rangle$,

$$\begin{aligned}\varphi_n^*(v_n(t)) + \int_0^t \langle w_n(s) - w(s), u_n(s) - u(s)\rangle\, ds = \varphi_n^*(v_{0n})\\ + \int_0^t \langle v'(s), u_n(s) - u(s)\rangle + \langle v_n'(s), u(s)\rangle + \langle f_n(s) - f(s), u_n(s) - u(s)\rangle\, ds\ .\end{aligned}\tag{5.4}$$

The limit of the right side exists and is equal to

$$\begin{aligned}&\lim_{n\to\infty}\Big[\varphi_n^*(v_{0n}) + \int_0^t \langle v'(s), u_n(s) - u(s)\rangle + \langle v_n'(s), u(s)\rangle +\\ &\qquad \langle f_n(s) - f(s), u_n(s) - u(s)\rangle\, ds]\\ &= \varphi^*(v_0) + \int_0^t \langle v'(s), u(s)\rangle\, ds\\ &= \varphi^*(v_0) + (\varphi^*(v(t)) - \varphi^*(v_0)) = \varphi^*(v(t))\ .\end{aligned}$$

Now consider the limit of the left side:

$$\begin{aligned}&\lim_{n\to\infty} \varphi_n^*(v_n(t)) + \int_0^t \langle w_n(s) - w(s), u_n(s) - u(s)\rangle\, ds\\ &\geq \liminf_{n\to\infty} \varphi_n^*(v_n(t)) + \limsup_{n\to\infty} \int_0^t \langle w_n(s) - w(s), u_n(s) - u(s)\rangle\, ds\ .\end{aligned}\tag{5.5}$$

We study the first term on the right first. Since v_n converges weakly in $H^1(0,T;\mathcal{V}^*)$, $\{v_n\}$ is a bounded subset of $L^\infty(0,T;\mathcal{V}^*)$. In particular, $\{v_n(t)\}$ is bounded and φ^* is lower semicontinuous, so

$$\begin{aligned}\liminf_{n\to\infty} \varphi_n^*(v_n(t)) &= \liminf_{n\to\infty} \varphi_n^*(v_n(t)) - \varphi^*(v_n(t)) + \varphi^*(v_n(t)) \\ &= \liminf_{n\to\infty} \varphi^*(v_n(t)) \geq \varphi^*(v(t)) \ .\end{aligned}$$

These estimates of the limits of the right and left sides of (5.4) imply that

$$0 \geq \limsup_{n\to\infty} \int_0^t \langle w_n(s) - w(s), u_n(s) - u(s)\rangle \, ds \ .$$

Since the integrand is nonnegative, the limit of the integral exists and is zero. The remainder of the proposition now follows from (5.5). ■

We now consider three special cases for which Proposition 2 can be used to deduce strong convergence. The first is the case in which $\mathcal{B}$ is strongly-monotone in $L^2(0,T;\mathcal{V})$. In this case, it follows from the proposition that $u_n \to u$ in $L^2(0,T;\mathcal{V})$.

The second case corresponds to the situation $\varphi^*(v) = \|v\|_{\mathcal{V}^*}^2$. In this case, the weak convergence of v_n to v and the convergence of the norms is sufficient to guarantee that $v_n \to v$ in $\mathcal{V}^*$.

The final case occurs when the convergence of $\varphi^*(v_n)$ implies the convergence of u_n. As an example, consider the function

$$\varphi(u) = \tfrac{1}{2}\|u\|_{L^2(\Omega)}^2 + \|u\|_{L^1(\Omega)} \ ,$$

with subgradient

$$\partial\varphi(u) = u + \operatorname{sgn}(u) \ .$$

The *signum* relation $\operatorname{sgn}(u)$ is defined to be $+1$ wherever $u > 0$, -1 wherever $u < 0$ and any (measurable) selection from the interval $[-1,1]$ on the set where $u = 0$. This functional arises naturally in the study of the Stefan problem; cf. [21]. The dual functional is

$$\partial\varphi^*(v) = (\partial\varphi)^{-1}(v) = (|v|-1)^+\operatorname{sgn}(v) \ ,$$

and an appropriate antiderivative is

$$\varphi^*(v) = \int_\Omega \tfrac{1}{2}(|v|-1)^{+2} \ .$$

In the Stefan problem, u represents the temperature, while $v \in \partial\varphi(u)$ represents the enthalpy. Note that

$$\varphi^*(v) = \tfrac{1}{2}\|u\|^2_{L^2(\Omega)} \ ,$$

so that, roughly, convergence of $\varphi^*_n(v_n)$ is the same as the convergence of $\|u_n\|_{L^2(\Omega)}$. Since $u_n \rightharpoonup u$, we may use the convergence of norms to conclude that $u_n \to u$ in $L^2(\Omega)$.

To make the rough ideas above precise in a particular example, suppose the φ^*_n are given by the Yosida approximations

$$v_n \in \partial\varphi^*_{1/n}(u_n) = \begin{cases} \dfrac{1}{1+1/n}(u_n+1) & \text{if } u_n > 1/n \ , \\ u_n & \text{if } u_n \in [-1/n, 1/n] \ , \\ \dfrac{1}{1+1/n}(-u_n-1) & \text{if } u_n < -1/n \ . \end{cases}$$

The conclusion $\varphi^*(v_n) \to \varphi^*(v)$ means

$$\left\| \frac{1}{1+1/n}(|u_n| - 1/n)^+ \right\|_{L^2(\Omega)} \to \|u\|_{L^2(\Omega)} \, .$$

If Ω is a set of finite measure, then $\|u_n\|_{L^2(\Omega)} \to \|u\|_{L^2(\Omega)}$. From the convergence of norms and the weak convergence, we conclude that $u_n \to u$ in $L^2(\Omega)$. That is, the temperatures in the approximate Stefan problem converge strongly, even though the enthalpies may only converge weakly.

6. Applications to flow in partially fissured media.

We now apply the abstract results of Sections 4 and 5 to the models described in Section 2. Our emphasis is on the continuous dependence of solutions upon the choice of model, although we begin with a discussion of existence of solutions.

Let Ω be an open, bounded subset of $\mathbb{R}^n$ with sufficiently smooth boundary (C^1 suffices). Take $H_0 = H_1 = L^2(\Omega)$, $H_2 = (L^2(\Omega))^n$, and V a subspace of $H^1(\Omega)$ chosen to reflect the boundary conditions to be imposed on u_0. For example, $H^1_0 = V$ is appropriate for Dirichlet boundary values.

After substituting (2.5.d) into (2.5.a), we see that the formal operators $\mathcal{B}_i$ are

$$\mathcal{B}_0(u_0, u_1, \vec{u}_2) = \frac{1}{\alpha}(u_0 - u_1) - \vec{\nabla}\cdot(B_0\vec{\nabla}u_0) + \vec{\nabla}\cdot(B_2^T\vec{u}_2),$$

$$\mathcal{B}_1(u_0, u_1, \vec{u}_2) = \frac{1}{\alpha}\,(u_1 - u_0)\,,$$

and

$$\mathcal{B}_2(u_0, u_1, \vec{u}_2) = \frac{1}{\beta}\left(\vec{u}_2 + B_2\vec{\nabla}u_0\right).$$

for some positive semidefinite, bounded, matrix–valued $B_i(x)$, i = 0,1,2. If, for example, Dirichlet boundary conditions are imposed on the fissure flow, then we would define $\mathcal{B}_i$ by

$$\mathcal{B}_0(u_0, u_1, \vec{u}_2)(v_0) = \int_\Omega \left(\frac{1}{\alpha}(u_0 - u_1)v_0 + (B_0\vec{\nabla}u_0 - B_2^T\vec{u}_2)\cdot\vec{\nabla}v_0\right),$$

$$\mathcal{B}_1(u_0, u_1, \vec{u}_2)(v_1) = \int_\Omega \frac{1}{\alpha}(u_1 - u_0)v_1,$$

and

$$\mathcal{B}_2(u_0, u_1, \vec{u}_2)(\vec{v}_2) = \int_\Omega \frac{1}{\beta}\left(\vec{u}_2 + B_2\vec{\nabla}u_0\right)\cdot\vec{v}_2.$$

Other boundary conditions are handled in the usual way. Note that it is not appropriate to specify the boundary conditions on u_1 or $\vec{u}_2$. If we adopt the "time delay" interpretation of the model, this phenomenon is explained by the fact that the boundary values of u_1 and u_2 are just the delayed boundary values of u_0.

PROPOSITION 3. *Hypothesis* [B$_1$] *is satisfied provided* B_0 *is positive semidefinite a.e.. Hypothesis* [B$_2$] *is satisfied if* B_0 *is uniformly positive definite.*

PROOF. It is clear that $\mathcal{B}$ is bounded, $\mathcal{B}_1$ is Lipschitz continuous, and if $u_{\lambda_0} \rightharpoonup u_0$ in $V \subset H^1$, then $u_{\lambda_0} \to u_0$ in L^2, so $\mathcal{B}_1(u_{\lambda_0}, \cdot, \vec{u}_{\lambda_2}) \to \mathcal{B}_1(u_0, \cdot, \vec{u}_2)$ uniformly on bounded subsets of L^2. The only remaining hypothesis to check is that $\mathcal{B}$ is maximal monotone.

The verification that $\mathcal{B}$ is monotone is straightforward:

$$\begin{aligned}
&\left(\mathcal{B}(u_0, u_1, \vec{u}_2) - \mathcal{B}(\hat{u}_0, \hat{u}_1, \hat{\vec{u}}_2), (u_0 - \hat{u}_0, u_1 - \hat{u}_1, \vec{u}_2 - \hat{\vec{u}}_2)\right)_{\mathcal{V}} \\
&\quad = \int_\Omega \left(B_0\vec{\nabla}(u_0 - \hat{u}_0)\right)\cdot\vec{\nabla}(u_0 - \hat{u}_0) + \frac{1}{\alpha}|(u_0 - \hat{u}_0) - (u_1 - \hat{u}_1)|^2 + \frac{1}{\beta}|(\vec{u}_2 - \hat{\vec{u}}_2)|^2 \\
&\quad \geq 0.
\end{aligned}$$

Since $\mathcal{R} + \mathcal{B}$ is $\mathcal{V}$-coercive, $\mathcal{B}$ is maximal monotone. If B_0 is uniformly positive definite, then $\mathcal{B}$ is $\mathcal{V}$-coercive. ∎

Proposition 2 and Theorem 2′ can be used to prove the existence of solutions to (3.4) for quite general $\mathcal{A}$. For example, when B_0 is uniformly positive definite, the flow in the fissures and pores can be modeled by the classical porous medium equations with $\mathcal{A}_i(u) = u^\alpha$ for $\alpha \in (0,1]$ and $i = 1$ or 2. If the material undergoes a change of phase, the Stefan problem with $\mathcal{A}_i = I + L\, sgn$, where sgn is the signum operator defined in the last example of Section 5, is well posed. In both these cases, the flows in the fissures and pores are modeled using nonlinear $\mathcal{A}_0$ and $\mathcal{A}_1$, while the flux exchange term must always be modeled using linear $\mathcal{A}_2$. This model was developed in response to a suggestion made to us by J. I. Diaz.

If B_0 were always uniformly positive definite, we could satisfy ourselves with Theorem (2′). Suppose, however, that $\Omega \subset \mathbb{R}^3$ consists of sheets of porous rock parallel to the $x - y$ plane, and that fissures separate these sheets. (This provides a model for a *layered medium*; see [9] and [6] for alternative models.) Then

$$B_0 = \begin{pmatrix} b_{11} & b_{12} & 0 \\ b_{21} & b_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

where the 2×2 submatrix in the upper left corner is positive definite. The bridging between the blocks takes place in the vertical direction, so

$$B_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & b_{33} \end{pmatrix},$$

where b_{33} is positive. Then both $\vec{\nabla} \cdot (B_0 \vec{\nabla})$ and $\vec{\nabla} \cdot (B_2^T B_2 \vec{\nabla})$ fail to be H_0^1-coercive, and Proposition 3 is not applicable to Theorem 2′. This model gives rise to well posed *linear* problems, though, subject to Theorem 3. In this case, we take $a_i \in L^\infty(\Omega)$, $a_i \geq 0$ *a.e.*, and define

$$\varphi_i(u) = \int_\Omega \tfrac{1}{2} a_i(x) u^2(x)\, dx,$$

so

$$\partial\varphi_i(u)\, v = \int_\Omega a_i(x) u(x) v(x)\, dx.$$

Take V as in the previous discussion and $\mathcal{V} = V \times L^2(\Omega) \times \left(L^2(\Omega)\right)^3$. To apply Theorem 3, we need only show that $Rg(\mathcal{A} + \mathcal{B}) \supset \mathcal{V}^*$. Observe first that

$$-\vec{\nabla} \cdot \left[\left(B_0 + \frac{1}{1 + \beta a_2} B_2^T B_2 \right) \vec{\nabla} \right]$$

is V-coercive for any choice of β and a_2. It follows that for any $f_0 \in L^2(\Omega) \subset V^*$, $f_1 \in L^2(\Omega)$ and $\vec{f}_2 \in (L^2(\Omega))^3$, there is a unique solution $u_0 \in V$ to the elliptic problem

$$\left(a_0 + \frac{a_1}{1+\alpha a_1}\right) u_0 - \vec{\nabla} \cdot \left[\left(B_0 + \frac{1}{1+\beta a_2} B_2^T B_2\right) \vec{\nabla} u_0\right] = f_0 + \frac{1}{1+\alpha a_1} f_1 - \vec{\nabla} \cdot \left(\frac{\beta}{1+\beta a_2} B_2^T \vec{f}_2\right). \tag{6.1}$$

Having found u_0, compute

$$u_1 = \frac{1}{1+\alpha a_1}\left(u_0 + \alpha f_1\right) \tag{6.2}$$

and

$$\vec{u}_2 = -\frac{1}{1+\beta a_2}\left(B_2 \vec{\nabla} u_0 + \beta \vec{f}_2\right). \tag{6.3}$$

It follows from a direct computation that $(u_0, u_1, \vec{u}_2)$ is the solution to the problem $(\mathcal{A} + \mathcal{B})u = (f_0, f_1, \vec{f}_2)$, and hence Theorem 3 is applicable. Note that in this situation, u_0 may not be smooth enough to belong to $V \subset H^1(\Omega)$. It is therefore hopeless to attempt to apply Theorem 2$'$ to this model.

Let us now consider the dependence of the solutions of these problems upon the models. In problems with *linear* $\mathcal{A}$, the corollary to Theorem 3 gives the continuous dependence of solutions upon the spatial operator $\mathcal{B}$. Of special interest is the case $B_2 \to 0$, whereby $\vec{u}_2$ is uncoupled from the other two components. To apply the corollary, we must show that for all $g \in L^2(\Omega) \times L^2(\Omega) \times (L^2(\Omega))^n$, the solutions w_n to $(\mathcal{A} + \mathcal{B}_n)w_n \ni g$ converge to the solution w_0 of $(\mathcal{A}+\mathcal{B})w_0 \ni g$. When $\mathcal{B}$ has the form discussed above, then this condition is easy to verify when B_0 is positive definite, but we have an important example for which this assumption is inappropriate. We may, however, apply the corollary to Theorem 3.

Let $(u_0^n, u_1^n, \vec{u}_2^n)$ be solutions to (6.1) for a sequence of problems for which $\beta_n \to 0$ (and B_0 need not be positive definite). Let $(u_0, u_1, \vec{u}_2)$ be solution to (6.1) when $\beta = 0$. The inner product of (6.1) with u_0 (assuming Dirichlet boundary conditions, for example) gives the identity

$$\int_\Omega \left(a_0 + \frac{a_1}{1+\alpha a_1}\right) |u_0^n|^2 + \left(B_0 + \frac{1}{1+\beta_n a_2} B_2^T B_2\right) \vec{\nabla} u_0^n \cdot \vec{\nabla} u_0^n = \int_\Omega f_0 u_0^n + \frac{1}{1+\alpha a_1} f_1 u_0^n + \frac{\beta_n}{1+\beta_n a_2}\left(B_2^T \vec{f}_2\right) \cdot \vec{\nabla} u_0^n. \tag{6.4}$$

Since $B_0 + \frac{1}{\beta_n a_2} B_2^T B_2$ *is* uniformly positive definite, $\{u_0^n\}$ is bounded in $H_0^1(\Omega)$. It follows that some (hence, any) subsequence converges weakly in $H_0^1(\Omega)$ to u_0. It also follows from (6.4) that the $H_0^1(\Omega)$ norms of u_0^n converge to the $H_0^1(\Omega)$ norm of u_0, so $u_0^n \to u_0$ in $H_0^1(\Omega)$. It then follows from (6.2) and (6.3) that $u_1^n \to u_1$ in $L^2(\Omega)$ and $\vec{u}_2^n \to \vec{u}_2$ in $(L^2(\Omega))^3$.

This example is important because it illustrates the continuous dependence of solutions to (6.1) and the first order kinetic (or viscoelastic) equations (2.7) upon the parameter β. Continuous dependence of solutions of the classical fissured medium equation (2.8) also follows. Note that the conclusion of the corollary is the same as the conclusion $\varphi_n^*(v_n) \to \varphi^*(v)$ from Proposition 2.

Our final consideration is the behavior of solutions upon the operator $\mathcal{A}$. We are especially interested in the case $\mathcal{A}_0 \to 0$, because this is the condition which connects our system (2.5) with the Sobolev-type equation (2.9) and the first order kinetic or viscoelastic equations with the classical fissured medium equations.

We now verify that the hypotheses (5.2) of Proposition 2 are met when

1) $\mathcal{B}$ is $L^2(0,T;\mathcal{V})-$coercive,
2) There is an element $\hat{u}$ in the domain of $\mathcal{B}$ for which $\varphi_n(\hat{u})$ is bounded, and
3) $\mathcal{A}_n(u_n) \to \mathcal{A}(u)$ whenever $u_n \to u$ in $H_0 \times H_1^w \times H_2^w$, where the superscript w means that the space is endowed with the weak topology.

Note that this last condition is not particularly restrictive. For example, if $\mathcal{A}_{i_n}$ is defined by $\mathcal{A}_{i_n}(u) = a_{i_n} u$, where $0 \le i \le 2$ and the functions $a_{i_n} \in L^\infty(\Omega)$, then condition 3) requires only that the functions a_{i_n} converge in $L^\infty(\Omega)$. In the example discussed at the beginning of this section, condition 1) means B_0 is uniformly positive definite. In this case, an *a priori* bound on $\|u_n\|_{L^2(0,T;\mathcal{V})}$ exists. If $\hat{f} \in \mathcal{B}(\hat{u})$, then the duality pairing of

$$\frac{d(v_n - \hat{u})}{dt} + \mathcal{B}(u_n) - \mathcal{B}(\hat{u}) = f - \hat{f}$$

with $u_n - \hat{u}$ and integration gives

$$\begin{aligned} &\varphi^*(v_n(T)) - \langle v_n(T), \hat{u}\rangle + \int_0^T \langle \mathcal{B}(u_n) - \mathcal{B}(\hat{u}), u_n - \hat{u}\rangle \, dt \\ &= \varphi^*(v_n(0)) - \langle v_n(0), \hat{u}\rangle + \int_0^T \left\langle f_n - \hat{f}, u_n - \hat{u} \right\rangle \, dt. \end{aligned}$$

The first two terms on the left are bounded below by $-\varphi^*(\hat{u})$; thus, the sequence $\|u_n - \hat{u}\|_{L^2(0,T;\mathcal{V})}$ is bounded. This bound means that u_1 and $\vec{u}_2$ are bounded in, respectively, $L^2(0,T;L^2(\Omega))$ and $L^2(0,T;(L^2(\Omega)))^3$, and so have weakly convergent subsequences. The first component, u_0, is bounded in $L^2(0,T;H^1(\Omega))$, and, therefore, has a subsequence which is weakly convergent in $L^2(0,T;H^1(\Omega))$. It follows from the expressions for $\mathcal{B}_1$ and $\mathcal{B}_2$ that $\mathcal{B}_1(u_{1_n})$ and $\mathcal{B}_2(\vec{u}_{2_n})$ converge in the weak topologies on respectively, $L^2(0,T;L^2(\Omega))$ and $L^2(0,T;(L^2(\Omega)))^3$, to elements of, respectively, $\mathcal{B}_1(u_1)$ and $\mathcal{B}_2(\vec{u}_2)$. Similarly, $\mathcal{B}_0(u_{0_n})$ converges weakly in $L^2(0,T;V^*)$ to $\mathcal{B}_0(u_0)$. From the differential equation (5.1), dv/dt must have a weakly convergent subsequence, and the limits must satisfy the differential equation (3.4). Condition 3) is precisely what is needed to verify is that $v \in \mathcal{A}(u)$.

As indicated in the examples following Proposition 2, it follows that $u_{0_n} \to u_0$ in V, $u_{1_n} \to u_1$ in $L^2(\Omega)$ and $\vec{u}_{2_n} \to \vec{u}_2$ in $(L^2(\Omega))^3$. The first component can be, for example, the Yosida approximations to the Stefan or porous medium operators, as indicated in the last example of Section 5. Other examples include the promised case of $a_0 \to 0$, whereby continuous dependence of the models indicated in Section 2 is obtained.

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