DIFFUSION IN DEFORMING POROUS MEDIA

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Abstract. We report on some recent progress in the mathematical theory of nonlinear fluid transport and poro-mechanics, specifically, the design, analysis and application of mathematical models for the flow of fluids driven by the coupled pressure and stress distributions within a deforming heterogeneous porous structure. The goal of this work is to develop a set of mathematical models of coupled flow and deformation processes as a basis for fundamental research on the theoretical and numerical modeling and simulation of flow in deforming heterogeneous porous media.

Contents

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1. Introduction

Deformation of a saturated porous solid affects the flow of fluid through it, and the fluid pressure contributes to the mechanical behaviour of that structure. Since the load on such a porous structure is supported by both the solid matrix and the fluid in the pores, the coupling of the stress in the solid to the pressure of the fluid plays an essential role. It involves the *dilation* or *contraction* of the deforming matrix and the *pressure gradient* of the diffusing pore fluid. In the classical *consolidation* process, a load is initially shared with the pore fluid, and with time the pore fluid pressure dissipates and the load is increasingly born by the porous solid matrix. The diffusing pore fluid thereby has an important effect on the mechanical response of the matrix. That is, for more slowly applied loads the material response appears less stiff, since the fluid has relatively more time to diffuse away. Conversely, the dilations of the matrix modify the porosity and thereby enhance the fluid flow. The classical Biot model of this process was developed in *soil science*, and it has been refined considerably for the increasingly more demanding needs in engineering and geophysics [17], [44], [63]. The simplest model describes the evolution of the scalar field of *fluid pressure* $p(x,t)$ and the vector field of *solid displacement* $\mathbf{u}(x,t)$ from the position $x \in \Omega$ at time $t > 0$. For a homogeneous and isotropic medium the classical linear *poroelasticity system* takes the form

$$\rho u_{tt}(x,t) - (\lambda + \mu)\nabla(\nabla \cdot \mathbf{u}(x,t)) - \mu\Delta\mathbf{u}(x,t) + \alpha\nabla p(x,t) = f(x,t)\,, \tag{1a}$$

$$c_0 p_t(x,t) - \nabla \cdot k\nabla p(x,t) + \alpha\nabla \cdot u_t(x,t) = h(x,t)\,. \tag{1b}$$

The physical coefficient $c_0(x) \geq 0$ is related to the *compressibility* of the fluid as well as the *porosity* of the medium at $x \in \Omega$. Namely, it is a measure of the amount of fluid which can be forced into the medium by pressure increments with constant volume. Similarly, $k(x) \geq 0$ involves the *viscosity* of the fluid and the *permeability* of the medium at $x \in \Omega$ as a measure of the Darcy flow corresponding to a pressure gradient. The parameter $\alpha \geq 0$ accounts for the mechanical coupling of the fluid pressure and the porous solid. Namely, the term $\alpha\nabla \cdot \mathbf{u}(x,t)$ represents the additional fluid content due to the *dilation* of the structure, and $\alpha\nabla p(x,t)$ is the additional stress within the structure due to the fluid pressure. The coefficient $\rho(x) \geq 0$ is the local *density*, and we obtain the Lamé constant λ and shear modulus μ from elasticity of the medium. The system (1) was obtained by Biot [10] from a phenomenological approach, and it has been subsequently derived by techniques of homogenization theory [3], [4], [13] and mixture theory [12], [38], [63]. The fundamental basis of the system is widely accepted [17]. Of particular interest is the *quasi-static* case $\rho = 0$ which results from negligible inertia effects and describes the slow deformations associated with consolidation and the associated seepage of fluid. The *incompressible* case $c_0 = 0$ leads to yet another source of degeneracy in the system.

1.1. **Applications.** Various refinements and extensions of this model are well suited for specific applications in geomechanics, but they frequently lack precision in prediction. Not only is there limited agreement on which models are appropriate for specific problems, but the recently developed sensor technologies continue to deliver large amounts of very detailed data which will be useful only with appropriate models that can assimilate this additional information. Natural geologic materials are rarely homogeneous on the scale of interest, and this is especially true for soils and particularly for naturally fractured reservoirs or aquifiers. This lack of homogeneity on the scale of interest often leads to highly singular problems and to spatial nonlocality of constitutive models for flow. Modern applications require models containing various sources of nonlinearity such as capillary effects in the smaller pores, faster flow in the larger fissures or macropores, hysteresis effects due to partial saturation of the fluid or to plasticity of the material, and the transport of multiphase fluids. The complexity of the fully coupled models and the general lack of theoretical background available for the qualitative behavior of the solutions have complicated the process of selecting appropriate models for specific applications and have delayed the development of rigorous numerical error studies [39], [44], [60].

Related systems arise in many diverse areas. For example, the *Biot deformation-diffusion system* (1) is formally equivalent to the classical linearized *thermoelasticity* system which describes the flow of heat through an elastic structure. In that context, $p(x,t)$ denotes the *temperature*, $c_0 > 0$ is the *specific heat* of the medium, and $k > 0$ is the *conductivity*. Then $\alpha \nabla p(x,t)$ arises from the *thermal stress* in the structure, and the term $\alpha \nabla \cdot u_t(x,t)$ corresponds to the *internal heating* due to the *dilation rate*. We have *not* made the uncoupling assumption in which this term is deleted from the diffusion equation. For the theory of the *fully-dynamic* system (1) with $\rho > 0$ in the context of thermoelasticity, see the fundamental work of Dafermos [19], the exhaustive and complementary accounts of Carleson [15] and Kupradze [32], and the development in the context of strongly elliptic systems by Fichera [27]. By contrast, very few references are to be found in the thermoelasticity literature for well-posedness of even the simplest linear problem for the *coupled quasi-static* case of (1) in which the system degenerates to a mixed elliptic-parabolic type. Such a system in one spatial dimension is developed by classical methods in the book of Day [22]. One can extend to this system many of the results for the classical diffusion equation; see the comprehensive book of Cannon [14]. For the nonlinear contact problem for coupled quasi-static thermoelasticity in one dimension see Allegreto-Cannon-Lin [1]. In Shi-Xu [46] the problem on the 2D disk is developed by decoupling the displacement to get a single integral-differential equation. The existence of a solution for the general N dimensional contact problem was given in Shi-Shillor [45] under the assumption that the coupling coefficient is sufficiently small. This 'smallness' condition was removed by Xu [62]. According to a scaling argument in Boley-Wiener [11], it appears that the reasons for taking $\rho = 0$ apply as well to simultaneously delete the term $\alpha \nabla \cdot \dot{u}(t)$ and thereby uncouple the system, so these two assumptions are frequently taken together in the thermoelasticity context; also see Esham-Weinacht [25] for the behaviour as $\rho \to 0$. However the coupling plays an essential role in poroelasticity where $\alpha \approx 1$.

In recent work of Preziosi-Farina [26], [42], we find that similar models arise in the description of *composite materials manufacturing* processes to describe resin transfer molding and structural resin injection molding. These consist of the injection of a liquid into a porous medium made of reinforcing elements. Here the infiltration is coupled with both the rheological properties of the liquid (thermal variation and curing) and the mechanical properties of the solid form. The resulting highly nonlinear problem consists of conservation equations for mass, momentum, and energy together with a polymerization or *cure* equation for the saturated solid-liquid mixture. It differs from the corresponding geomechanical problem of soil consolidation with regard to the range of parameters of interest and in the addition of an equation that describes the cure process of the liquid. By contrast, the plastic behaviour of the solid is more complicated in soils.

Highly nonlinear systems of the form (1) arise in the *biomechanics* of soft tissues; see the discussions of Lai-Hou-Mow [33] and the extension of Huyghe-Janssen [31] to large-deformation models. Biological porous media such as tissues and cartilage in contact with changing salt concentrations exhibit swelling or shrinking that depend on a combination of electrostatic forces and hydration forces similar to that in clays and shales and gels. Here the characteristic pore size can be close to the molecular level and, in addition to the pressure, also chemical concentrations and electrical gradients are necessary to describe the fluid flow, deformation, and ion flow; see the works of Cushman-Murad [18], [38]. Swelling colloidal systems are much more complicated than granular or well-structured geologic media. In these biological applications, the electrostatic forces are often dominant. The theory of *multiporous media*, as originally developed for the mechanics of naturally fractured reservoirs, has found applications to the description of *blood perfusion*. The coronary vascular system is an example of a pore structure inside a deforming solid, namely, the heart muscle. The muscle is subject to large deformations and the pore structure is highly organized on differing scales. The resulting scales of porosity correspond to arteries, arterioles, capillaries, venules and veins. Each has a characteristic pore size, blood velocity and mechanical properties. The coupling between the blood perfusion and the deformation of the muscle tissue has been well documented [56, 40].

Although we shall focus our following discussion on the applications in geomechanics, one should remain aware that the mathematical theory may be used as well in the development of many other important topics.

1.2. **Objectives.** It has been our intention to develop the mathematical theory of various coupled deformation and flow systems and investigate their suitability for corresponding applications. We shall begin with the theory of the basic Biot system for granular media and then extend this to include *multiple phases* and *multiple components*, *non-Darcy flow* in regions of higher fluid velocities, *viscous* and *hysteresis* effects appropriate for certain soils and rock types, nonlinear material *constitutive relations* which permit visco-elastic-plastic deformation of the matrix, and *multiscale effects* arising from upscaling.

The well posedness of the systems of partial differential equations that serve as appropriate models, both quantitative and qualitative properties of their solutions, and the effects of individual terms in the systems will be investigated. These require estimates or properties of solutions for comparative analysis of the appropriateness of each of these systems as a model for the intended application from a mathematical and numerical point of view. Selected quantitative results delivered by numerical simulations will be validated when possible against available field data or examples.

2. Recent Results

2.1. **The Linear Quasi-Static Case.** Although there is an enormous amount of literature concerning the applications of the coupled quasi-static case of the Biot system (1) to engineering and geophysics problems in poroelasticity, one finds very few references devoted to the fundamental mathematical issues, even for this simplest system. The first results on well-posedness of the quasi-static case of the basic system (1) appeared in the fundamental work of J.-L Auriault and Sanchez-Palencia [3]. There the meaning of the various coefficients was given by means of homogenization. The later paper of Zenisek [64] deals with existence of solutions, and there one has not only $\rho = 0$ but also the additional degeneracy of the *incompressible* case, $c_0 = 0$. These seem to be the only such references which address the fundamental well-posedness for the coupled quasi-static case, but additional issues of analysis and approximation of this case were already raised in [39], [38], [63]. We started our study with the analysis of existence, uniqueness and regularity properties of solutions to the linear quasi-static Biot problem [48]. The model was extended to include the *exposed pore fraction* on the boundary and secondary viscosity effects. If we denote the *characteristic function* of the traction boundary, Γ_t by χ_t, the initial boundary value problem takes the form

$$-(\lambda + \mu)\nabla(\nabla \cdot u(t)) - \mu\Delta u(t) + \nabla p(t) = \mathbf{0} \text{ and} \tag{2a}$$

$$\frac{\partial}{\partial t}(c_0 p(t) + \nabla \cdot \mathbf{u}(t)) - \nabla \cdot k\nabla(p(t)) = h_0(t) \text{ in } \Omega\,, \tag{2b}$$

$$\mathbf{u}(t) = \mathbf{0} \text{ on } \Gamma_0, \tag{2c}$$

$$\sigma_{ij}(\mathbf{u}(t))n_j - p(t)\, n_i\, \beta = 0,\ 1 \le i \le 3,\ \text{ on } \Gamma_t\,, \tag{2d}$$

$$-\frac{\partial}{\partial t}(\mathbf{u}(t) \cdot \mathbf{n})\,(1 - \beta)\chi_t + k\frac{\partial p(t)}{\partial n} = h_1(t)\,\chi_t \text{ on } \Gamma\,, \tag{2e}$$

$$\lim_{t \to 0^+} (c_0 p(t) + \nabla \cdot \mathbf{u}(t)) = v_0 \text{ in } L^2(\Omega)\,, \tag{2f}$$

$$\lim_{t \to 0^+} (1 - \beta)(\mathbf{u}(t) \cdot \mathbf{n}) = v_1 \text{ in } L^2(\Gamma_t)\,. \tag{2g}$$

The partial differential equations (2a), (2b) comprise the quasi-static case of the Biot system (1). The boundary conditions (2c), (2d) are the complementary pair consisting of null displacement on the *clamped boundary*, Γ_0, and a balance of forces on the *traction boundary*, Γ_t, and (2e) requires a balance of fluid mass. The function $\beta(\cdot)$ is defined on that portion of the boundary Γ_t which is neither drained nor clamped, and it specifies the surface fraction of the pores which are *sealed* along Γ_t. Here the hydraulic pressure contributes to the total stress within the structure. The remaining portion $1 - \beta(\cdot)$ of the pores are *exposed* along Γ_t, and these contribute to the flux. On any portion of Γ_t which is completely exposed, that is, where $\beta = 0$, only the *effective* or elastic component of stress is specified, since there the fluid pressure does not contribute to the support of the matrix. On the entire boundary there is a transverse flow that is given by the input $h_1(\cdot)$ and the relative normal displacement of the structure. This input could be specified in the form $h_1(t) = -(1-\beta)\mathbf{v}(t)\cdot\mathbf{n}$, where $\mathbf{v}(t)$ is the given velocity of fluid or boundary flux on Γ_t. The first term and right side of this flux balance is null where $\beta = 1$, so the same holds for the second terms in (2e), that is, we have the *impermeable* conditions $k\frac{\partial p(t)}{\partial n} = 0$ on a completely sealed portion of Γ_t. We also note that in (2e) the first term on the left side and the right side of the equation are null on Γ_0, so the same necessarily holds for the second term on the left side. That is, we always have the *null flux* condition $k\frac{\partial p}{\partial n} = 0$ on Γ_0.

We proved in [48] that the initial-boundary-value problem (2) is essentially a *parabolic* system which has a *strong* solution under minimal smoothness requirements on the initial data and source $h(\cdot)$. In particular, the dynamics of the problem corresponds to an *analytic semigroup* in L^2 for *strong* solutions and in H^{-1} for *weak* solutions.

2.2. **Composite Deformable Porous Media.** The simplest and most frequently used model of flow in a *rigid* fully–saturated but heterogeneous medium with several distinct spatial scales which allows for qualitatively different properties is the *Barenblatt double–diffusion model*. This consists of the combined effects of two distinct components in parallel. Both components occur locally in any representative volume element, and they behave as two independent diffusion processes which are coupled by a distributed exchange term that is proportional to the difference in pressure between fluids in the two components. In the special case which is used to model naturally fractured media, the first component of the model is the highly developed fracture system and the second is the porous matrix structure. See Barenblatt *et al* [6], Bai *et al* [5], Bear [7], Warren-Root [59]. In the more complex models of double–diffusion combined with deformation, both of the pressure fields contribute to the stress field of the structure, so it is necessary to incorporate Biot's concepts into the Barenblatt model. The momentum equations contain contributions to total stress from each of the two pressure fields, and the two equations of fluid transport follow from the continuity of fluid mass and consideration of the effects of *dilation* of the structure on the flow in both of the components. The fluid transport within this composite deformable porous medium is described by a pair of pressure equations for diffusion in the respective components of the medium together with an exchange term. This simplistic combination of the Barenblatt double–diffusion model with the Biot diffusion–deformation model has been developed and used extensively in the engineering literature, and it takes the form

$$-(\lambda+\mu)\nabla(\nabla\cdot\mathbf{u}(x,t)) - \mu\Delta\mathbf{u}(x,t) + \alpha_1\nabla p_1(t) + \alpha_2\nabla p_2(t) = 0\,, \tag{3a}$$

$$c_1\dot{p}_1(t) - \nabla\cdot k_1\nabla p_1(t) + \alpha_1\nabla\cdot\dot{u}(t) + \gamma(p_1(t) - p_2(t)) = h_1(t)\,, \tag{3b}$$

$$c_2\dot{p}_2(t) - \nabla\cdot k_2\nabla p_2(t) + \alpha_2\nabla\cdot\dot{u}(t) + \gamma(p_2(t) - p_1 t)) = h_2(t)\,. \tag{3c}$$

This describes consolidation processes in a fluid-saturated double-diffusion model of fractured rock. See Beskos-Aifantis [9], Wilson-Aifantis [61], Huyakorn-Pinder [30], Valliappin-Khalili-Naghadeh [57], Berryman-Wang [8].

The mathematical aspects of the *Barenblatt-Biot model* (3) for elastic deformation and laminar flow in a heterogeneous porous medium were developed in work with Momken [51]. This includes the treatment of various relevant degenerate cases, such as incompressible constituents or totally fissured components, and it includes the boundary conditions arising from partially exposed pores. The quasi-static initial-boundary problem is shown to have a unique weak solution, and this solution is strong when the data are smoother. The dynamics is shown to be *analytic* only in the non-degenerate case. Of course, we know from the rigid case that this is essentially the best that could be expected.

2.3. **The Deforming Dam.** The coupling of geomechanics with *multiphase* flow and transport problems arises in such areas as reservoir simulations, environmental studies, soil science, and the modeling of sediment regions [34], [44]. For air–water systems, Zienkiewicz and co-workers [63] have provided a derivation of such models by hybrid mixture theory. This direction has extremely important applications to dam behavior during earthquakes, and it is generally relevant to water transport in the vadose zone of soils. Such models must account for variations in the saturation, density, pressure and permeability, and this leads to a highly nonlinear system of partial differential equations for multi-phase flow.

Our recent work with Su [53] on the *deformable dam problem* contains the first proof of existence for a *deforming partially saturated* medium. This includes the classical free-boundary problem with a *phreatic surface* in which the saturation $S(\cdot)$ is given by a continuous monotone function that increases from near zero to unity in the vicinity of the capillary tension. Diffusion of the slightly compressible fluid is through a partially saturated porous and *elastic* medium $\Omega \subset \mathbb{R}^3$. Denote the fluid *density* by $\rho(x,t)$ and its *pressure* by $p(x,t)$ for $x \in \Omega$. The fluid is *barotropic*, *i.e.*, the density and pressure are related by a *state equation* $\rho = \rho(p)$, with a non-decreasing constitutive function $\rho(\cdot)$ that characterizes the type of fluid. For a homogeneous and isotropic medium the *partially saturated consolidation problem* takes the form

$$-(\lambda+\mu)\nabla(\nabla\cdot\mathbf{u}) - \mu\Delta\mathbf{u} + \nabla(\chi(p)\,p) = \mathbf{F}(x,t)\,, \tag{4a}$$

$$\frac{\partial}{\partial t}(\phi(p)S(p)\rho(p) + \nabla\cdot\mathbf{u}) + \nabla\cdot(\rho(p)\,\mathbf{q}) = F(x,t)\,, \tag{4b}$$

$$\mathbf{q} = -k(p)(\nabla p + \rho(p)\mathbf{g})\,. \tag{4c}$$

This system consists of the *equilibrium equation* for momentum conservation, the *storage equation* for mass conservation, and *Darcy's law* for the filtration velocity, $\mathbf{q}$. The function $\phi(\cdot)$ is *porosity*, $S(\cdot)$ is *saturation*, and $k(\cdot)$ is the *permeability* for the laminar flow in the medium. All of these functions are non-negative and pressure dependent. The (linearized) *strain* tensor $\varepsilon_{kl}(\mathbf{u}) \equiv \frac{1}{2}(\partial_k u_l + \partial_l u_k)$ provides a measure of the local deformation of the body, and the term $\nabla\cdot\mathbf{u} = \varepsilon_{kk}(\mathbf{u})$ represents the *fluid content* due to the local volume dilation. The *total stress* σ_{ij}

is the sum the *effective stress* of the of the purely elastic *isotropic* structure given by *Hooke's law* and *effective pressure* stress of the fluid on the structure, hence,

$$\sigma_{ij} = \lambda\delta_{ij}\varepsilon_{kk} + 2\mu\varepsilon_{ij} - \delta_{ij}\chi(p)\, p\,,$$

with positive Lamé constant λ and shear modulus μ. The *Bishop function* $\chi(\cdot)$ is a measure of the fraction of pore surface in contact with the fluid. Let the negative pressure $p_0 < 0$ denote the *capillary tension.* The saturation function $S(\cdot)$ is monotone with $S(p) = 1$ for $p \geq p_0$, and the Bishop function is well approximated in many situations by $\chi(p) \approx S(p)$. There is a *fully saturated region,* $\{x \in \Omega :\ p(x,t) > p_0\}$, while in the *capillary fringe,* $\{x \in \Omega :\ p(x,t) < p_0\}$, the medium is only partially saturated. The *phreatic surface* $\{x \in \Omega :\ p(x,t) = p_0\}$ is the unknown *free surface* that separates these regions. The boundary of Ω is given by the disjoint union of the parts Γ_D and Γ_{fl}, and Γ_{fl} is further written as the disjoint union of Γ_N and Γ_U. The part Γ_{fl} is the *flux boundary.* On its complement, Γ_D, the value of pressure is given by the depth below the surface:

$$p(x,t) = d(x_3), \quad x = (x_1, x_2, x_3) \in \Gamma_D, \tag{4d}$$

where $d(\cdot) > 0$. On Γ_N there is no flow, so we have a null normal flux:

$$\rho(p)\,\mathbf{q}\cdot\mathbf{n} = 0, \qquad x \in \Gamma_N, \tag{4e}$$

where $\mathbf{n}$ is the unit outward normal on the boundary, $\partial\Omega$. On Γ_U we have

$$p \leq 0, \quad \rho(p)\mathbf{q}\cdot\mathbf{n} \geq 0, \quad p\,\rho(p)\mathbf{q}\cdot\mathbf{n} = 0, \qquad x \in \Gamma_U. \tag{4f}$$

This implies that the fluid pressure on the boundary cannot exceed the outside null pressure of air, and there can be no flow into Ω. Also, $p = 0$ on the *seepage surface* which is that part of Γ_U where $\mathbf{q}\cdot\mathbf{n} > 0$, and there is no flow from the boundary above that, where $p < 0$. The boundary conditions on $\partial\Omega$ also involve the displacement or the *tractions* $\sigma_{ij}(x,t)n_j$ on $\partial\Omega$, namely,

$$u_i = 0 \text{ on } \Gamma_0\,, \quad \sigma_{ij}(x,t)n_j = t_i \text{ on } \Gamma_{tr}\,, \quad 1 \leq i \leq 3, \tag{4g}$$

where Γ_0 and Γ_{tr} are given complementary subsets of the boundary. Finally, the initial value of the water content $\theta_0(\cdot)$ is specified,

$$\phi(p(x,0))S(p(x,0))\rho(p(x,0)) + \nabla\cdot\mathbf{u}(x,0) = \theta_0(x)\,, \quad x \in \Omega\,, \tag{4h}$$

where the initial displacement satisfies the constraint

$$-(\lambda+\mu)\nabla(\nabla\cdot\mathbf{u}(x,0)) - \mu\Delta\mathbf{u}(x,0) + \nabla(\chi(p(x,0))\,p(x,0)) = \mathbf{F}(x,0)$$

together with the boundary conditions (4g).

This work has more recently been extended in [54] to the Barenblatt-Biot system (3). This covers the multi-phase multi-component situation of a composite medium of two components, each of which can be independently partially-saturated. Thus, the model tracks the partial saturation in both the fracture system and in the small scale pore system, thereby leading to a *pair* of free surfaces.

3. Current Progress

3.1. **Visco-Plastic Media.** Linearization of a model for a swelling clay with *secondary consolidation*, as derived by Cushman and co-workers [38] using hybrid mixture theory, yields the system

$$\begin{aligned} -\mu^* \nabla(\nabla \cdot u_t(x,t)) - (\lambda+\mu)\nabla(\nabla \cdot \mathbf{u}(x,t)) \\ -\mu\Delta\mathbf{u}(x,t) + \alpha\nabla p(x,t) = f(x,t)\,, \end{aligned}$$

$$c_0 p_t(t) - \nabla \cdot k\nabla p(t) + \alpha\nabla \cdot u_t(t) = h(x,t)\,.$$

The new term $\mu^* > 0$ introduces a *viscosity* effect into the quasi-static form of (1), and this system is another interesting example of a degenerate implicit evolution equation [16]. The theory developed in [48] included this situation, and it was shown there that the addition of μ^* *decreases* the regularizing effects of the dynamics. That is, the addition of this viscosity term serves to decrease or delay the dissipation in the system.

More important are the dissipation processes that lead to *hysteresis* phenomena. For example, saturation and (to a lesser extent) permeability exhibit a hysteretic dependence on the pressure. Visco-elastic and elasto-plastic behaviors of the material are common, especially in soils. Plasticity is an essential element for the description of soils and generally for materials of interest in geomechanics. Various approaches to include appropriate constitutive equations have been developed, and many numerical codes include plasticity models [17], [37], [44], [63]. Another example arises from the irreversible fluid content that corresponds to the *plastic porosity* of the porous medium. Such examples illustrate the need for the inclusion of *hysteresis* in models of deforming porous medium.

We had previously studied such memory dependent phenomena as secondary viscosity and hysteresis effects in models of flow and transport in (rigid) porous media by *nonlinear semigroup techniques* as in [41], [50], [52], [58]. In recent work with Ulisse Steffanelli [55], we have developed a very general model for Darcy flow through a *viscous-plastic* medium as the coupled system of partial differential and functional equations

$$\frac{\partial}{\partial t}\big(c_0\, p + \alpha\nabla \cdot \mathbf{u}\big) - \nabla \cdot k(\nabla p) = c_0^{1/2} h_0, \tag{5a}$$

$$\rho\frac{\partial^2}{\partial t^2}\mathbf{u} - \nabla\mu^*\big(\tfrac{\partial}{\partial t}(\nabla \cdot \mathbf{u})\big) - \nabla \cdot \sigma + \alpha\nabla p = \rho^{1/2}\mathbf{f}_0, \tag{5b}$$

$$\sigma = \mathcal{H}(\varepsilon(\mathbf{u})), \tag{5c}$$

in the cylindrical domain $\Omega\times(0,T)$, where Ω is a non-empty bounded and open set in $\mathbb{R}^3$ with smooth boundary $\Gamma \equiv \partial\Omega$, and $(0,T)$ is the time interval of interest. Also, $h_0 : \Omega\times(0,T) \longrightarrow \mathbb{R}$ and $\mathbf{f}_0 : \Omega\times(0,T) \longrightarrow \mathbb{R}^3$ are suitably given functions. The system (5) is complemented with suitable boundary and initial conditions. We introduce a pair of partitions of the boundary Γ into complementary sets $\{\Gamma_d, \Gamma_f\}$ and $\{\Gamma_c, \Gamma_t\}$. Set $\Gamma_s \equiv \Gamma_t \cap \Gamma_f$ and let the measurable function $\beta : \Gamma_s \longrightarrow [0,1]$ be prescribed as before. We seek a solution of (5) that satisfies the boundary conditions

$$p = 0 \quad \text{on } \Gamma_d, \tag{6a}$$

$$k(\nabla p)\cdot \mathbf{n} - \alpha\beta\,\frac{\partial}{\partial t}\big(\mathbf{u}\cdot\mathbf{n}\big) = 0 \quad \text{on } \Gamma_f, \tag{6b}$$

$$\mathbf{u} = \mathbf{0} \quad \text{on } \Gamma_c, \tag{6c}$$

$$\mu^*\big(\tfrac{\partial}{\partial t}(\nabla\cdot\mathbf{u})\big)\mathbf{n} + \sigma\mathbf{n} - \alpha(1-\beta)p\mathbf{n} = \mathbf{0} \quad \text{on } \Gamma_t. \tag{6d}$$

Here $\mathbf{n}$ stands for the unit outward normal vector to Γ, and $(\sigma\mathbf{n})_i = \sigma_{ij}n_j$ is the *normal stress.* Finally, we require the solution to satisfy the initial conditions

$$c_0 p(\cdot,0) = c_0 p_0(\cdot) \text{ on } \Omega, \tag{7a}$$

$$\mathbf{u}(\cdot,0) = \mathbf{u}_0(\cdot), \quad \rho\mathbf{u}_t(\cdot,0) = \rho\mathbf{v}_0(\cdot) \text{ on } \Omega, \tag{7b}$$

$$\sigma(\cdot,0) = \sigma_0(\cdot) \text{ on } \Omega, \tag{7c}$$

where p_0, $\mathbf{u}_0, \mathbf{v}_0$, and σ_0 are suitably given functions.

The system (5) consists of the diffusion equation for the pressure, the conservation equation for momentum, and a constitutive relation for the deformation response of the medium, respectively. The constitutive relation (5c) involves the stress σ and the *small strain* tensor $\varepsilon(\mathbf{u})$, and $\mu^* \geq 0$ arises from *secondary consolidation* effects. We briefly comment on the boundary conditions. First of all, the fluid is *drained* on the portion Γ_d and the medium is *clamped* along Γ_c. The relations (6b) and (6d) are constraints on *fluid flux* and *traction*, respectively. On the set Γ_s, where neither p nor $\mathbf{u}$ is prescribed, the function β comes into play. This function specifies the fraction of the pores of the medium that are *exposed* along Γ_s. Indeed, for these pores, the motion of the solid adds their contents to the fluid flux through the term $\beta\frac{\partial}{\partial t}\big(\mathbf{u}\cdot\mathbf{n}\big)$ in (6b). In the remaining portion, the *sealed* pores, the hydraulic pressure contributes to the total stress within the structure, and this is the origin of the normal pressure term $(1-\beta)p\mathbf{n}$ in (6d).

Let us emphasize that a very extensive variety of models is included in the system (5). Moreover, the theory developed in [55] permits highly degenerate situations in which some (or even all) of the parameters $c_0(\cdot), \rho(\cdot)$, $\mu^*(\cdot)$, and $k(\cdot)$ may vanish! Specifically, we include any combination of the *quasi-static* case, $\rho = 0$, the *incompressible* case, $c_0 = 0$, the *uncoupled* case, $\alpha = 0$, and even the *impermeable* case, $k = 0$. The model for the porous solid is a very general rheological material made up of the parallel combination of elementary components of various types, elastic, viscous, and plastic, with combinations of kinematic and isotropic hardening. Such a construction requires the introduction of *internal variables* [28], [58]. In addition, the system is generalized to include *quasilinear diffusion* $k(\cdot)$ and *nonlinear dissipation* $\mu^*(\cdot)$. We are able to prove existence and uniqueness of a *strong solution* of this nonlinear and degenerate system without any coercive-type assumptions on any of the operators in (5a) or (5b)! Rather, all of the essential assumptions are restricted to the constitutive relation (5c), and these consist of a variant of the *safe load condition.* Additional work currently underway includes the extension to composite structures with multi-porous characteristics and coupled fluid flow and heat conduction within the deforming solid, as well as such geomechanic coupling with *non–Darcy flow* models.

3.2. **Multi-Scale Problems.** The parallel structure of the Barenblatt-Biot system (3) limits both the property types of materials and the fine-scale geometry of the two-component models that can be represented. *Distributed microstructure systems* can assimilate into the model the detailed fine-scale geometry and the multiple scales, and they can better quantify the exchange of fluid and momentum across the intricate interface between the components. Such systems frequently arise as the limit by *homogenization* of corresponding exact but highly singular partial differential equations with rapidly oscillating coefficients. This provides not only a derivation of the model systems, but shows also the relation with the classical but singular problem on the microscale, and it provides a method for directly computing the *effective coefficients* which represent averaged material properties. Homogenization techniques have been used to identify and develop more realistic models of multi-porous or multi-permeable composite media in the rigid case, and one can consult [2], [24], [29] for representative results. Homogenization methods were also used by Auriault–Sanchez-Palencia [3] and by Burridge-Keller [13] to derive the Biot system. One starts on the microscale with the Navier elasticity system for the solid deformation coupled to a Stokes flow system for the fluid and obtains the Biot system (1) in the limit as the spatial scale goes to zero. For various situations, Auriault *et al* [4], [29] used appropriately scaled coefficients to construct the distributed microstructure models which led to differing macroscale behaviors.

With the intent of gaining more experience with the types of structures that can be obtained from various geometries and scalings of the microstructure of a deformable porous medium, we have applied homogenization methods to upscale from the micro-scale or an intermediate meso-scale to the macro-scale in a number of cases. The components on the micro- or meso-scale are described by either a Stokes, a Darcy or a Biot system for the first component, the *macropores* or fractures, and either a Navier, a Darcy or a Biot system for the second component, which comprises the solid or microporous structure, with coefficients appropriately scaled, the choice being dependent on the specific application and range of scales anticipated. An important technical aspect for each case is the appropriate set of boundary conditions to use at the interface between the various systems.

To illustrate with a relatively simple example the types of systems that emerge, we describe the *highly heterogeneous micro-model* and the limiting form of the corresponding *Darcy-Biot distributed microstructure system* which is the *macro-model* for the composition of a *rigid* porous and permeable system intertwined on a fine-scale with a periodic array of very compliant *elastic inclusions* of much lower permeability. Thus, the permeability is scaled by ε^2 in the inclusions, just as in the *fractured medium* model of Arbogast-Douglas-Hornung [2], and we scale the elasticity similarly. We expect small deformations to be activated by the high-frequency pressure gradients.

Let's begin with a description of the geometry of the microstructure. The macroporous and permeable structure with local inclusions is periodically distributed in a domain Ω in $\mathbb{R}^N$ with period εY, where $\varepsilon > 0$ and $Y = [0,1]^N$ is the unit cube. Let Y be given in complementary parts, Y_1 and Y_2, which determine the local geometry of the porous structure and the inclusions, respectively. Denote by $\chi_m(y)$

the characteristic function of Y_m for $m = 1, 2$, extended Y-periodically to all of $\mathbb{R}^N$. Thus, $\chi_1(y) + \chi_2(y) = 1$. We shall assume that the set $\{y \in \mathbb{R}^N : \chi_1(y) = 1\}$ is smooth and connected. The domain Ω is thus partitioned into the two subdomains

$$\Omega_m^\varepsilon = \left\{ x \in \Omega : \chi_m\left(\frac{x}{\varepsilon}\right) = 1 \right\} , \qquad m = 1, 2 .$$

Let $\Gamma_{12}^\varepsilon \equiv \partial\Omega_1^\varepsilon \cap \partial\Omega_2^\varepsilon \cap \Omega$ be that part of the interface of Ω_1^ε with Ω_2^ε that is interior to Ω, and let $\Gamma_{12} \equiv \partial Y_1 \cap \partial Y_2 \cap Y$ be the corresponding interface in the cell Y. Likewise, let $\Gamma_{22} \equiv Y_2 \cap \partial Y$ and denote by Γ_{22}^ε its periodic extension which forms the interface between those parts of the second component Ω_2^ε which lie within neighboring εY-cells. These are the *local inclusions* and we denote them by Y_2^ε. The second component Ω_2^ε may be connected, but this is not required.

The flow in the porous structure Ω_1^ε is described by a Darcy system that is coupled across the interface Γ_{12}^ε to a Biot system for the slow flow and deformation in the inclusions Ω_2^ε. In the region Ω_2^ε we scale the permeability by ε^2. Thus, we shall denote by c_m and $\varepsilon^{2(m-1)}\kappa_m$ the *compressibility* and the *permeability*, respectively, in Ω_m^ε, $m = 1, 2$. The fluid *pressure* in the macroporous region Ω_1^ε is denoted by $p_1^\varepsilon(x, t)$ and the corresponding *flux* there is given by $-\kappa_1 \nabla p_1^\varepsilon$. The pressure in the region Ω_2^ε is $p_2^\varepsilon(x, t)$ with scaled flux $-\varepsilon^2 \kappa_2 \nabla p_2^\varepsilon$. It is the resulting very high frequency spatial variations in the pressure gradients in the second component which lead to local storage and corresponding local deformations. These are described by the (small) *displacement* $\mathbf{u}(x, t)$ from the position $x \in \Omega_2^\varepsilon$, and $\varepsilon_{kl}(\mathbf{u}) \equiv \frac{1}{2}(\partial_k u_l + \partial_l u_k)$ is the (linearized) *strain* tensor. The *stress* $\sigma(\mathbf{u})$ is given by the *generalized Hooke's law* $\sigma_{ij}(\mathbf{u}) = a_{ijkl}^m \, \varepsilon_{kl}(\mathbf{u})$ with the positive definite symmetric *elasticity* tensor a_{ijkl} for a general anisotropic material. The boundary conditions will involve the surface density of forces or *traction* $\sigma_{ij} n_j$. The normal will be directed *out* of Ω_2^ε. The elastic structure is described by the bilinear form

$$e(\mathbf{u}, \mathbf{v}) \equiv \int_{\Omega_2^\varepsilon} a_{ijkl} \, \varepsilon_{kl}(\mathbf{u}) \partial_j v_i \, dx = \int_{\Omega_2^\varepsilon} a_{ijkl} \, \varepsilon_{kl}(\mathbf{u}) \varepsilon_{ij}(\mathbf{v}) \, dx$$

on the space $\mathbf{V}^\varepsilon \equiv \{\mathbf{v} \in H^1(\Omega) : \ \mathbf{v} = 0 \text{ on } \Omega_1^\varepsilon\}$ of *admissable displacements*. The local operator is obtained by means of Stokes' theorem, that is,

$$e(\mathbf{u}, \mathbf{v}) = \int_{\Omega_2^\varepsilon} \mathcal{E}(\mathbf{u}(x)) \cdot \mathbf{v}(x) \, dx$$

where the formal operator is given by $\mathcal{E}(\mathbf{u})_i = -\partial_j a_{ijkl} \, \varepsilon_{kl}(\mathbf{u}) \, , \ 1 \le i \le 3$, whenever $\mathbf{u}, \ \mathbf{v} \in \mathbf{V}$ and $\mathcal{E}(\mathbf{u}) \in [L^2(\Omega_2^\varepsilon)]^3$.

The Darcy flow in the first component and the Biot system for the second component are given by the *highly heterogeneous system*

$$c_1 \dot{p}_1^\varepsilon - \nabla \cdot (\kappa_1 \nabla p_1^\varepsilon) = F_1 \qquad \text{in } \Omega_1^\varepsilon \, ,$$

$$p_1^\varepsilon = p_2^\varepsilon \, , \quad \kappa_1 \frac{\partial p_1^\varepsilon}{\partial n} = \varepsilon^2 \kappa_2 \frac{\partial p_2^\varepsilon}{\partial n} \, , \quad \mathbf{u}^\varepsilon = \mathbf{0} \quad \text{on } \Gamma_{12}^\varepsilon \, ,$$

$$\begin{aligned} c_2 \dot{p}_2^\varepsilon - \nabla \cdot (\varepsilon^2 \kappa_2 \nabla p_2^\varepsilon) + b_\varepsilon \nabla \cdot \dot{\mathbf{u}}^\varepsilon = F_2 \qquad & \text{and} \\ \rho_2 \ddot{\mathbf{u}}^\varepsilon + \varepsilon^2 \mathcal{E}(\mathbf{u}^\varepsilon) + b_\varepsilon \nabla p_2^\varepsilon = \mathbf{f}_2 \, , \qquad & \text{in } \Omega_2^\varepsilon \, , \end{aligned}$$

which is the *exact* model of the fine-scale structure. It is supplemented with appropriate initial and boundary conditions, and then it follows that there is a unique solution $p_1^\varepsilon(x,t),\ p_2^\varepsilon(x,t),\ \mathbf{u}^\varepsilon(x,t),\ x \in \Omega$ for each $\varepsilon > 0$. Moreover, the three sequences of functions have *two-scale* limits as $\varepsilon \to 0$, and these are given by the triple of functions $p_1(x,t),\ P_2(x,y,t),\ \mathbf{U}(x,y,t),\ x \in \Omega,\ y \in Y_2$. Then we show that this triple is a solution of a Darcy-Biot distributed microstructure system which we now describe.

The global flow in the porous structure is determined on the macro-scale by the *macroscopic Darcy equation*

$$\tilde{c}_1 \dot{p}_1(x,t) - \frac{\partial}{\partial x_j} A^1_{ij} \frac{\partial p_1(x,t)}{\partial x_i} + \int_{\Gamma_{12}} \kappa_2 \frac{\partial P_2(x,s,t)}{\partial n}\, ds = F_1(x,t)\,, \quad x \in \Omega\,,$$

where $\int_{\Gamma_{12}} \kappa_2 \frac{\partial P_2(x,s,t)}{\partial n}\, ds$ is the exchange term representing the flow into the local inclusion Y_2^ε at the point $x \in \Omega$. The flow and deformation within the re-scaled Y_2^ε are given by the *local Biot system*

$$c_2 \dot{P}_2(x,y,t) - \nabla_y \cdot \kappa_2 \nabla_y P_2(x,y,t) + b_0 \nabla_y \cdot \dot{\mathbf{U}}(x,y,t) = F_2(x,y,t)\,,\ \ y \in Y_2\,,$$

$$P_2(x,s,t) = p_1(x,t)\,, \quad s \in \Gamma_{12}\,,$$

$$P_2(x,s,t) \text{ and } \kappa_2 \nabla_y P_2(x,s,t) \text{ are } Y\text{-periodic on } \Gamma_{22}\,,$$

$$\rho \ddot{U}_i(x,y,t) - \frac{\partial}{\partial y_j}\left(\sigma^y_{ij}(\mathbf{U}(x,y,t)) - \delta_{ij} b_0 P_2(x,y,t)\right) = \mathbf{0}\,, \quad y \in Y_2\,,$$

$$\mathbf{U}(x,s,t) \text{ and } \sigma^y_{ij}(\mathbf{U}(x,s,t))n_j - b_0 P_2(x,s,t) n_i \text{ are } Y\text{-periodic on } \Gamma_{22}\,,$$

$$\mathbf{U}(x,s,t) = \mathbf{0}\,, \quad s \in \Gamma_{12}\,.$$

The subscript y on the gradient indicates that the derivative is taken with respect to the local variable y. The solution $P_2(\cdot,\cdot),\ \mathbf{U}(\cdot,\cdot)$ of the local system depends on the global pressure $p_1(\cdot)$ at the point $x \in \Omega$. Because of the small size of the cells, this pressure is assumed to be well approximated by the "constant" value $p_1(x,t)$ on the interface Γ^ε_{12}. Note that if the deformation is suppressed in this system, *i.e.*, if $\mathbf{U}(x,y,t) = \mathbf{0}$, then this is precisely the model of Arbogast *et al* [2] for single phase flow in a doubly-porous medium.

This Darcy-Biot model is a very special case, intended only to suggest the structure of the limiting initial-boundary-value problems that arise, and the micromodels that come from the particular applications always lead to considerably more complicated distributed microstructure models. One can use Biot systems for each component and scale the parameters for each component in a wide variety of ways, the choice being dependent on the situation. Also, one can start with a Biot system for the structure coupled to a fluid flow model either of Stokes type or of slightly compressible flow type and then investigate the limiting form of the composite for various scalings of the parameters. Similar *Biot-Biot* models have been constructed for the mechanics of soft biological tissues. These are based on the hypothesis that tissue can be regarded as a *composite cellular poroelastic* material composed of a poroelastic extracellular matrix in which poroelastic cells are embedded. These models account for the difference between the local and global cellular fluid pressures, and these depend on the various coefficients and characteristic length scales of cells and tissue. This behaviour is similar to the secondary consolidation phenomenon sometimes observed in geomechanics. Our experience suggests that the resulting distributed microstructure systems provide accurate models which include the fine scales and geometry appropriate for many situations, but we have to balance our need for such detail with the tolerance for extra effort required to solve the system or to simulate solutions.

4. Conclusion

Our long range objective is to develop a set of comprehensive models incorporating many of the preceding cases as components in a modular format and to combine them with available subsurface flow and transport simulators as a basis to perform basic research on the theoretical and numerical modeling and simulation of *deforming heterogeneous porous media*. The models will be designed to apply to specific application areas. They will build on established cases and be enhanced to include previously neglected effects, which are important in various emerging applications. They must account for complex *nonlinear* behavior and media *heterogeniety* as described above. The fundamental issues of *scale* arise in the consideration of multi-component media, since porosity, permeability and compliance often occur on several distinct spatial scales in both natural and manufactured materials. These may be fractures or simply regions of extremely high permeability with large correlation lengths. The goal is to develop models whose components are simple enough to be analytically or numerically tractable but sufficiently detailed to capture the nonlinear effects and fine scale interface geometry that complicate the competition between long and short time behavior of transport and consolidation processes.

Although this modular format will include many of the cases above, not all of the components or extensions need be used for any specific application. Thus it will be necessary to determine which components are relevant for an individual application, depending on such things as the range of the variables, the time scale of the motion, and the spatial scale relevant for the application. This information, together with that obtained from the analysis of the various components, will be used to decide which of the modules are appropriate for a specific application. Each of the models will have an established mathematical basis, and we are extending these to include the composite poro-mechanical systems that are constructed to meet the needs of the specific applications.

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