JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS 47, 563–572 (1974)

The Final Value Problem for Evolution Equations

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Submitted by J. L. Lions

1. INTRODUCTION

Let A be an unbounded linear operator with dense domain $D(A)$ in a complex Hilbert space H, and consider the problem of finding a solution of the evolution equation

$$u'(t) + Au(t) = 0, \qquad t \in (0, T) \tag{1.1}$$

satisfying the prescribed final-value, $u(T) = f$. We assume A is maximal accretive, so the problem is generally not well-posed.

Assume in addition that A^2 is accretive. We shall demonstrate that there is at most one solution of this problem, and we give a constructive quasi-reversibility method of constructing solutions of (1.1) which approximately satisfy the final condition. In particular, one lets $\alpha > 0$ and solves backward the "reversible" approximation to (1.1) given by

$$v'(t) + \alpha Av'(t) + Av(t) = 0, \tag{1.2}$$

with $v(T) = f$. Then $v(0)$ is used as the initial value for a solution u_α of (1.1). We prove below that $\lim_{\alpha \to 0} u_\alpha(T) = f$ for any $f \in H$ and the method is stable in the sense that $\| u_\alpha(T)\| \leqslant \| f \|$ for all $\alpha > 0$. If there actually exists a solution u of the final value problem, then as $\alpha \to 0$, the approximations u_α and their derivatives $u_\alpha^{(m)}$ converge uniformly on compact subsets of $(0, T]$ to u and its derivative, $u^{(m)}$, respectively. Finally, we obtain estimates on the degree of approximation of the solution u by the approximations u_α, both in the norm of H and in stronger norms determined by powers of A.

The general method of quasireversibility was introduced by Lattes and

* Research supported in part by National Science Foundation grant GP-34261.

Lions [13] for the solution of non-well-posed problems. They approximated (1.1) by the equation

$$w'(t) + Aw(t) - \alpha A^2 w(t) = 0, \tag{1.3}$$

and assumed A is self-adjoint and positive, a condition more restrictive than ours. As before, (1.3) is solved backwards subject to the final condition $w(T) = f$. Then $w(0)$ is the initial condition for a solution w_α of (1.1). The authors asked only that $w_\alpha(T)$ approximate f but did not consider $w_\alpha(t)$ for $t < T$. When the inverse of A is compact, an elementary computation shows that $\| u_\alpha(T) - f \| \leqslant \| w_\alpha(T) - f \|$, so approximation of (1.1) by (1.2) is at least as good as the corresponding approximation by (1.3) [7, 15].

The idea of approximating (1.1) by (1.2) is due to Yosida and is the basis for his proof of the generation theorem for semigroups of operators [19]. When A is a realization of a partial differential operator, (1.2) is a pseudo-parabolic or Sobolev partial differential equation [16]. Such equations arise from certain models of fluid flow in fissured material [1], heat conduction [2], shear in second order fluids [3, 10], consolidation of clay [18], and others [6] in which the coefficient α has the dimensions of viscosity. This writer and Ting [17] pointed out that Yosida's proof of the generation theorem shows that the parabolic equation (1.1) can be approximated by the pseudo-parabolic (1.2). Such approximations have also been useful in nonlinear problems, e.g. [4, 12], and may be viewed as a method of "vanishing viscosity." Hence, we have a motivation from the physical models above to use (1.2).

The plan of the paper is as follows. In Section 2 we review the basic results on the generation of semigroups of operators and their relation to the evolution equation (1.1). The method of quasireversibility suggests the construction in Section 3 of a special class of semigroups. We characterize when these are contractions in terms of the operators A and A^2. These contraction semigroups are shown to converge to the identity, in an appropriate sense, and this leads to the major results of the paper in Section 4. There we show that our method converges if and only if there exists a solution. Uniqueness of a solution of the final value problem is verified, and we give estimates on the convergence of our approximations and their derivatives to a solution and its corresponding derivatives.

The results we give are intentionally far from "best possible" in any sense. Rather, we have restricted our attention to Hilbert space (rather than, e.g. reflexive Banach space), and to the simplest evolution equation (1.1) which is irreversible (rather than, e.g. the situation in [12].) We also choose to give the elementary proofs available rather than to obtain the corresponding results from well-known theorems in the literature. (For example, certain results of Sections 3 and 4 can be obtained from standard results on the convergence of semigroups.) However, these restrictions permit a self-contained and elementary presentation of results which cover most interesting applications.

2. Generation of Semigroups

We first recall the Hille–Yosida theorem. Let $\mathscr{L}(H)$ denote the set of continuous linear operators on H. By a "semigroup" on H we mean a function S: $[0, \infty) \to \mathscr{L}(H)$ such that $S(t + s) = S(t)\,S(s)$ for all $t, s \geqslant 0$, $S(0) = I$, and $S(\cdot)\,x$: $[0, \infty) \to H$ is continuous for each $x \in H$. S is a "contraction semigroup" on H if, in addition, $\| S(t)\| \leqslant 1$ for each $t \geqslant 0$. The (infinitesimal) "generator" of the semigroup S is the operator B defined by

$$Bx = \lim_{t\to 0} t^{-1}(S(t)\,x - x),$$

the domain being the set of all x for which the limit exists.

Theorem [9, 19]. *The operator B is the generator of a contraction semigroup if and only if B is closed, densely defined, each $\lambda > 0$ is in the resolvent set of B, and $\| \lambda(\lambda - B)^{-1} \| \leqslant 1$ for all $\lambda > 0$.*

Corollary. *The operator $-A$ is the generator of a contraction semigroup if and only if for every $\alpha > 0$ the operator $J_\alpha = (I + \alpha A)^{-1}$ is a contraction in $\mathscr{L}(H)$.*

The condition on A in this corollary easily implies that $\mathrm{Re}(Ax, x)_H \geqslant 0$ for all $x \in D(A)$. Such operators are "accretive," and we call A "m-accretive" if $-A$ generates a contraction semigroup [11].

Since it motivates our major results of the next section, we outline Yosida's elegant proof of the corollary above. The operator A is m-accretive, so we can define J_α as above and $A_\alpha = AJ_\alpha$ for each $\alpha > 0$. From the identity

$$A_\alpha = \alpha^{-1}(I - J_\alpha) \tag{2.1}$$

and the J_α being contractions, it follows that each A_α is accretive. But $A_\alpha \in \mathscr{L}(H)$ so we can define a "group" of linear operators by

$$S_\alpha(t) \equiv \exp(-tA_\alpha), \qquad -\infty < t < \infty,$$

where we use the power series to define the exponential function. Hence, each A_α is m-accretive and $S_\alpha(t)$ is a contraction if $t \geqslant 0$. On $D(A)$ we have $A_\alpha = J_\alpha A$; this identity and (2.1) give, respectively,

$$\| A_\alpha x \| \leqslant \| Ax \|, \qquad \| J_\alpha x - x \| \leqslant \alpha \| Ax \|, \qquad x \in D(A). \tag{2.2}$$

From (2.1) and (2.2) we have

$$\| A_\alpha x - Ax \| \leqslant \alpha \| A^2 x \|, \qquad x \in D(A^2), \tag{2.3}$$

so J_α approximates the identity and A_α approximates A on $D(A)$ for small α. These facts and the uniform boundedness of $\{S_\alpha(t): t \geqslant 0, \alpha > 0\}$ are used to demonstrate the existence of the strong limit $S(t)\, x = \lim_{\alpha \to 0} S_\alpha(t)\, x$, $x \in H$. This limit is the desired semigroup generated by A.

Our interest in the semigroup $S(t)$ arises from the fact that for any $\xi \in D(A)$, the function defined by $u(t) = S(t)\, \xi$ is differentiable and satisfies (1.1) for every $t \geqslant 0$. The special class of semigroups called "holomorphic" and described below have the property that $S(t)$ maps all of H into the domain of every power of A for each $t > 0$, so the $u(t)$ defined above is a solution of (1.1) for every $\xi \in H$.

An unbounded operator A on H is called "sectorial" with semiangle θ if all of the complex numbers (Ax, x), $x \in D(A)$, belong to the sector $\{z: | \arg(z)| \leqslant \theta\}$. Thus, A is accretive if it is sectorial with semiangle $\pi/2$. If A is m-accretive and sectorial, we call it "m-sectorial."

THEOREM [11]. *If A is m-sectorial with semiangle θ, where $0 \leqslant \theta < \pi/2$, then S is a holomorphic semigroup. For each $t > 0$ and $x \in H$, $S(t)\, x \in D(A)$ and $AS(t) \in \mathscr{L}(H)$ with $AS(t) \leqslant M/t$. The identity $S(t) = S(t/m)^m$ shows that $S(t)$ maps H into $D(A^m)$ for $t > 0$ and integer $m \geqslant 1$, and also we have*

$$A^m S(t) \leqslant (M/t)^m. \tag{2.4}$$

There is an intimate connection between solutions of (1.1) and the semigroup S generated by $-A$.

DEFINITION. A solution of (1.1) on the interval $[a, b]$ is a

$$u \in C([a, b], H) \cap C^1((a, b), H)$$

such that for all $t \in (a, b)$, $u(t) \in D(A)$ and (1.1) holds.

If A is accretive, then for any solution u of (1.1) on $[a, b]$ we have

$$(d/dt) \| u(t)\|^2 = 2\mathrm{Re}(u'(t), u(t)) = -2\mathrm{Re}(Au(t), u(t)) \leqslant 0$$

for $t \in (a, b)$, so $\| u(t)\| \leqslant \| u(a)\|$. Applying this to the difference of two solutions shows that they are uniquely determined by and depend continuously on $u(a)$. Hence, if for each $\xi \in D(A)$ there is a solution of (1.1) on $[a, b]$ with $u(a) = \xi$, then there is for each such ξ a unique solution u on $[0, \infty)$ with $u(0) = \xi$. Thus, by defining $S(t)\, \xi = u(t)$, we obtain linear maps $S(t)$ of $D(A)$ into itself. If $D(A)$ is dense in H, these can be extended to obtain the contraction semigroup on H generated by $-A$. Conversely, if A is m-accretive, the solution of (1.1) on $[a, b]$ with $u(a) = \xi$ is given by $u(t) = S(t - a)\,\xi$ for each $\xi \in D(A)$.

Let A be m-accretive. If u is a solution of (1.1) on $[a, b]$ then for each $t \in (a, b]$ we have $(d/ds)\,(S(t - s)\,u(s)) = 0$ for $s \in (a, t)$, hence

$$u(t) = S(t - a)\,u(a).$$

Since u is differentiable at t exactly when $u(t) \in D(A)$, we see that u is a solution of (1.1) on $[a, b]$ if and only if $u(t) = S(t - a)\,u(a)$ for all $t \in [a, b]$, and $u(t) \in D(A)$ for all $t \in (a, b)$. This suggests the following.

Definition. A weak solution of (1.1) on $[a, b]$ is a function of the form $S(t - a)\,\xi$ for some $\xi \in H$.

It follows easily that there exists a weak solution u of (1.1) on $[T - \delta, T]$ taking the final value $u(T) = f \in H$, where $0 < \delta$, if and only if $f = S(\delta)\,\xi$ for some $\xi \in H$. Also, if $-A$ generates a holomorphic semigroup, then the notions of solution and weak solution coincide.

3. The QR-Semigroup

Let A be m-accretive and consider the final-value problem for (1.1). From our remarks at the end of Section 2, it follows that we should find an initial vector ξ such that $S(T)\,\xi = f$. Since the operators $S_\alpha(t)$ form a group (and hence are defined for $t < 0$), and since they approximate the semigroup $S(t)$ at those $t \geqslant 0$ when $\alpha > 0$ is small, a natural candidate for an initial condition for which the solution to (1.1) arrives close to the final value f is the vector $S_\alpha(-T)f$. The corresponding solution is given by

$$u_\alpha(t) = S(t)\,S_\alpha(-T)f, \qquad t \in [0, T]. \tag{3.1}$$

We want to show (at least) that $\lim_{\alpha\to 0} u_\alpha(T) = f$, so we are led to examine the operators

$$E_\alpha(t) = S(t)\,S_\alpha(-t), \qquad \alpha > 0, \quad t \geqslant 0. \tag{3.2}$$

Lemma 1. *For each $\alpha > 0$, E_α is a semigroup on H and $-(A - A_\alpha)$ is the generator.*

Proof. Since $S(t)$ and $S_\alpha(-t)$ commute, E_α is clearly a semigroup on H, and we denote its generator by B. Differentiation of $E_\alpha(t)\,x$ for $x \in D(A)$ shows that $x \in D(B)$ and B is an extension of $A_\alpha - A$. But $S(t) = E_\alpha(t)\,S_\alpha(t)$ shows likewise that $D(B) = D(A)$, so $B = A_\alpha - A$.

DEFINITION. For each $\alpha > 0$, E_α is a "QR-semigroup" for the m-accretive operator A. The collection of QR-semigroups is "stable" if each is a contraction semigroup.

Since the stability of the QR-semigroups is essential in the development below, we shall characterize it in terms of A. For any $\alpha > 0$, Lemma 1 implies that E_α is a contraction exactly when $A - A_\alpha$ is accretive. But from (2.1), $A - A_\alpha = \alpha A^2 J_\alpha$ on $D(A)$, so $A - A_\alpha$ is accretive if and only if $\operatorname{Re}(A^2x, x + \alpha Ax) \geqslant 0$ for all $x \in D(A^2)$. This gives the following.

LEMMA 2. *The QR-semigroups are stable if and only if A^2 is accretive.*

We examine the condition in Lemma 2. Suppose first that $B \in \mathscr{L}(H)$ is accretive, hence m-accretive. The polar decomposition $B = X + iY$ expresses B in terms of self-adjoint bounded operators, and the real part, X, is nonnegative. The real part of B^2 is given by $X^2 - Y^2$, and this is nonnegative exactly when $-X \leqslant | Y | \leqslant X$. Thus, B^2 is accretive if and only if B is sectorial with semiangle $\pi/4$.

Let A be an m-accretive (possibly unbounded) operator on H. For each $\epsilon > 0$, $(\epsilon + A)^{-1} \in \mathscr{L}(H)$ and is accretive. Also,

$$\operatorname{Re}((\epsilon + A)^{-2} x, x) = \epsilon^2 \| z \|^2 + \epsilon \operatorname{Re}(Az, z) + \operatorname{Re}(A^2z, z)$$

where $z = (\epsilon + A)^{-2} x \in D(A^2)$, so it follows by applying the preceding result with $B = (\epsilon + A)^{-1}$ that A^2 is accretive if and only if $(\epsilon + A)^{-1}$ is sectorial with semiangle $\pi/4$ for every $\epsilon > 0$. An easy computation shows that this is equivalent to A being sectorial with semiangle $\pi/4$, so we have the following result.

LEMMA 3. *The m-accretive operator A is sectorial with semiangle $\pi/4$ if and only if A^2 is accretive.*

Consider the QR-semigroups $\{E_\alpha : \alpha > 0\}$. Let $x \in D(A)$ and $t \geqslant 0$. For $\alpha, \beta > 0$ it follows by the fundamental theorem of calculus that

$$\begin{aligned} E_\alpha(t)\, x - E_\beta(t)\, x &= \int_0^t \frac{d}{ds} \{E_\alpha(s)\, E_\beta(t-s)\, x\}\, ds \\ &= \int_0^t E_\alpha(s)\, E_\beta(t-s) \{A_\beta x - A_\alpha x\}\, ds. \end{aligned}$$

If the QR-semigroups are stable, then we obtain

$$\| E_\alpha(t)\, x - E_\beta(t)\, x \| \leqslant t \| A_\beta x - A_\alpha x \|,$$

so $\lim_{\alpha \to 0} E_\alpha(t)\, x$ exists for all $x \in D(A)$. Since the operators $\{E_\alpha(t), \alpha > 0\}$ are uniformly bounded and $D(A)$ is dense in H, the limit exists for all $x \in H$

and we denote it by $E(t)\,x$. The estimate shows that the convergence of $E_\alpha(t)\,x$ to $E(t)\,x$ is uniform on bounded intervals, so we may take the limit in the identity

$$E_\alpha(t)\,x = x - \int_0^t E_\alpha(s)\,(Ax - A_\alpha x)\,ds,$$

to obtain $E(t)\,x = x$ for $x \in D(A)$. But $E(t)$ is a contraction and $D(A)$ is dense, so $E(t) = I$ for all $t \geqslant 0$. Thus, we have proved the following fundamental result.

THEOREM 1. *Let A be an m-accretive in H and define the QR-semigroups, E_α, by* (3.2). *Then the QR-semigroups are stable if and only if A is sectorial with semiangle $\pi/4$. This is equivalent to A^2 being accretive, and in that case we have $\lim_{\alpha\to 0} E_\alpha(t)\,x = x$ for each $x \in H$, uniformly on bounded intervals, and the following estimates hold*:

$$\|E_\alpha(t)\,x - x\| \leqslant t\,\|Ax - A_\alpha x\|, \qquad x \in D(A);$$
$$\|E_\alpha(t)\,x - x\| \leqslant t\alpha\,\|A^2 x\|, \qquad x \in D(A^2).$$

Remarks. Since the semigroup generated by A is holomorphic, the final value problem is well-posed only if A is bounded [8]. Also, the condition in Theorem 1 on A is satisfied if $A = cB$, where B is symmetric and c is complex with $|\arg(c)| \leqslant \pi/4$.

4. Existence, Uniqueness, and Approximation

Hereafter assume A is an m-sectorial operator with semiangle $\pi/4$, so the results of Theorem 1 apply. We first show that there is at most one solution of the final value problem for (1.1) on $[0, T]$. This is the problem of backward uniqueness for (1.1) and by linearity is equivalent to showing that the kernel of $S(T)$ consists only of the zero vector. An easy computation shows that the kernel of $S(T)$ is the orthogonal complement of the range of the adjoint, $S^*(T)$, so we need to show that the range of $S^*(T)$ is dense in H.

The adjoint A^* of the m-accretive A is also m-accretive, and $-A^*$ is the generator of the contraction semigroup $\{S^*(t)\colon t \geqslant 0\}$ [19]. Since $S(t)$ commutes with $S_\alpha(-t)$, we have $E_\alpha^*(t) = S^*(t)\,S_\alpha^*(-t)$ for $\alpha > 0$ and $t \geqslant 0$, so Lemma 1 shows that E_α^* is generated by $-(A^* - A_\alpha^*)$. But the adjoint of a bounded operator has the same norm, and each E_α is a contraction semigroup, so E_α^* is a contraction semigroup for each $\alpha > 0$. Theorem 1 then shows that for every $x \in H$, $S^*(T) S_\alpha^*(-T) x \to x$ as $\alpha \to 0$, so the range of $S^*(T)$ is dense in H.

We noted in Section 2 that a given $f \in H$ is the final value of a solution of (1.1) on $[T - \delta, T]$, where $0 \leqslant \delta \leqslant T$, if and only if $f = S(\delta)\,\xi$ for some $\xi \in H$. By our uniqueness result above, ξ is uniquely determined by f. Moreover, Theorem 1 shows that $\xi = \lim_{\alpha \to 0} E_\alpha(\delta)\,\xi = \lim_{\alpha \to 0} S_\alpha(-\delta) f$. Conversely, if $\lim_{\alpha \to 0} S_\alpha(-\delta) f = \xi$, then, since each $S_\alpha(\delta)$ is a contraction, we have $\lim_{\alpha \to 0} S_\alpha(\delta) S_\alpha(-\delta) f = S(\delta)\,\xi$. But Theorem 1 implies that this limit is just f, so we have proved the following.

THEOREM 2. *Let A be m-sectorial with semiangle $\pi/4$. For each $f \in H$, there is at most one solution u of* (1.1) *on* $[0, T]$ *with $u(T) = f$. If $0 \leqslant \delta \leqslant T$, there is a solution u of* (1.1) *on* $[T - \delta, T]$ *with $u(T) = f$ if and only if* $\lim_{\alpha \to 0} S_\alpha(-\delta) f$ *exists in H, and then this limit is the vector $u(T - \delta)$.*

Let u be the solution of (1.1) on $[T - \delta, T]$ with $u(T) = f$. Then we have the representations

$$u(t) = S(t + \delta - T)\,\xi, \qquad t \geqslant T - \delta,$$

where ξ is determined by $f = S(\delta)\,\xi$, and

$$u_\alpha(t) = S(t)\, S_\alpha(-T) f = S(t + \delta - T)\, E_\alpha(T)\,\xi, \qquad t \geqslant T - \delta, \qquad \alpha > 0.$$

Thus, we obtain

$$u_\alpha(t) - u(t) = S(t + \delta - T)\{E_\alpha(T)\,\xi - \xi\}, \qquad \alpha > 0, \quad t \geqslant T - \delta. \tag{4.1}$$

Since S is a contraction semigroup, (4.1) shows that $u_\alpha \to u$, uniformly on $[T - \delta, T]$, as $\alpha \to 0$. Furthermore, the semigroup S is holomorphic on $(0, \infty)$, so we can differentiate (4.1) to get

$$u_\alpha^{(m)}(T) - u^{(m)}(t) = (-A)^m\, S(t + \delta - T)\{E_\alpha(T)\,\xi - \xi\},$$
$$\alpha > 0, \qquad t > T - \delta, \qquad m \geqslant 0.$$

The estimate (2.4) then gives

$$\begin{gathered} \| u_\alpha^{(m)}(t) - u^{(m)}(t)\| \leqslant [M/(t + \delta - T)]^m \, \| E_\alpha(T)\,\xi - \xi \|, \\ \alpha > 0, \qquad t > T - \delta, \qquad m \geqslant 0. \end{gathered} \tag{4.2}$$

We can also estimate the dependence of $u_\alpha - u$ on α. If ξ satisfies a "smoothness" assumption, $\xi \in D(A^2)$, then Theorem 1 shows that the last term in (4.1) is bounded by $T\alpha A^2 \xi$. But such an assumption on the initial vector is unnecessary, since the semigroup S is holomorphic. In particular, if $0 < \epsilon \leqslant \delta$, then $S(\epsilon)\,\xi \in D(A^2)$ and we can write (4.1) in the form

$$u_\alpha(t) - u(t) = S(t + \delta - T - \epsilon)\{E_\alpha(T)\,S(\epsilon)\,\xi - S(\epsilon)\,\xi\}, \qquad t \geqslant T - \delta + \epsilon.$$

Differentiation of this identity and (2.4) give the estimate

$$\begin{gathered} \| u_\alpha^{(m)}(t) - u^{(m)}(t) \| \leqslant [M/(t + \delta - T - \epsilon)]^m T\alpha \, \| A^2 S(\epsilon)\xi \|, \\ t \geqslant T - \delta + \epsilon, \qquad 0 < \epsilon \leqslant \delta, \qquad \alpha > 0, \end{gathered} \tag{4.3}$$

so we have proved the following result.

THEOREM 3. *Let A be m-sectorial with semiangle $\pi/4$ and $f = S(\delta)\,\xi$ for some $\delta \in [0, T]$. Let u be the solution of* (1.1) *on* $[T - \delta, T]$ *with $u(T) = f$ and let u_α be the solution of* (1.1) *on* $[0, T]$ *defined by* (3.1). *Then the estimates* (4.2) *and* (4.3) *hold.*

Remark. By restricting consideration to solutions which satisfy a prescribed global bound, one can use the logarithmic convexity of solutions to (1.1) to "stabilize" the final value problem [5, 14].

Finally, we note that the estimates above hold in the stronger norms induced by powers of A. For each integer $p \geqslant 0$, $(I + A)^p$ is a bijection of $D(A^p)$ onto H and the norm $\| x \|_p = \|(I + A)^p\, x\|$ makes $D(A^p)$ a Hilbert space. We have $\| x \|_p \leqslant \| x \|_q$ for all $x \in D(A^q) \subset D(A^p)$, where $0 \leqslant p \leqslant q$. Also, A commutes with each A_α, $S_\alpha(t)$, and $S(t)$, so (4.3) holds with the H-norm replaced by the p-norm. Recall that $S(t)$ maps H into every $D(A^p)$, $p \geqslant 0$, $t > 0$, since S is holomorphic.

COROLLARY. *In the situation of Theorem* 3, *the estimate* (4.3) *holds with the H-norm replaced by the stronger p-norm for every integer $p \geqslant 0$.*

This result is particularly useful when A is a realization of a regular elliptic operator of order $2q$ on $H = L^2(G)$, where G is an open subset of Euclidean n-space with smooth boundary. For then Sobolev's lemma shows that $D(A^p)$ is contained in the space of functions uniformly continuous on G, where p is chosen with $2qp$ as large as the integer part of $n/2$, and (4.3) is then a uniform estimate over the region G. A similar result holds for spatial derivatives of the solution.

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