Distributed Microstructure Models of Porous Media

Ralph E. Showalter

Abstract. Laminar flow through through fissured or otherwise highly inhomogeneous media leads to very singular initial-boundary-value problems for equations with rapidly oscillating coefficients. The limiting case (by homogenization) is a continuous distribution of model cells which represent a valid approximation of the finite (singular) case, and we survey some recent results on the theory of such systems. This is developed as an application of continuous direct sums of Banach spaces which arise rather naturally as the energy or state spaces for the corresponding (stationary) variational or (temporal) dynamic problems. We discuss the basic models for a totally fissured medium, the extension to include secondary flux in partially fissured media, and the classical model systems which are realized as limiting cases of the microstructure models.

1. Introduction.

Fractured or fissured porous media are commonly modelled as a composite material consisting of two components for which the internal flow is described by a pair of partial differential equations, one acting in each of the components, and a coupling that describes the interface between these components. For any region which consists of two finely interspersed materials, one can consider averaged properties of both materials as existing at every point in the region. and this leads to various classes of *double porosity* models. The classical example of this approach is the parabolic system

$$\begin{aligned} &\frac{\partial}{\partial t}(au_1) - \vec{\nabla}\cdot(A\vec{\nabla}u_1) + \frac{1}{\delta}(u_1 - u_2) = f_1 \\ &\frac{\partial}{\partial t}(bu_2) - \vec{\nabla}\cdot(B\vec{\nabla}u_2) + \frac{1}{\delta}(u_2 - u_1) = f_2 \ . \end{aligned} \tag{1}$$

discussed in [7] for which u_1 represents the density of fluid in the first material and u_2 the density in the second. Similarly $a(x)$ and $A(x)$ are porosity and permeability of

the first material, respectively, while $b(x)$ and $B(x)$ are corresponding properties of the second material. The third term is an attempt to quantify the exchange of fluid between the two components. (See [41] for a corresponding heat conduction model.) Since the two components are treated symmetrically, such a double porosity model is said to be of *parallel flow* type. This symmetric treatment of the two components is a real limitation of these classical double porosity models. For a fractured medium such a representation is particularly inappropriate, since the porous and permeable cells within the structure have flow properties radically different from those of the surrounding highly developed system of fissures. Moreover the geometry of the individual cells and the corresponding interface is lost in the averaging process leading to such models. Another class of double porosity models consists of the *distributed microstructure* models. At each point in the region there is given a representative model cell, the flow within each such local cell is described by an initial-boundary-value problem, and the boundary values on the cells are coupled to a single global initial-boundary-value problem which describes the global flow in the region. Thus we have a continuum of partial differential equations to describe the local flow on the micro-scale, and these are coupled to a single partial differential equation for the macro-scale flow. This concept occurred in a heat conduction problem in [34] and has arisen in a variety of applications which we mention below.

We shall illustrate these two classes of double porosity models in each of two models of fissured media. We use both types to describe first the *totally fissured* case in which the cells are individually and completely isolated from each other by the fissure system. In these models the cells act as storage sites only, and there is no direct diffusion from cell to cell, as they are connected only indirectly through the surrounding fissure system. Then we introduce corresponding models for the *partially fissured* case in which there is some fluid flow induced through the cell structure by the pressure gradient in the fissure system. This flow through the cells contributes an additional component to the velocity field in the fissure system which we call the *secondary flux*. Finally, we show the form of the functional differential equations that arise for the global flow when the local problems are eliminated from the system by Green's operator representations.

2. Totally Fissured Media.

A fractured medium consists of an ensemble of small porous and permeable cells which are surrounded by a highly developed system of fractures. The bulk of the

flow occurs in the highly permeable fracture system, and most of the storage of fluid is in the system of cells which accounts for almost all of the total volume. One approach to constructing a model of such a medium is to regard the fissure system as the first component and the cell system as the second component of a *parallel flow* model obtained from (1) by adjusting the coefficients appropriately. In order to specialize the system (1) to a totally fissured medium in which the individual cells are isolated ¿from each other and no direct cell-to-cell flow is possible, one sets $B = 0$. The resulting system of parabolic-ordinary type differential equations

$$\frac{\partial}{\partial t}(au_1) - \vec{\nabla} \cdot (A\vec{\nabla}u_1) + \frac{1}{\delta}(u_1 - u_2) = f \tag{2.a}$$

$$\frac{\partial}{\partial t}(bu_2) + \frac{1}{\delta}(u_2 - u_1) = 0 \tag{2.b}$$

is called the *first-order kinetic* model, since the cell storage is regarded as an added kinetic storage perturbation of the global fracture system. See [1], [13], [51], [11], [33], [21], [14], [9], [17], [32], [46] for applications and mathematical developments of such models.

Two essential limitations of the parallel-flow models are the suppression of the geometry of the cells and their corresponding interfaces on which the coupling occurs and the lack of any distinction between the space and time scales of the two components of the medium. The introduction of *distributed microstructure* models represents an attempt to recognize the geometry and the multiple scales in the problem as well as to better quantify the exchange of fluid across the intricate interface between the components. The global flow in the fracture system is described in the macro-scale x by

$$\frac{\partial}{\partial t}\big(a(x)u(x,t)\big) - \vec{\nabla} \cdot A(x)\vec{\nabla}u + q(x,t) = f(x,t) \ , \qquad x \in \Omega \ , \tag{3.a}$$

where $q(x,t)$ is the exchange term representing the flow into the cell Ω_x. The flow within the local cell Ω_x is described in the micro-scale variable y by

$$\frac{\partial}{\partial t}\big(b(x,y)U(x,y,t)\big) - \vec{\nabla}_y \cdot B(x,y)\vec{\nabla}_y U = F(x,y,t) \ , \qquad y \in \Omega_x \ . \tag{3.b}$$

Because of the smallness of the cells within the global region, the fissure pressure is assumed to be well approximated by the "constant" value $u(x,t)$ at every point of the cell boundary, so the effect of the fissures on the cell pressure is given by the interface condition

$$B(x,s)\vec{\nabla}_y U \cdot \nu + \frac{1}{\delta}(U - u) = 0 \ , \qquad s \in \Gamma_s \ , \tag{3.c}$$

where ν is the unit outward normal on Γ_x. (When $\delta = \infty$, this becomes (and converges to) the "matched" boundary condition, $u(x,t) = U(x,s,t)$ on Γ_x.) Finally, the amount of fluid flux across the interface scaled by the cell size determines the remaining term in (3.a) by

$$q(x,t) = \frac{1}{|\Omega_x|} \int_{\Gamma_x} B(x,s) \vec{\nabla}_y U \cdot \nu \, ds \ , \tag{3.d}$$

where $|\Omega_x|$ denotes the Lebesgue measure on Ω_x, and this contributes to the *cell storage*. Thus, the system (3) comprises a double-porosity model of "distributed microstructure" type for a totally fissured medium; it need be supplemented by appropriate boundary conditions for the global pressure $u(x,t)$ and initial conditions for $u(x,0)$ and $U(x,y,0)$ in order to comprise a well-posed problem. See [34], [40], [39], [15], [50], [8], [20], [5], [6], [2], [4], [15], [16], [18], [19], [22], [23], [24], [25], [26], [27], [28], [30], [29], [37], [48], [47], [49], [45], [44], [43] for applications and mathematical theory for (3) and various related problems.

Finally, we remark that the system (3) can be rewritten as a single equation of functional-differential type. By applying Gauss' theorem to (3.b) we obtain from (3.d)

$$\frac{\partial}{\partial t} \int_{\Omega_x} bU \, dy = \int_{\Gamma_x} B \frac{\partial U}{\partial \nu} \, ds + \int_{\Omega_x} F \, dy$$

Use the Green's function for the problem (3.b) to represent the solution $U(x,y,t)$ as an integral over Γ_x of $u(x,t)$ and substitute this in (3.a) to get the implicit convolution evolution equation

$$\frac{\partial}{\partial t} \left\{ a(x) u(x,t) + \int_0^t k(x, t-\tau) u(x,\tau) \, d\tau \right\} - \vec{\nabla} \cdot A(x) \vec{\nabla} u = f(x,t) \ . \tag{4}$$

The convolution term represents a storage effect with memory. See [31] for a direct treatment and the very recent work of [38], where this equation forms the basis for an independent theoretical and numerical analysis.

3. Partially Fissured Media.

Next we present a model for a partially fissured medium, that is, a fissured medium in which there are some flow paths directly joining the cells in addition to the predominate connection with the surrounding fissure system. Thus the cells are not completely isolated from one another by the fissure system. In this situation one must account for the effect of the gradient of the global flow on the local flow within

the cells, for it is this fissure pressure gradient which necessarily provides the driving force for this cell-to-cell transport. In order to implement this in a model of parallel flow type, we introduce into the first order kinetic model (2) a secondary flux $\vec{u}_3$ through the cell system. This flux is assumed to respond to the fissure pressure gradient with a delay analogous to that of the cell storage in response to the value of fissure pressure in (2). The model system is then of the form

$$\frac{\partial}{\partial t}(au_1) - \vec{\nabla} \cdot (A\vec{\nabla}u_1) + \frac{1}{\delta}(u_1 - u_2) + \vec{\nabla} \cdot C^*\vec{u}_3 = f \tag{5.a}$$

$$\frac{\partial}{\partial t}(bu_2) + \frac{1}{\delta}(u_2 - u_1) = 0 \tag{5.b}$$

$$\frac{\partial}{\partial t}(c\vec{u}_3) + \frac{1}{\beta}(\vec{u}_3 + C\vec{\nabla}u_1) = 0 \tag{5.c}$$

where we assume the responses of the cell structure at a point to the value and to the gradient of fissure pressure are additive, an assumption that is valid for cell structures which are symmetric with respect to coordinate directions. The third and fourth terms in (5.a) give the distributed mass flow rate into the cells from the fissure system at a point. According to (5.b), the first of these goes toward the storage of fluid in the cells. Fluid from the fissure system enters the cell system at a point of higher pressure, it flows through the cell matrix to a point of lower pressure, and then it exits the cell back into the fissure system according to the second of these exchange terms in (5.a). This results in a *secondary flux* $\vec{u}_3$ which follows the fissure pressure gradient according to (5.c). The matrix C^*C arises from the bridging of the cells and it distinguishes the *partially fissured* model (5) from the *fully fissured* model (2). See [42] for a discussion and development of such models with multiple nonlinearities.

In order to obtain a *distributed microstructure* model of a partially fissured medium, we need to recognize that the cell system in (5) responds additively to the value and the gradient of fissure pressure, that is, to the best linear approximation of the fissure pressure at the cell location. As before the global fluid flow in the fissures is described by

$$\frac{\partial}{\partial t}\big(a(x)u(x,t)\big) - \vec{\nabla} \cdot A(x)\vec{\nabla}u + q(x,t) = f(x,t)\ , \qquad x \in \Omega\ , \tag{6.a}$$

and the local flow in the cell at each point x is given by

$$\frac{\partial}{\partial t}\big(b(x,y)U(x,y,t)\big) - \vec{\nabla}_y \cdot B(x,y)\vec{\nabla}_y U = F(x,y,t)\ , \qquad y \in \Omega_x\ . \tag{6.b}$$

Our assumption that the cell pressure on the boundary is driven by the best linear approximation of the fissure pressure leads to the boundary condition

$$B(x,s)\vec{\nabla}_y U\cdot\nu+\frac{1}{\delta}(U-u-\vec{\nabla}u\cdot s)=0\ ,\qquad s\in\Gamma_s\ , \tag{6.c}$$

Finally, the exchange term q in (6.a) consists of two parts, the average amount flowing into the cell to be stored and the divergence of the secondary flux flowing through the cell structure. The total exchange is given by

$$q(x,t)=\frac{1}{|\Omega_x|}\int_{\Gamma_x}B(x,s)\vec{\nabla}_y U\cdot\nu\,ds-\frac{1}{|\Omega_x|}\vec{\nabla}\cdot\left(\int_{\Gamma_x}B(x,y)\vec{\nabla}_y U\cdot\nu s\,ds\right)\ . \tag{6.d}$$

The system (6) comprises the distributed microstructure model for a partially fissured medium. This model was introduced in [10] to describe the highly anisotropic situation in layered media and developed in [12] for more general media. See [3] for a discrete version and numerical work.

When the cells Ω_x are symmetric in coordinate directions, one can separate the effects of storage from those of the secondary flux. Specifically, the storage can then be expressed in terms of the value of the fissure pressure at the point over a time interval through a convolution integral obtained as before from a Green's function representation of the cell problem, and the secondary flux and its corresponding contribution to the global flow are expressed likewise in terms of the global flux. This leads just as before to a functional partial differential equation of the form

$$\begin{aligned}&\frac{\partial}{\partial t}\big(a(x)u(x,t)+k_1(x,\cdot)*u(x,t)\big)\\&-\vec{\nabla}\cdot\left(A(x)\vec{\nabla}u(x,t)+\frac{\partial}{\partial t}k_{12}(x,\cdot)*\vec{\nabla}u(x,t)+k_2(x,\cdot)*\vec{\nabla}u(x,t)\right)\\&\qquad=f(x,t)\ ,\qquad x\in\Omega\ ,\ t>0\ .\end{aligned} \tag{7}$$

which is known as *Nunziato's equation*. This equation was presented in [36] without any physical or philosophical justification as an interesting generalization of heat conduction with memory models due to Gurtin and Chen. See [35] for mathematical development of these equations.

4. Remarks.

The basic distributed microstructure model (3) is obtained as the limit by homogenization of a corresponding exact but highly singular partial differential equation with rapidly oscillating coefficients. This provides not merely another derivation of

the model equations, but shows also the relation with the classical but singular case of a single diffusion equation, and it provides a method for directly computing the coefficients in (3) which necessarily represent averaged material properties. A similar result of convergence of a classical system to (6) is to be expected.

The first order kinetic model (2) is the limiting case of (3) as the permeability coefficient B tends to infinity. In this limit the cell behaves as a single point, equivalently, the function U is independent of the local variable y, and the geometry of the cell is lost. The system (6) converges to (5) likewise as B tends to infinity, and when δ tends to zero in (3) or (6), the limiting problem has the boundary conditions (3.c) or (6.c) replaced by the corresponding "matched" conditions of Dirichlet type.

An interesting open problem is the determination of the coefficients in the system (3) from measurements of data on the boundary of the global region. It would be particularly interesting to obtain information on the cell geometry from such boundary measurements.

The systems (1) and (2) comprise parabolic and degenerate parabolic dynamical systems, respectively, in the product space $L^2(\Omega) \times L^2(\Omega)$. The functional differential equations (4) and (7) lead to dynamical systems in $L^2(\Omega)$, but these are governed by C_0 semigroups without regularizing effects, and the estimates and techniques for these are comparatively difficult. These equations lack the parabolic structure one seeks in such models. However the systems (3) and (6) do retain all of the parabolic structure and corresponding estimates and regularity of classical parabolic systems when they are posed on the spaces $L^2(\Omega) \times L^2(\Omega, L^2(\Omega_x))$.

Experience suggests that the distributed microstructure models are conceptually easy to work with, they provide accurate models which include the fine scales and geometry appropriate for many problems, and their theory can be developed in a straightforward manner using conventional techniques. The numerical analysis of these systems provides a natural application of parallel methods.

References

[1] A. Anzelius, *Über Erwärmung vermittels durchströmender Medien*, Zeit. Ang. Math. Mech., **6** (1926), 291–294.

[2] T. Arbogast, *Analysis of the simulation of single phase flow through a naturally fractured reservoir*, SIAM J. Numer. Anal., **26** (1989), 12–29.

[3] T. Arbogast, *The double porosity model for single phase flow in naturally fractured reservoirs*, Inst. Math. Appl., **316** (1987).

[4] T. Arbogast, *Gravitational forces in dual-porosity models of single phase flow*,

[5] T. Arbogast, J. Douglas, U. Hornung, *Derivation of the double porosity model of single phase flow via homogenization theory*, SIAM J. Math. Anal., **21** (1990), 823–836.

[6] T. Arbogast, J. Douglas, U. Hornung, *Modeling of naturally fractured petroleum reservoirs by formal homogenization techniques*, in "Frontiers in Pure and Applied Mathematics" (R. Dautray, ed.), Elsevier, Amsterdam, 1991, 1–19.

[7] G.I. Barenblatt, I.P. Zheltov, I.N. Kochina, *Basic concepts in the theory of seepage of homogeneous liquids in fissured rocks*, J. Appl. Math. and Mech., **24** (1960), 1286–1303.

[8] J.A. Barker, *Block-geometry functions characterizing transport in densely fissured media*, J. of Hydrology, **77** (1985), 263–279.

[9] E. Charlaix, J.P. Hulin, T.J. Plona, *Experimental study of tracer dispersion in sintered glass porous materials of variable compation*, Phys. Fluids, **30** (1987), 1690–1698.

[10] G.W. Clark and R.E. Showalter, *Fluid flow in a layered medium*, Quarterly Appl. Math., to appear.

[11] K.H. Coats, B.D. Smith, *Dead-end pore volume and dispersion in porous media*, Trans. Soc. Petr. Eng., **231** (1964), 73–84.

[12] J.D. Cook and R.E. Showalter, *Microstructure diffusion models with secondary flux*, preprint.

[13] H.A. Deans, *A mathematical model for dispersion in the direction of flow in porous media*, Trans. Soc. Petr. Eng., **228** (1963), 49–52.

[14] E. DiBenedetto and R.E. Showalter, *A free-boundary problem for a degenerate parabolic system*, Jour. Diff. Eqn., **50** (1983), 1–19.

[15] P.F. Diesler, Jr. and R.H. Wilhelm, *Diffusion in beds of porous solids: measurement by frequency response techniques*, Ind. Eng. Chem., **45** (1953), 1219–1227.

[16] J. Douglas, P.J. Paes Leme, T. Arbogast, T. Schmitt, *Simulation of flow in naturally fractured reservoirs*, in Proceedings, Ninth SPE Symposium on Reservoir Simulation, Society of Petroleum Engineers, Dallas, Texas, 1987, pp.271–279, Paper SPE 16019.

[17] C.J. van Duijn and P. Knabner, *Solute transport through porous media with slow adsorption*, to appear.

[18] A. Friedman, P. Knabner, *A transport model with micro- and macro-structure*,

[19] A. Friedman, A. Tzavaras, *A quasilinear parabolic system arising in modelling of catalytic reactors*, Jour. Diff. Eqns., **70** (1987), 167–196.

[20] M.Th. van Genuchten, F.N. Dalton, *Models for simulating salt movement in*

aggregated field soils, Geoderma, **38** (1986), 165–183.

[21] M.Th. van Genuchten, P.J. Wierenga, *Mass transfer studies in sorbing porous media I. Analytical Solutions*, Soil Sci. Soc. Amer. J., **40** (1976), 473–480.

[22] U. Hornung, *Miscible displacement in porous media influenced by mobile and immobile water*, in "Nonlinear Partial Differential Equations" (P. Fife, P. Bates, eds.), Springer, New York, 1988.

[23] U. Hornung, *Homogenization of Miscible Displacement in Unsaturated Aggregated Soils*, in "Composite Media and Homogenization Theory, ICTP, Triest 1990" (G. Dal Mase, G.F. Dell'Antonio, eds.) Progress in Nonlinear Differential Equations and Their Applications, Birkhäuser, Boston, 1991, 143–153.

[24] U. Hornung, *Miscible displacement in porous media influenced by mobile and immobile water*, Rocky Mtn. Jour. Math., **21.2** (1991), 645–669; corr. 1153–1158.

[25] U. Hornung, *Modellierung von Stofftransport in aggregierten und geklüfften Böden*, Wasserwirtschaft **1** (1992), to appear.

[26] U. Hornung, *Applications of the Homogenization Method to Flow and Transport through Porous Media*, in "Flow and Transport in Porous Media. Summer School, Beijing 1988" (Xiao Shutie, ed.), World Scientific, Singapore, to appear.

[27] U. Hornung and W. Jäger, *A model for chemical reactions in porous media*, in "Complex Chemical Reaction Systems. Mathematical Modeling and Simulation" (J. Warnatz, W. Jäger, eds.), Chemical Physics, **47**, Springer, Berlin, 1987, 318–334.

[28] U. Hornung and W. Jäger, *Diffusion, convection, adsorption, and reaction of chemicals in porous media*, J. Diff. Equations, **92** (1991), 199–225.

[29] U. Hornung and W. Jäger, *Homogenization of Reactive Transport through Porous Media*, in "EQUADIFF 1991" (C. Perelló, ed.), World Scientific Publishing, Singapore, 1992, submitted.

[30] U. Hornung, W. Jäger and A. Mikelić, *Reactive Transport through an Array of Cells with Semi-Permeable Membranes*, in preparation.

[31] U. Hornung and R.E. Showalter, *Diffusion models for fractured media*, Jour. Math. Anal. Appl., **147** (1990), 69–80.

[32] P. Knabner, Mathematishche Modelle für den Transport gelöster Stoffe in sorbierenden porösen Medien, Univ. Augsburg, no.121, 1989.

[33] F.T. Lindstrom, M.N.L. Narasimham, *Mathematical theory of a kinetic model for dispersion of previously distributed chemicals in a sorbing porous medium*, SIAM J. Appl. Math., **24** (1973), 496–510.

[34] A.N. Lowan, *On the problem of the heat recuperator*, Phil. Mag., **17** (1934), 914–933.

[35] R.K. Miller, *An integrodifferential equation for rigid heat conductors with memory*, J. Math. Anal. Appl. **66** (1978), 313–332.

[36] J.W. Nunziato, *On heat conduction in materials with memory*, Quarterly Appl. Math. **29** (1971), 187–204.

[37] L. Packer and R.E. Showalter, *Distributed capacitance microstructure in conductors*, to appear.

[38] M. Peszenska, *Mathematical analysis and numerical approach to flow through fissured media*, Dissertation, Augsburg, May, 1992.

[39] J.B. Rosen, *Kinetics of a fixed bed system for solid diffusion into spherical particles*, J. Chem. Phys., **20** (1952), 387–394.

[40] J.B. Rosen and W.E. Winshe, *The admittance concept in the kinetics of chromatography*, J. Chem. Phys., **18** (1950), 1587–1592.

[41] L.I. Rubinstein, *On the problem of the process of propagation of heat in heterogeneous media*, Izv. Akad. Nauk SSSR, Ser. Geogr., **1** (1948).

[42] J. Rulla and R.E. Showalter, *Diffusion in partially fissured media and implicit evolution equations*, in preparation.

[43] R.E. Showalter, *Implicit evolution equations*, "Differential Equations and Applications," Vol.II, Proc. of International Conf. on Theory and Applications of Differential Equations, Columbus, OH, University Press, Athens, 1989, pp.404–411.

[44] R.E. Showalter, *Diffusion in a Fissured Medium with Micro-Structure*, Free Boundary Problems, Montreal, 1990.

[45] R.E. Showalter, *Diffusion models with micro-structure*, Transport in Porous Media, **6** (1991), 567–580.

[46] R.E. Showalter and N.J. Walkington, *A diffusion system for fluid in fractured media*, Differential and Integral Eqns., **3** (1990), 219–236.

[47] R.E. Showalter and N.J. Walkington, *Micro-structure models of diffusion in fissured media*, Jour. Math. Anal. Appl., **155** (1991), 1–20.

[48] R.E. Showalter and N.J. Walkington, *Elliptic systems for a medium with microstructure*, in "Geometric Inequalities and Convex Bodies," I.J. Bakelman, ed., Marcel Dekker, New York.

[49] R.E. Showalter and N.J. Walkington, *Diffusion of fluid in a fissured medium with micro-structure*, SIAM Jour. Math. Anal., **22** (1991), 1702–1722.

[50] Ch. Vogt, *A homogenization theorem leading to a Volterra integro-differential*

equation for permeation chromotography, preprint #155, SFB **123**, Heidelberg, 1982.

[51] J.E. Warren, P.J. Root, *The behavior of naturally fractured reservoirs*, Soc. Petr. Eng. J., **3** (1963), 245–255.

Author's Address:

Ralph E. Showalter
Department of Mathematics
The University of Texas at Austin
Austin, Texas 78712-1082, USA