

Convergence of Diffusion with Concentrating Capacity

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The diffusion of electric charge on a thin-film dielectric is described by an initial-boundary-value problem for a parabolic partial differential equation on a planar region. Consider the situation where conductivity on a given subregion increases to large values and the corresponding limiting problem has the total capacitance of that subregion all concentrated on the boundary of the complementary region. As the conductivity increases, the convergence of the solutions to that of the limiting problem is established, and convergence rates are obtained. The additional effect of deleting this concentrated capacitance is also estimated. © 1989 Academic Press, Inc.

1. INTRODUCTION

Consider the voltage distribution $u(x_1, x_2, t)$ as a function of time $t > 0$ and position $x = (x_1, x_2)$ in the planar resistive layer Ω of a thin film RC structure shown in Fig. 1. A voltage gradient (= electric field) induces a current \mathbf{J} given by Ohm's law, $\mathbf{J} = -G(x) \nabla u$. The voltage difference across the dielectric layer induces a charge of magnitude $Q = C(x) u$. These two equations define the (distributed) capacitance $C(x)$ of the dielectric layer and the (distributed) conductance $G(x)$ of the resistive layer; these are dependent on the materials and their thickness in the structure.

Let Ω_0 be a part of Ω . Since the sum of the current coming into Ω_0 through its boundary $\partial\Omega_0$ plus any outside charge sources of density $F(x, t)$ is equal to the rate at which charge accumulates in Ω , the conservation of charge requires that

$$-\int_{\partial\Omega_0} \mathbf{J} \cdot \mathbf{n} + \iint_{\Omega_0} F(x, t) dx = \frac{d}{dt} \iint_{\Omega_0} Q dx.$$

If \mathbf{J} is smooth we obtain from the divergence theorem

$$\iint_{\Omega_0} \left(\frac{\partial Q}{\partial t} + \nabla \cdot \mathbf{J} - F \right) dx = 0$$

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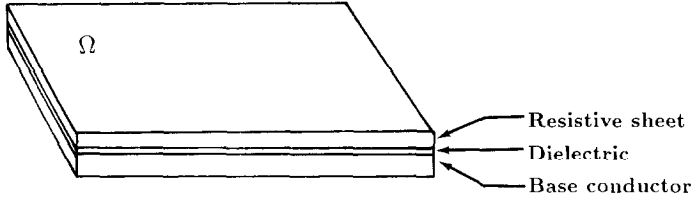


FIGURE 1

for every such $\Omega_0 \subset \Omega$ and, hence,

$$\frac{\partial}{\partial t} (C(x) u(x, t)) - \nabla \cdot G(x) \nabla u(x, t) = F(x, t)$$

as the PDE for the voltage distribution in Ω . Typical boundary conditions on $\partial\Omega$ are to specify at each point either the voltage, $u(s, t)$, or the normal current flow, $\mathbf{J} \cdot \mathbf{n} = -G(x)(\partial u / \partial n)$, \mathbf{n} being the unit outward normal on $\partial\Omega$. Such boundary conditions together with the initial charge distribution $Q_0(x) = C(x) u(x, 0)$ lead to a well-posed initial-boundary-value problem for the PDE above. Finally, we remark that if there is an interior curve S in Ω along which $G(x)$ is not smooth then we have the *interface conditions* on voltage and current,

$$u|_{S^+} = u|_{S^-}, \quad G(x) \frac{\partial u}{\partial n} \Big|_{S^+} = G(x) \frac{\partial u}{\partial n} \Big|_{S^-},$$

where \mathbf{n} is a consistent normal vector on S and S^+ , S^- denote limits from the respective sides of S . Such interfaces arise when two such RC networks are joined in series as indicated in order to fulfill a design requirement for an abrupt transition in conductance or capacitance [7, 8].

In the construction of such networks it is common to place over a portion Ω_0 of the resistive layer Ω a very highly conductive material. For

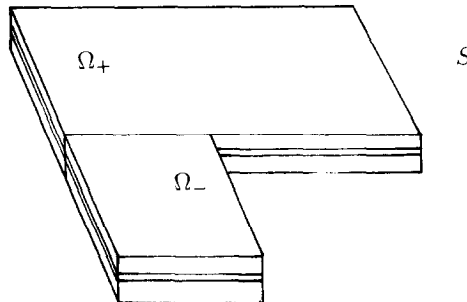


FIGURE 2

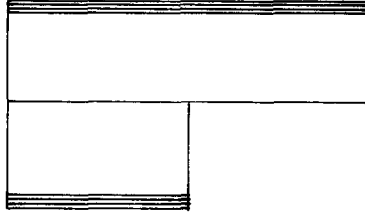


FIGURE 3

example, to fabricate effective contact points at the top and bottom of the network in Fig. 2, highly conductive strips are added as shown in Fig. 3 thereby creating equipotential lines along top and bottom as contact points [8, Chap. 7].

Furthermore, as a second example, note that the analysis of the circuit of Fig. 3 can be significantly simplified if it can be reduced to a problem with one spatial dimension, i.e., $u(x, t) = u(x_2, t)$ is independent of the horizontal displacement x_1 . This symmetry can be essentially achieved by placing a highly conductive strip along the extended interface as indicated in Fig. 4.

In the analysis of the networks that result from the addition of such highly conductive layers it is convenient (and common in practice) to assume that the conductivity of the added layer is infinite and to ignore the capacitance under this layer, all of which is effectively concentrated on a submanifold of lower dimension. Our objective is to study the original initial-boundary-value problem containing a region Ω_0 with a layer of additional conductivity $1/\varepsilon$, $\varepsilon > 0$ small, its approximation by a related limiting problem in which the capacitance of Ω_0 has been concentrated on points corresponding to the components of Ω_0 , and then the additional approximation obtained by deleting this concentrated capacitance. Similar problems in dimension 3 arise in diffusion of a slightly compressible fluid through a region with a singular permeability due to fracturing of the medium [2, 11].

Our plan is to present in Section 2 the stationary case as an example of perturbation for the operator equation $(1/\varepsilon) Au^\varepsilon + Bu^\varepsilon = f^\varepsilon$ in a Hilbert

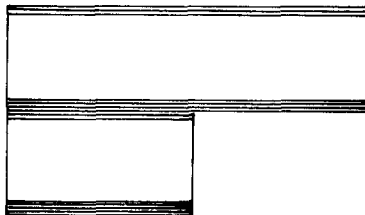


FIGURE 4

space. It will be shown that u^ε converges as $\varepsilon \rightarrow 0$ to the solution u of $Bu=f$ and $Au=0$, and we shall give successively sufficient conditions for weak convergence, strong convergence, and for a linear rate of convergence. The corresponding results for the evolution problem are presented in Section 3. Here there arises a technical difficulty due to the lack of regularity of the solution of the limiting problem. The effect of deleting the concentrated capacity from the evolution problem will be estimated in Section 4.

The presentation in Section 2 will show that the corresponding stationary problems are related to eigenvalue problems in which the eigenvalue parameter appears in the boundary condition. Such problems have been discussed in [5, 6, 11, 12, 13]. Convergence results for both stationary and evolution problems as above were given in [1] as an example of strong resolvent convergence and the associated convergence of the semigroups generated by self-adjoint Sturm–Liouville operators in Hilbert space. As we see below, such problems arise naturally as limits of standard diffusion models with increasing conductivity or permeability on a sub-region.

2. THE STATIONARY PROBLEM

We begin by stating a model boundary-value problem that arises from the situation described in the Introduction. This is stated in an abstract form as a problem for operators in Hilbert space and we describe the convergence results in this context. Then we recover our model problem by making an appropriate choice of Sobolev spaces and operators.

The model problem is given on a bounded domain Ω in \mathbb{R}^2 which is written as a disjoint union $\Omega = \Omega_1 \cup S \cup \Omega_0$ where Ω_1 and Ω_0 are subdomains and $S \subset \partial\Omega_1 \cap \partial\Omega_0$. The case where Ω_0 is not connected is an easy but relevant modification of the discussion. Let $\Gamma_j \subset \partial\Omega_j$ for $j=0, 1$ and assume Γ_0 and Γ_1 do not intersect S . Let $C(x)$ and $G(x)$ be positive real-valued functions on Ω and suppose that for each $\varepsilon > 0$ we are given $F^\varepsilon(x)$, $x \in \Omega$, and $g^\varepsilon(s)$, $s \in \Gamma_0$. We are concerned with the boundary-value problems

$$\lambda C(x) u^\varepsilon(x) - \nabla \cdot G(x) \nabla u^\varepsilon(x) = F^\varepsilon(x), \quad x \in \Omega_1, \quad (2.1a)$$

$$\lambda C(x) u^\varepsilon(x) - \nabla \cdot \left(G(x) + \frac{1}{\varepsilon} \right) \nabla u^\varepsilon(x) = F^\varepsilon(x), \quad x \in \Omega_0, \quad (2.1b)$$

$$u^\varepsilon|_{S_1} = u^\varepsilon|_{S_0}, \quad G \frac{\partial u^\varepsilon}{\partial n} \Big|_{S_1} = \left(G + \frac{1}{\varepsilon} \right) \frac{\partial u^\varepsilon}{\partial n} \Big|_{S_0}, \quad \text{on } S \quad (2.1c)$$

$$u^\varepsilon|_{\Gamma_1} = 0, \quad \frac{\partial u^\varepsilon}{\partial n} = 0 \text{ on } \partial\Omega \sim \Gamma_0 \sim \Gamma_1, \quad \left(G + \frac{1}{\varepsilon} \right) \frac{\partial u^\varepsilon}{\partial n} = g^\varepsilon \text{ on } \Gamma_0, \quad (2.1d)$$

where $\lambda \geq 0$ is specified. The interface condition (2.1c) refers to the values on S obtained as limits from Ω_1 or Ω_0 and denoted by S_1 and S_0 , respectively.

Our intention is to show that the solutions converge to that of the problem

$$\lambda C(x) u(x) - \nabla \cdot G(x) \nabla u(x) = F(x), \quad x \in \Omega_1, \quad (2.2a)$$

$$u(x) = u_0, \quad x \in S, \quad (2.2b)$$

$$\lambda \int_{\Omega_0} C(x) dx u_0 + \int_S G(s) \frac{\partial u}{\partial n} ds = \int_{\Omega_0} F(x) dx + \int_{\Gamma_0} g(s) ds,$$

$$u|_{\Gamma_1} = 0, \quad G \frac{\partial u}{\partial n} = 0 \quad \text{on } \partial\Omega_1 \sim \Gamma_1 \sim S. \quad (2.2c)$$

The non-local boundary condition (2.2b) states that there is a common value u_0 for the voltage distribution on S (due to the very high conductivity on Ω_0) and that the current flux is balanced by a charge on the effective capacitance in Ω_0 all concentrated on S .

For the abstract problem [3, 12] we have a Hilbert space V with norm $\|\cdot\|$ and we denote its dual by V' . Let A and B be continuous linear operators from V into V' and assume that A is symmetric and non-negative and that B is V -coercive. That is, there is a constant $\alpha > 0$ such that

$$Bv(v) \geq \alpha \|v\|^2, \quad v \in V.$$

Denote by V_0 the kernel of A ; since A is symmetric it follows that $V_0 = \{v \in V: Av(v) = 0\}$. Weak and strong convergence will be denoted by \rightharpoonup and \rightarrow , respectively.

For each $\varepsilon > 0$ let $f^\varepsilon \in V'$ be given and consider the problem

$$u^\varepsilon \in V: \left(\frac{1}{\varepsilon} A + B \right) u^\varepsilon = f^\varepsilon \quad \text{in } V'. \quad (2.3)$$

Note that since $(1/\varepsilon) A + B$ is V -coercive there exists a unique solution u^ε of (2.3).

THEOREM 1. (a) Suppose $\{f^\varepsilon\}$ is bounded in V' and $f^\varepsilon \rightharpoonup f_0$ in V'_0 . Then $u^\varepsilon \rightharpoonup u$ in V where u is the unique solution of

$$u \in V_0: Bu(v) = f_0(v), \quad v \in V_0. \quad (2.4)$$

(b) If $f^\varepsilon \rightarrow f$ in V' (and so f_0 is the restriction of f to V_0), then $u^\varepsilon \rightarrow u$ in V .

(c) *If also the range of A , $\text{Rg}(A)$, is closed (and so it equals the annihilator V_0^\perp of V_0 in V') and if $\|f^\varepsilon - f\|_{V'} = \mathcal{O}(\varepsilon)$ then $\|u^\varepsilon - u\| = \mathcal{O}(\varepsilon)$.*

Proof. (a) If we apply (2.3) to u^ε and use the V -coercive property of B there follows the estimate $\alpha\|u^\varepsilon\| \leq \|f^\varepsilon\|$, so $\|u^\varepsilon\|$ is bounded in V . This implies there is a subsequence, which we denote also by $\{u^\varepsilon\}$, such that $u^\varepsilon \rightharpoonup u$ in V . Also, $Au^\varepsilon = \varepsilon(f^\varepsilon - Bu^\varepsilon) \rightarrow 0$ so $Au = 0$ and $Bu^\varepsilon \rightharpoonup Bu$ in V' . From (2.3) follows

$$Bu^\varepsilon(v) = f^\varepsilon(v), \quad v \in V_0,$$

so letting $\varepsilon \rightarrow 0$ leads to (2.4). This solution of (2.4) is unique so it follows that the original sequence u^ε converges weakly to u . Note that the existence and uniqueness of u in (2.4) are a direct consequence of B being V_0 -coercive.

(b) This follows directly from the estimate

$$\alpha\|u^\varepsilon - u\|^2 \leq \left(\frac{1}{\varepsilon} A + B\right)(u^\varepsilon - u)(u^\varepsilon - u) = (f^\varepsilon - Bu)(u^\varepsilon - u)$$

because the last term converges to zero.

(c) From (2.4) we have $f - Bu \in V_0^\perp$, and this is just $\text{Rg}(A)$ so there exists a

$$u_1 \in V: Au_1 = f - Bu.$$

With this we compute the functional

$$\left(\frac{1}{\varepsilon} A + B\right)(u^\varepsilon - u - \varepsilon u_1) = f^\varepsilon - f - \varepsilon Bu_1.$$

This functional is applied to $u^\varepsilon - u - \varepsilon u_1$ to obtain

$$\alpha\|u^\varepsilon - u - \varepsilon u_1\|^2 \leq (\|f^\varepsilon - f\| + \varepsilon\|Bu_1\|) \|u^\varepsilon - u - \varepsilon u_1\|.$$

From here we immediately see

$$\|u_\varepsilon - u\| \leq \varepsilon\|u_1\| + (1/\alpha)(\|f^\varepsilon - f\| + \varepsilon\|Bu_1\|),$$

and this establishes the rate of convergence.

For an example of the application of Theorem 1 we return to the model problem (2.1) above. With the notation introduced there, we let $H^1(\Omega)$ be the Sobolev space [9, 12] of (equivalence classes of) functions v in $L^2(\Omega)$ for which each distribution derivative $\partial_j v = \partial v / \partial x_j$ belongs to $L^2(\Omega)$ for

$j = 1, 2$. Denote by $\nabla v = (\partial_1 v, \partial_2 v)$ the gradient, by \mathbf{n} the unit outward normal on $\partial\Omega$ and on $\partial\Omega_1$, and the corresponding directional derivative by $\partial_n v = \nabla v \cdot \mathbf{n}$. Let Γ_1 be a subset of $\partial\Omega_1$ with positive measure (or capacity) and define V to be that subspace of $H^1(G)$ consisting of those functions whose trace vanishes on Γ_1 . Define $B: V \rightarrow V'$ by

$$Bu(v) = \int_{\Omega} (G(x) \nabla u \cdot \nabla v + \lambda C(x) uv) dx, \quad u, v \in V,$$

where $C, G \in L^\infty(\Omega)$, $C(x) \geq 0$ and $G(x) \geq \alpha_0 > 0$ for $x \in G$, and $\lambda \geq 0$. Note that on the subspace V , $\|\nabla v\|_{L^2(\Omega)}$ and $\|v\|_{H^1(\Omega)}$ are equivalent norms, so B is V -coercive. Next define $A: V \rightarrow V'$ by

$$Au(v) = \int_{\Omega_0} \nabla u \cdot \nabla v dx, \quad u, v \in V,$$

where Ω_0 is the above specified subdomain of Ω . Then the kernel of A is $V_0 = \{v \in V: v_0 = \text{constant}\}$ where v_0 denotes the restriction, $v_0 = v|_{\Omega_0}$, and we see $\text{Rg} A \subset V_0^\perp$, the annihilator of V_0 in V' . In order to apply part (c) of Theorem 1 we check the following.

LEMMA 1. $\text{Rg} A = V_0^\perp$ in V' .

Proof. Let $\mathcal{P}: H^1(\Omega_0) \rightarrow V$ be a continuous and linear extension operator: for each $v \in H^1(\Omega)$ the restriction v_0 satisfies $v_0 = (\mathcal{P}v_0)_0$. Let $f \in V_0^\perp$. Then $f \circ \mathcal{P} \in H^1(\Omega_0)'$ and $f \circ \mathcal{P} \in \{1\}^\perp$ since if $v_0 = 1$ on Ω_0 then $\mathcal{P}v_0 \in V_0$ and $f(\mathcal{P}v_0) = 0$. By the solvability of the Neuman problem on Ω_0 , there exists $u_0 \in H^1(\Omega_0)$: $Au_0(w) = f(\mathcal{P}w)$ for $w \in H^1(\Omega_0)$. For any $v \in H^1(\Omega)$ we have

$$Au_0(v) = Au_0(v_0) = f(\mathcal{P}v_0) = f(v)$$

since $v - \mathcal{P}v_0 \in V_0$. Thus, $A(\mathcal{P}u_0) = f$ in V' so $f \in \text{Rg}(A)$.

The remaining data f^ε in (2.3) is given as follows. Let $F^\varepsilon \in L^2(\Omega)$ and $g^\varepsilon \in L^2(\Gamma_0)$ for each $\varepsilon > 0$, and define

$$f^\varepsilon(v) = \int_{\Omega} F^\varepsilon(x) v(x) dx + \int_{\Gamma_0} g^\varepsilon(s) v(s) ds, \quad v \in V,$$

where $v(s)$ denotes the trace of v in $L^2(\Gamma_0)$. Note that in V_0 we have

$$f^\varepsilon(v) = \int_{\Omega_1} F^\varepsilon(x) v(x) dx + \left(\int_{\Omega_0} F^\varepsilon(x) dx + \int_{\Gamma_0} g^\varepsilon(s) ds \right) v_0, \quad v \in V_0.$$

It is a standard exercise [12] to check that (2.3) and (2.4) are precisely the

weak or variational formulations of the boundary-value problems (2.1) and (2.2), respectively. To apply Theorem 1 we shall assume $\{F^\varepsilon\}$ and $\{g^\varepsilon\}$ are bounded in $L^2(\Omega)$ and $L^2(\Gamma_0)$, respectively.

(a) Assume also that $F^\varepsilon \rightarrow F$ in $L^2(\Omega)$ and that $\int_{\Gamma_0} g^\varepsilon ds \rightarrow \tilde{g}$ in \mathbb{R} . Then $f^\varepsilon \rightarrow f_0$ in V'_0 where

$$f_0(v) = \int_{\Omega_1} F(x) v(x) dx + \left(\int_{\Omega_0} F(x) dx + \tilde{g} \right) v_0, \quad v \in V_0,$$

so we obtain $u^\varepsilon \rightarrow u$ in V , where u is the weak solution of (2.2) with $\int_{\Omega_0} g ds$ replaced by \tilde{g} in (2.2b).

(b) If in addition $F^\varepsilon \rightarrow F$ in $L^2(\Omega)$ and $g^\varepsilon \rightarrow g$ in $L^2(\Gamma_0)$, then $u^\varepsilon \rightarrow u$ in V .

(c) Finally, if $\|F^\varepsilon - F\|_{L^2(\Omega_0)} + \|g^\varepsilon - g\|_{L^2(\Gamma_0)} = \mathcal{O}(\varepsilon)$, then $\|u^\varepsilon - u\|_V = \mathcal{O}(\varepsilon)$.

Remarks. For applications it is the *rate* of convergence which is most useful. The limit problem (2.2) is used to approximate (2.1) with a very small $\varepsilon > 0$, where $1/\varepsilon$ is proportional to the thickness of the added conductive layer.

Nothing in the problem depends on the restriction to dimension 2; it all holds in any dimension, although the intended application is relevant only in the plane. See [2] for applications in higher dimensions.

The "eigenvalue" parameter occurs in the boundary on S in (2.2). This shows explicitly that such a problem is a limit of a "standard" diffusion model (2.1).

3. THE EVOLUTION PROBLEM

Here we study the convergence of the solution of the Cauchy problem

$$u'_\varepsilon(t) + \left(\frac{1}{\varepsilon} A + B \right) u_\varepsilon(t) = f^\varepsilon(t), \quad 0 < t < T, \quad u_\varepsilon(0) = u_0^\varepsilon,$$

as $\varepsilon \rightarrow 0$ where A and B are given as in Section 2. Results are obtained as in Section 2 for weak convergence and for strong convergence. However, we are able to establish the linear rate of convergence only under additional conditions which imply the smoothness of the limiting solution. For background on the solvability of implicit evolution equations that occur in the following, it is sufficient to consult [4, 10, 11, 12].

Let the Hilbert spaces V , V_0 and operators A , B be given as in Section 2. Also let H be a Hilbert space containing V such that the imbedding $V \hookrightarrow H$

is dense and continuous so $H' \subset V'$ by restriction. Let $C: H \rightarrow H'$ be the Riesz isomorphism. For the evolution problem we shall need the Hilbert space $\mathcal{V} = L^2(0, T; V)$ of square Bochner-integrable functions and its dual $\mathcal{V}' \cong L^2(0, T; V')$. We shall use the same notation for $A: V \rightarrow V'$ and its realization $A: \mathcal{V} \rightarrow \mathcal{V}'$ given by $(Av)(t) = A(v(t))$ at a.e. $t \in [0, T]$. For each $\varepsilon > 0$ let $f^\varepsilon \in \mathcal{V}'$ and $u_0^\varepsilon \in H$. Since $(1/\varepsilon)A + B$ is V -coercive, there exists a unique solution u_ε of

$$u_\varepsilon \in \mathcal{V}: \frac{d}{dt}(Cu_\varepsilon) + \left(\frac{1}{\varepsilon}A + B\right)u_\varepsilon = f^\varepsilon \quad \text{in } \mathcal{V}', \quad u_\varepsilon(0) = u_0^\varepsilon. \quad (3.1)$$

This can be found in [11]; also see [4] for related results. The sense in which the initial value is attained in (3.1) should be noted. Specifically, let W be the image of V under the operator C ; it is a Hilbert space with the norm $\|w\|_W = \|C^{-1}v\|_V$ and $C: V \rightarrow W$ is an isomorphism. It has a continuous dual $C': W' \rightarrow V'$ for which

$$\langle C'h, v \rangle = \langle h, Cv \rangle = \langle Ch, v \rangle = (h, v)_H, \quad h \in H, \quad v \in V.$$

It follows that C' is an extension of $C: H \rightarrow H'$ and that for each $u \in H'$

$$\sup\{ |(u, w)_H| : \|w\|_W \leq 1 \} = \sup\{ |\langle u, v \rangle| : \|v\|_V \leq 1 \} = \|u\|_{V'}$$

so H' is the "pivot" space between W and V' . That is, $|(u, w)_H| \leq \|u\|_{V'} \|w\|_W$, $u \in H'$, $w \in W$. This implies that for a solution u of (3.1), for which we have $Cu \in L^2(0, T; W)$ and $(d/dt)(Cu) \in L^2(0, T; V')$, it follows [3, 9] that Cu is continuous from $[0, T]$ into H' . This is equivalent to $u: [0, T] \rightarrow H$ being continuous, so the initial condition in (3.1) is meaningful.

We state our first results for (3.1) as follows.

THEOREM 2. (a) Suppose $\{f^\varepsilon\}$ is bounded in \mathcal{V}' and $f^\varepsilon \rightarrow f_0$ in $\mathcal{V}'_0 = L^2(0, T; V'_0)$, and that $\{u_0^\varepsilon\}$ is bounded in H with $u_0^\varepsilon \rightarrow u_0$ in H_0 , where H_0 is the closure in H of V_0 . Then $u_\varepsilon \rightarrow u$ in \mathcal{V} , where u is the unique solution of

$$u \in \mathcal{V}_0 = L^2(0, T; V_0): \frac{d}{dt}(C_0 u) + Bu = f_0 \quad \text{in } \mathcal{V}', \quad u(0) = u_0, \quad (3.2)$$

and C_0 is the Riesz isomorphism of H_0 onto H'_0 .

(b) If $f^\varepsilon \rightarrow f$ in \mathcal{V}' and $u_0^\varepsilon \rightarrow u_0$ in H , then $f|_{\mathcal{V}'_0} = f_0$, $u_\varepsilon \rightarrow u$ in \mathcal{V} , and $u_\varepsilon(t) \rightarrow u(t)$ in H at every $t \in [0, T]$.

Proof. We shall identify H with H' and H_0 with H'_0 by way of C and C_0

in the following; hence, we replace C and C_0 by the identity for the proof. See below for additional remarks.

(a) Apply (3.1) to u_ε and thereby obtain bounds on $\|u_\varepsilon\|_{\mathcal{V}}$, $\|u_\varepsilon\|_{L^\infty(0, T; H)}$, $\|Bu_\varepsilon\|_{\mathcal{V}'}$, $\|u'_\varepsilon\|_{\mathcal{V}'_0}$, and $(1/\varepsilon) \int_0^T Au_\varepsilon(u_\varepsilon) dt$. Thus, there is a subsequence, denoted again by $\{u_\varepsilon\}$, such that $u_\varepsilon \rightharpoonup u$ in \mathcal{V} , $Bu_\varepsilon \rightharpoonup Bu$ in \mathcal{V}' , $u_\varepsilon(T) \rightharpoonup u^T$ in H , and $u'_\varepsilon \rightharpoonup u'$ in \mathcal{V}'_0 , but *not* in \mathcal{V}' . By lower semi-continuity we obtain $\int_0^T Au(u) ds = 0$ so there follows

$$u \in \mathcal{V}'_0: u' + Bu = f_0 \quad \text{in } \mathcal{V}'_0.$$

For each $v \in C^1(0, T; V_0)$ we have

$$(u_\varepsilon(T), v(T))_H - (u_\varepsilon(0), v(0))_H = \int_0^T u'_\varepsilon(s)(v(s)) ds + \int_0^T v'(s)(u_\varepsilon(s)) ds,$$

so letting $\varepsilon \rightarrow 0$ yields

$$\begin{aligned} (u^T, v(T))_H - (u_0, v(0))_H &= \int_0^T (u'(v) + v'(u)) ds \\ &= (u(T), v(T))_H - (u(0), v(0))_H \end{aligned}$$

for all such v . This implies that $u(0) = u_0$ in H_0 and $u(T) = \text{Proj}_{H_0}(u^T)$, the indicated projection of H onto H_0 . This shows u is the (unique) solution of (3.2) and, hence, the original sequence converges weakly to u in \mathcal{V} .

(b) First we verify

$$\begin{aligned} &\int_0^T \left(\frac{1}{\varepsilon} A + B \right) (u_\varepsilon - u)(u_\varepsilon - u) \\ &= \int_0^T (f^\varepsilon - Bu - u'_\varepsilon)(u_\varepsilon - u) \\ &= \int_0^T (f^\varepsilon - Bu)(u_\varepsilon - u) + \int_0^T u'_\varepsilon(u) + \left(\frac{1}{2}\right)|u_\varepsilon(0)|_H^2 - \left(\frac{1}{2}\right)|u_\varepsilon(T)|_H^2. \end{aligned}$$

Note that $u_\varepsilon - u \rightharpoonup 0$ in \mathcal{V}' , $u'_\varepsilon \rightharpoonup u'$ in \mathcal{V}'_0 , $u_\varepsilon(0) \rightarrow u(0)$ in H , and $|u(T)|_H \leq |u^T|_H \leq \liminf |u_\varepsilon(T)|_H$ by weak lower semi-continuity, so there follows

$$\limsup \int_0^T \left(\frac{1}{\varepsilon} A + B \right) (u_\varepsilon - u)(u_\varepsilon - u) \leq \int_0^T u'(u) + \left(\frac{1}{2}\right)|u(0)|_H^2 - \left(\frac{1}{2}\right)|u(T)|_H^2 = 0.$$

Since B is coercive this shows $u_\varepsilon \rightarrow u$ in \mathcal{V} as desired. Moreover, this shows

$\lim_{\varepsilon \rightarrow 0} \frac{1}{2} |u_\varepsilon(T)|_H^2 = \frac{1}{2} |u(T)|_H^2$ so we obtain $u_\varepsilon(T) \rightarrow u(T)$ in H . This holds on every interval $[0, t]$, $0 < t \leq T$, so $u_\varepsilon(t) \rightarrow u(t)$ in H for all $t \in [0, T]$.

In order to obtain a *rate* estimate on the convergence we consider the following sufficient condition on the limiting *solution*.

LEMMA 2. Assume $u' \in \mathcal{V}'$ and that there exists a

$$u_1 \in \mathcal{V} : u'_1 \in \mathcal{V}' \quad \text{and} \quad Au_1 = f - Bu - u' \quad \text{in } \mathcal{V}'. \quad (3.3)$$

Then it follows that

$$\begin{aligned} & |u_\varepsilon(T) - u(T) - \varepsilon u_1(T)|_H^2 + \alpha \|u_\varepsilon - u - \varepsilon u_1\|_{\mathcal{V}}^2 \\ & \leq (|u_0^\varepsilon - u_0|_H + \varepsilon |u_1(0)|_H)^2 + \frac{1}{\alpha} (\|f^\varepsilon - f\|_{\mathcal{V}'} + \varepsilon \|u'_1 + Bu_1\|_{\mathcal{V}'})^2. \end{aligned} \quad (3.4)$$

Proof. First we compute in \mathcal{V}' that

$$(u'_\varepsilon - u' - \varepsilon u'_1) + \left(\frac{1}{\varepsilon} A + B \right) (u_\varepsilon - u - \varepsilon u_1) = f^\varepsilon - f - \varepsilon (u'_1 + Bu_1).$$

Then we apply this to $u_\varepsilon - u - \varepsilon u_1$ and integrate to obtain

$$\begin{aligned} & \frac{1}{2} |u_\varepsilon(T) - u(T) - \varepsilon u_1(T)|_H^2 + \alpha \|u_\varepsilon - u - \varepsilon u_1\|_{\mathcal{V}}^2 \\ & \leq \frac{1}{2} (|u_0^\varepsilon - u_0|_H + \varepsilon |u_1(0)|_H)^2 \\ & \quad + (\|f^\varepsilon - f\|_{\mathcal{V}'} + \varepsilon \|u'_1 + Bu_1\|_{\mathcal{V}'}) \|u_\varepsilon - u - \varepsilon u_1\|_{\mathcal{V}}. \end{aligned}$$

Finally we give conditions on the *data* which lead to the preceding conditions on the solution.

THEOREM 2(c). In addition to the assumptions in (b), we assume $\text{Rg}(A)$ is closed in V' , $f_0, f'_0 \in L^2(0, T; H_0)$, $u_0 \in V_0$, and that $f_0(0) - Bu_0 \in V_0$. Then (3.4) holds, so if $\|f^\varepsilon - f\|_{\mathcal{V}'} + |u_0^\varepsilon - u_0|_H = \mathcal{O}(\varepsilon)$, then

$$\sup_{0 \leq t \leq T} |u_\varepsilon - u|_H + \|u_\varepsilon - u\|_{\mathcal{V}} = \mathcal{O}(\varepsilon).$$

Proof. There is a unique

$$w \in \mathcal{V}_0 : w' + Bw = f'_0 \text{ in } L^2(0, T; H_0), \quad w(0) = f_0(0) - Bu_0 \in V_0.$$

Set $u(t) \equiv u_0 + \int_0^t w(s) ds$; then integrate the equation for w to obtain

$$w(t) + B \int_0^t w(s) ds = f_0(t) - f_0(0) - w(0).$$

It follows that

$$u, u' \in \mathcal{V}_0: u' + Bu = f_0 \text{ in } L^2(0, T; H_0), \quad u(0) = u_0,$$

and this is the same u that is the solution of (3.2). It follows that $u'' \in L^2(0, T; H_0)$, $Bu' \in L^2(0, T; H_0)$, and that

$$\frac{d}{dt}(f_0 - Bu - u') \in L^2(0, T; H_0) \cap L^2(0, T; V_0^\perp) \subset L^2(0, T; V')$$

since $H_0 \subset H \hookrightarrow V'$. Finally, $A: V \rightarrow V'$ factors into an isomorphism of V/V_0 onto V_0^\perp , so (3.3) follows with $u_1, u'_1 \in \mathcal{V}$; hence, $u'_1 \in \mathcal{V}'$.

Remarks. The Riesz isomorphism $C: H \rightarrow H'$ is determined by the scalar product on H . Each closed subspace H_0 of H determines a decomposition of $H = H_0 \oplus H_0^\perp$, where H_0^\perp is the orthogonal complement, and a corresponding decomposition of $C = C_0 P_0 + C_\perp (I - P_0)$, where $P_0: H \rightarrow H_0$ is the projection, C_\perp is the restriction of C to H_0^\perp , and $C_0 h$ is the restriction of Ch to H_0 for each $h \in H_0$. Thus $C_0: H_0 \hookrightarrow H'_0$ is the Riesz isomorphism of H_0 and C_\perp is an isomorphism of the orthogonal complement H_0^\perp in H onto the annihilator H_0^\perp of H_0 in H' . Conversely, from such a decomposition of C we recover H_0 as the kernel of the second operator.

4. PERTURBATION OF CAPACITY

In this final part we show the effect of deleting altogether the concentrated capacity from the limiting problem (3.2). Specifically, the contribution to the solution decreases linearly with the total perturbation in concentrated capacity. The result of Theorem 3 is similar to known results on singular perturbation of evolution equations; the situation below requires special regularity conditions on the solution of the limiting problem in order to obtain the *rate* estimates. Then we illustrate the implications of Section 3 and Theorem 3 with an initial-boundary-value problem corresponding to the model problem of Section 2.

For our last result we consider the effect of perturbing a part of the leading operator C_0 in the Cauchy problem (3.2). Thus, we set $C_0 = C_1 + \delta C_2$, $\delta > 0$, and let $\delta \rightarrow 0$.

THEOREM 3. *Let V_0 be a Hilbert space and $B: V_0 \rightarrow V'_0$ be continuous, linear, and coercive. Let C_1 and C_2 be continuous, linear, symmetric, and non-negative from V_0 to V'_0 ; denote by V_1 the space V_0 with the semi-scalar*

product $C_1 u(v)$. Assume $f_0 \in \mathcal{V}_0'$ and $u_0 \in V_0$ are given with $f_0' \in \mathcal{V}_0'$ and $Bu_0 - f_0(0) \in V_1'$. Then for each $\delta > 0$ the solution of

$$\begin{aligned} u_\delta \in \mathcal{V}_0: \frac{d}{dt} (C_1 + \delta C_2) u_\delta + Bu_\delta &= f_0 \text{ in } \mathcal{V}_0', \\ (C_1 + \delta C_2) u_\delta(0) &= (C_1 + \delta C_2) u_0 \end{aligned} \quad (4.1)$$

and the solution of

$$u \in \mathcal{V}_0: \frac{d}{dt} C_1 u + Bu = f_0 \text{ in } \mathcal{V}_0', \quad C_1 u(0) = C_1 u_0, \quad (4.2)$$

satisfy the estimates

$$\|u_\delta - u\|_{\mathcal{V}_0} = \mathcal{O}(\delta), \quad (4.3a)$$

$$\sup_{0 \leq t \leq T} \langle C_1(u_\delta(t) - u(t)), u_\delta(t) - u(t) \rangle^{1/2} = \mathcal{O}(\delta), \quad (4.3b)$$

$$\sup_{0 \leq t \leq T} \langle C_2(u_\delta(t) - u(t)), u_\delta(t) - u(t) \rangle^{1/2} = \mathcal{O}(\delta^{1/2}). \quad (4.3c)$$

Proof. The existence and uniqueness of solutions of (4.1) and (4.2) follow from [11] under more general conditions on the data f_0 and u_0 . However, we need the solution of (4.2) to satisfy the additional conditions

$$\frac{d}{dt} (C_2 u) \in \mathcal{V}_0', \quad C_2 u(0) = C_2 u_0. \quad (4.4)$$

In this case we subtract (4.2) from (4.1), apply this difference to $u_\delta - u$, and integrate to obtain for $0 < t \leq T$

$$\begin{aligned} & \frac{1}{2} \langle (C_1 + \delta C_2)(u_\delta - u), u_\delta - u \rangle + \int_0^t B(u_\delta - u)(u_\delta - u) \, ds \\ & \leq \delta \int_0^t |\langle (C_2 u)', u_\delta - u \rangle| \, ds \\ & \leq \left(\frac{\delta}{2}\right) \left(\frac{\delta}{\alpha} \|(C_2 u)'\|_{\mathcal{V}_0'}^2 + \frac{\alpha}{\delta} \|u_\delta - u\|_{\mathcal{V}_0}^2\right). \end{aligned}$$

Since B is V_0 -coercive this leads directly to

$$\langle (C_1 + \delta C_2)(u_\delta - u), u_\delta - u \rangle(T) + \alpha \|u_\delta - u\|_{\mathcal{V}_0}^2 \leq \delta^2 / \alpha \|(C_2 u)'\|_{\mathcal{V}_0'}^2$$

and, hence, the estimates (4.3).

It remains to verify (4.4) for the solution of (4.2). Since $f(0) - Bu_0 \in V'_1$, there is a solution of

$$v \in \mathcal{V}_0: \frac{d}{dt} (C_1 v) + Bv = f'_0 \text{ in } \mathcal{V}'_0, \quad C_1 v(0) = f_0(0) - Bu_0. \quad (4.5)$$

Set $u(t) \equiv u_0 + \int_0^t v$ and integrate (4.5) to obtain

$$C_1 v(t) + B(u(t) - u_0) = f_0(t) - f_0(0) + C_1 v(0).$$

Then it is clear that $v = u' \in \mathcal{V}_0$ so u is the solution of (4.2) and in addition satisfies (4.4) as required.

We shall show by an example how our abstract results in Theorem 2 and Theorem 3 apply to an initial-boundary-value problem which describes the diffusion models considered above. As in Section 2, assume we are given Ω and the functions $G(x)$, $C(x)$, but also assume $C(x) \geq \alpha > 0$, $x \in \Omega$, and set $H = L^2(\Omega)$. The spaces V , V_0 and the operator A are given as before; then we have $H_0 = L^2(\Omega_1) \oplus \mathbb{R}$, a subspace of $L^2(\Omega_1) \oplus L^2(\Omega_0) = H$. The scalar product and Riesz map of H are given by

$$Cu(v) = \int_{\Omega} C(x) u(x) v(x) dx, \quad u, v \in H,$$

and we define $B: V \rightarrow V'$ by

$$Bu(v) = \int_{\Omega} G(x) \nabla u(x) \cdot \nabla v(x) dx, \quad u, v \in V.$$

For each $\varepsilon > 0$ let there be given

$$u_0^\varepsilon \in L^2(\Omega), \quad F^\varepsilon \in L^2(\Omega \times (0, T)), \quad g^\varepsilon \in L^2(\Gamma_0 \times (0, T)),$$

with corresponding bounds independent of ε , and define $f^\varepsilon \in \mathcal{V}'$ by

$$f^\varepsilon(t)(v) = \int_{\Omega} F^\varepsilon(x, t) v(x) dx + \int_{\Gamma_0} g^\varepsilon(s, t) v(s) ds, \quad v \in V, t \in [0, T].$$

Then the Cauchy problem (3.1) is just a generalized formulation [3, 12] of the problem

$$\frac{\partial}{\partial t} (Cu_\varepsilon) - \nabla \cdot (G \nabla u_\varepsilon) = F^\varepsilon \quad \text{in } \Omega_1 \times (0, T), \quad (4.6a)$$

$$\frac{\partial}{\partial t} (Cu_\varepsilon) - \nabla \cdot \left(\left(G + \frac{1}{\varepsilon} \right) \nabla u_\varepsilon \right) = F^\varepsilon \quad \text{in } \Omega_0 \times (0, T), \quad (4.6b)$$

$$u_\varepsilon|_{S_1} = u_\varepsilon|_{S_0}, \quad G \frac{\partial u_\varepsilon}{\partial n} \Big|_{S_1} = \left(G + \frac{1}{\varepsilon} \right) \frac{\partial u_\varepsilon}{\partial n} \Big|_{S_0} \quad \text{on } S \times (0, T), \quad (4.6c)$$

$$u_\varepsilon|_{\Gamma_1} = 0, \quad \frac{\partial u_\varepsilon}{\partial n} \Big|_{\partial\Omega \sim \Gamma_1 \sim \Gamma_0} = 0, \quad \left(G + \frac{1}{\varepsilon} \right) \frac{\partial u_\varepsilon}{\partial n} \Big|_{\Gamma_0} = g^\varepsilon \quad \text{on } (0, T), \quad (4.6d)$$

$$u_\varepsilon(\cdot, 0) = u_0^\varepsilon \quad \text{on } \Omega. \quad (4.6e)$$

Note that the restrictions of $f^\varepsilon(t)$ to V_0 are given by

$$\begin{aligned} f^\varepsilon(t)(v) &= \int_{\Omega_1} F^\varepsilon(x, t) v(x) dx \\ &\quad + \left(\int_{\Omega_0} F^\varepsilon(x, t) dx + \int_{\Gamma_0} g^\varepsilon(s, t) ds \right) v_0, \quad v \in V_0, \end{aligned}$$

where v_0 is the (constant) value of v on Ω_0 . Assume $F^\varepsilon \rightarrow F$ in $L^2(\Omega \times (0, T))$, $\int_{\Gamma_0} g^\varepsilon(s, \cdot) ds \rightarrow \tilde{g}(\cdot)$ in $L^2(0, T)$, $u_0^\varepsilon \rightarrow u_0$ in $L^2(\Omega_1)$, and $\int_{\Omega_0} u_0^\varepsilon dx \rightarrow \tilde{u}$ in \mathbb{R} . Then Theorem 2.a implies that $u_\varepsilon \rightarrow u$ in $\mathcal{V} = L^2(0, T; V)$ where u is the solution of (3.2) with

$$f_0(t)(v) = \int_{\Omega_1} F(x, t) v(x) dx + \left(\int_{\Omega_0} F(x, t) dx + \tilde{g}(t) \right) v_0, \quad v \in V_0.$$

The Cauchy problem (3.2) is just the generalized form [3, 12] of the problem

$$\frac{\partial}{\partial t} (Cu) - \nabla \cdot (G \nabla u) = F \quad \text{in } \Omega_1 \times (0, T), \quad (4.7a)$$

$$u|_S = u_0(t), \quad \int_{\Omega_0} C(x) dx u'_0(t) + \int_S G \frac{\partial u(t)}{\partial n} ds = \int_{\Omega_0} F(x, t) + \tilde{g}(t), \quad (4.7b)$$

$$u|_{\Gamma_1} = 0, \quad G \frac{\partial u}{\partial n} = 0 \quad \text{on } \partial\Omega_1 \sim \Gamma_1 \sim S \quad \text{on } (0, T), \quad (4.7c)$$

$$u(\cdot, 0) = u_0 \quad \text{on } \Omega_1, \quad u_0(0) = \tilde{u} / \int_{\Omega_0} C(x) dx. \quad (4.7d)$$

If we assume $F^\varepsilon \rightarrow F$ in $L^2(\Omega \times (0, T))$, $g^\varepsilon \rightarrow g$ in $L^2(\Gamma_0 \times (0, T))$, and $u_0^\varepsilon \rightarrow u_0$ in $L^2(\Omega)$, so then $\tilde{g}(t) = \int_{\Gamma_0} g(s, t) ds$ and $\tilde{u} = \int_{\Gamma_0} u_0(x) dx$ above, then it follows by Theorem 2.b that we have strong convergence $u_\varepsilon \rightarrow u$ in \mathcal{V} and $u_\varepsilon(t) \rightarrow u(t)$ in $L^2(\Omega)$ at every $t \in [0, T]$. If in addition we assume the limiting data satisfy $\partial F / \partial t \in L^2(\Omega \times (0, T))$, $\tilde{g}' \in L^2(0, T)$, and

$$u_0 \in V_0: Bu_0 \in f_0(0) + V_0 \quad (4.8)$$

then from Theorem 2.c it follows that linear rate assumptions on the convergence of F^ε , g^ε , and u_0^ε will imply a linear rate of convergence estimate on u_ε in the corresponding spaces above. Thus, Theorem 2 describes the approximation of (4.6) by (4.7).

Likewise we can apply Theorem 3 to describe the effect of deleting the time-derivative term in the boundary condition (4.7b). This term is due to the concentrated capacity $\int_{\Omega_0} C(x) dx$ on the boundary S . Define

$$C_1 u(v) = \int_{\Omega_1} C(x) u(x) v(x) dx, \quad C_2 u(v) = \int_{\Omega_0} C(x) u(x) v(x), \quad u, v \in V_0,$$

so that $V'_1 = L^2(\Omega_1)$. For $0 < \delta \leq 1$ problem (4.1) corresponds to problem (3.2) and hence to (4.7) with $C(x)$, $x \in \Omega_0$, replaced by $\delta C(x)$, $x \in \Omega_0$. Likewise, problem (4.2) corresponds to (4.7) with $C(x)$, $x \in \Omega_0$, replaced by zero. The conditions of Theorem 3 are already met with the assumptions above so it follows that estimates (4.3) are obtained. This shows explicitly the dependence of the solution of (4.7) on the magnitude of the concentrated capacity.

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