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Micro-Structure Models of Diffusion in Fissured Media*

R. E. SHOWALTER AND N. J. WALKINGTON

Department of Mathematics, The University of Texas, Austin, Texas 78712

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Two diffusion models are developed which recognize the local geometry of the individual cells or storage sites and the exchange of flux on the micro-scale of these cells. The Cauchy problems for these model systems are shown to be resolved by holomorphic semigroups, and various classical models are obtained as limits of these disributed micro-structure models.

1. FISSURED MEDIUM MODELS

We shall begin with a review of some classical models of diffusion through composite media, especially the case of a fissured medium, in which the double-porosity concept is useful. Then we introduce two models which describe also the micro-structure of the medium, the geometry of the individual cells or pores in the medium, and the flux across the intricate interface which separates them from the global medium or matrix structure in which they are imbedded. In Section 2 we shall make precise these two micro-structure diffusion models in a variational formulation which arises naturally from the formal classical setting presented here. In Section 3 we show that these Cauchy problems in Hilbert space are well-posed, they are resolved by holomorphic semigroups in Hilbert space, and the solutions vary smoothly with the parameters and coefficients. Certain singular limits are particularly relevant in order to relate the micro-structure models to the simpler classical and first-order kinetic models. Although various generalizations are within easy reach of our results, for example, certain extensions to nonlinear or convection-dominated equations, we have chosen to present our results here within a simple but relevant setting in order to enhance the exposition and to emphasize the concept of a

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distributed micro-structure. This concept has a long but disjointed history, and we believe it to be very effective and useful.

A fissured medium consists of a matrix of porous and permeable blocks or cells which are separated from each other by a highly developed system of fissures or bulk flow paths through which the majority of diffusion occurs. In the simplest homogeneous model one introduces the (locally averaged) experimental flow characteristics of the composite medium. For a fluid with *density* u and *pressure* p which is assumed to be slightly compressible, hence, the state equation $u = u_0 \exp(cp) = S(p)$ holds, we have the Darcy law for the *flux*,

$$\mathbf{J} = -(k/\mu)u\,\nabla p = -(k/c\mu)\,\nabla u.$$

Here k is the *permeability* of the medium and μ is *viscosity* of the fluid. The conservation of fluid mass then yields the *classical diffusion equation*

$$\frac{\partial}{\partial t}(mu) - \nabla \cdot \frac{k}{c\mu}\nabla u = f \tag{1.1}$$

in which m is effective *porosity* and f is the distributed source.

For fluid flow through a general heterogeneous medium consisting of two components, such as a matrix of cells and fissures, one can consider *double-porosity* models. The idea is to introduce at each point in space *two* densities u_1, u_2, two pressures p_1, p_2 (etc.), each obtained by averaging in the respective medium over a generic neighborhood sufficiently large to include many cells. When one assumes the flow rate of fluid exchange between the two components is proportional to the product of the pressure difference and the average density over that pressure interval,

$$\rho = \frac{1}{p_2 - p_1}\int_{p_1}^{p_2} S(p)\,dp = \frac{1}{p_2 - p_1}\frac{1}{c}(u_2 - u_1),$$

then, with an obvious change of notation we obtain the parabolic system

$$\frac{\partial}{\partial t}(au_1) - \nabla \cdot A\,\nabla u_1 + \frac{1}{\delta}(u_1 - u_2) = f \tag{1.2.a}$$

$$\frac{\partial}{\partial t}(bu_2) - \nabla \cdot B\,\nabla u_2 + \frac{1}{\delta}(u_2 - u_1) = 0 \tag{1.2.b}$$

as the *double-porosity parallel model* for flow in a general two-component medium.

The system (1.2) can be modified to reflect the special characteristics of a fully fissured medium. The predominant characteristic is that the cells are

isolated from one another by the fissures, so there is no direct cell-to-cell flow. In order to model this phenomenon, we set $B=0$ in (1.2) and obtain thereby the *first-order kinetic model*

$$\frac{\partial}{\partial t}(au_1) - \nabla \cdot (A \nabla u_1) + \frac{1}{\delta}(u_1 - u_2) = f \tag{1.3.a}$$

$$\frac{\partial}{\partial t}(bu_2) + \frac{1}{\delta}(u_2 - u_1) = 0. \tag{1.3.b}$$

A second feature of fissured media is that the fissures occupy a much smaller volume than the blocks, so that $a \ll b$ in (1.3). In particular, the first term, or rate at which fluid is stored in the fissures, is usually not a significant contribution to the equation, so we set $a=0$ in (1.3.a) to obtain the *fissured medium equation*

$$\frac{\partial}{\partial t} b(u_1 - \delta \nabla \cdot A \nabla u_1 - \delta f) - \nabla \cdot (A \nabla u_1) = f \tag{1.4}$$

by formally eliminating u_2.

The double-porosity parallel models described above are based on the assumption that the exchange flux, q, has a spatially distributed density proportional to a pressure difference, and they thereby provide a rather simplistic approximation to the true dynamics of the flux exchange. Next we develop a pair of double-porosity models which recognize the local geometry of the cell matrix, or a generic *cell model* at each point in the macro-scale of the fissure system and thereby reflect more accurately the flux exchange on the micro-scale of the individuals cells. This will be achieved by a mixed coupling whereby fissures affect cells through their boundaries on the micro-scale of the true interface, but cell effects are distributed on the macro-scale of the fissures. This partial-averaging is justified by the smallness of the cells and the higher diffusion rates in the surrounding fissures. One thereby expects an essentially uniform pressure applied by the fissures to the boundary of an individual cell, and it will be necessary to use two spatial scales to implement this construction.

Now we describe our first *distributed micro-structure model.* This consists of a flow region Ω on which is prescribed a continuous distribution of (individually isolated) cells. Thus let Ω be a bounded domain in $\mathbb{R}^n$ with boundary $\Gamma = \partial\Omega$. As before, we let $u(x, t)$ denote the density in the fissure system at each $x \in \Omega$ and time $t > 0$. This fissure density satisfies the diffusion equation

$$\frac{\partial}{\partial t}(a(x)u) - \nabla_x \cdot (A(x) \nabla_x u) + q(x, t) = f(x, t), \qquad x \in \Omega \tag{1.5.a}$$

and appropriate boundary conditions on Γ where $q(x, t)$ is the density of the mass flow of fluid from the fissure system into the cell at x. For each such $x \in \Omega$, let there be given a cell Ω_x, a bounded domain in $\mathbb{R}^n$ with smooth boundary, $\Gamma_x = \partial\Omega_x$. For a function w on Ω_x we denote by $\gamma_x w = w|_{\Gamma_x}$ its restriction or *trace* on Γ_x. The fluid density in the cell Ω_x is given by $U(x, z, t)$, $z \in \Omega_x$, and this function satisfies a local diffusion equation

$$\frac{\partial}{\partial t}(b(x)U) - \nabla_z \cdot (B(x)\nabla_z U) = 0, \qquad z \in \Omega_x, \tag{1.5.b}$$

and boundary condition

$$\gamma_x U(x, s, t) = u(x, t), \qquad s \in \Gamma_x \tag{1.5.c}$$

for each $x \in \Omega$ and $t > 0$. The total flux that enters the cell is given by

$$q(x, t) = B(x) \int_{\Gamma_x} \frac{\partial U(x, s, t)}{\partial \nu} \, ds = b(x) \frac{\partial}{\partial t} \int_{\Omega_x} U(x, z, t) \, dz \tag{1.5.d}$$

according to (1.5.b) and Green's theorem, and this is the distributed sink in (1.5.a).

The system (1.5) constitutes our first double-porosity model with micro-structure. The fracture system is a porous medium whose grains are the individual cells Ω_x which are distributed over the region Ω. These blocks do not interact directly with each other. Each is a porous medium whose flow is governed by (1.5.b) on the micro-scale of Ω_x and whose pressure is matched by (1.5.c) to that of the surrounding fissures. Finally, the total flux across the block boundary Γ_x given by (1.5.d) determines a source density for the fissure system given on the macro-scale of Ω by (1.5.a). We shall refer to the system (1.5) as the *matched micro-structure model.* Of course it will be supplemented with initial conditions to get a well-posed problem.

In our second double-porosity model with micro-structure we relax the requirement that the pressures are exactly matched along the interface, and hence that each cell pressure is constant on the boundary, and replace it with the assumption that the cell boundary flux is proportional to the pressure difference on the boundary. As before we have the system of diffusion equations

$$\frac{\partial}{\partial t}(a(x)u) - \nabla_x \cdot (A(x)\nabla_x u) + q(x, t) = f(x, t), \qquad x \in \Omega \tag{1.6.a}$$

$$\frac{\partial}{\partial t}(b(x)U) - \nabla_z \cdot (B(x)\nabla_z U) = 0, \qquad z \in \Omega_x, \tag{1.6.b}$$

but they are coupled by the requirements

$$q(x, t) = \frac{1}{\delta} \int_{\Gamma_x} (u(x, t) - \gamma_x U(x, s, t))\, ds, \qquad x \in \Omega, \tag{1.6.c}$$

$$B(x) \frac{\partial U(x, s, t)}{\partial \nu} + \frac{1}{\delta} (\gamma_x U(x, s, t) - u(x, t)) = 0, \qquad s \in \Gamma_x. \tag{1.6.d}$$

The system (1.6) is the *regularized micro-structure model.* We shall show the Cauchy problem is well-posed.

The micro-structure models incorporate two averaging processes. The pressure in the fissures (u) was extended to the whole of the domain Ω, and the number of blocks went from a large (but finite) collection to an (uncountable) collection of cells, one at each point $x \in \Omega$. We next write down an exact model for the flow in the fissures and blocks, and indicate how the micro-structure models will result from a (formal) limiting process. One advantage of this approach is that it becomes clear how the coefficients in the micro-model relate to those in the exact model, a vital ingredient for problems that arise physically.

Let $\bar{\Omega} = \bar{\Omega}_f \cup \bigcup_{i=1}^{N} \bar{\tilde{\Omega}}_i$, where Ω_f represents the fissures, and $\tilde{\Omega}_i$ are the blocks. The blocks correspond to cells in the micro-model, accordingly we define $\varepsilon_i^n = |\tilde{\Omega}_i|$ and scale each block by $1/\varepsilon_i$ to give a cell $\Omega_i = (1/\varepsilon_i)\, \tilde{\Omega}_i$. The "exact" equations for flow in the fissures and blocks may then be written as

$$\frac{\partial}{\partial t} (au) - \nabla \cdot (\tilde{A} \nabla u) = f \qquad \text{in } \Omega_f \tag{1.7.a}$$

$$\frac{\partial}{\partial t} (b_i U_i) - \nabla \cdot \left(\frac{1}{\varepsilon_i^2} B_i \nabla U_i \right) = 0 \qquad \text{in } \Omega_i, \quad i = 1, 2, ..., N \tag{1.7.b}$$

$$\gamma_i U_i = u \qquad \text{and} \qquad \frac{1}{\varepsilon_i} B_i \frac{\partial U_i}{\partial \nu} = -\tilde{A} \frac{\partial u}{\partial \nu} \text{ on } \partial\Omega_i, \quad i = 1, ..., N, \tag{1.7.c.}$$

where we identify $\partial\Omega_i$ with $\partial\tilde{\Omega}_i$ in (1.7.c) as above. If the density in the fissures is assumed to be constant on each of the relatively small $\partial\tilde{\Omega}_i$, an exercise in calculus shows that Eqs. (1.7) become

$$\frac{\partial}{\partial t} (X_{\Omega_f} au) - \nabla \cdot (X_{\Omega_f} \tilde{A} \nabla u) + q = X_{\Omega_f} f \qquad \text{in } \Omega \tag{1.8.a}$$

$$\frac{\partial}{\partial t} (b_i U_i) - \nabla \cdot \left[\frac{1}{\varepsilon_i^2} B_i \nabla U_i \right] = 0 \qquad \text{in } \Omega_i, \quad i = 1, 2, ..., N \tag{1.8.b}$$

$$\gamma_i U_i = u \,|_{\partial\tilde{\Omega}_i}, \qquad i = 1, 2, ..., N, \tag{1.8.c}$$

where

$$q(x) = \sum_{i=1}^{N} \frac{1}{|\Omega_i|} \int_{\partial\Omega_i} \frac{1}{\varepsilon_i^2} B_i \frac{\partial U_i}{\partial \nu} ds \, \mathbf{X}_i(x) \tag{1.8.d}$$

and X_{Ω_f} is the characteristic function for Ω_f and $\mathbf{X}_i$ is the characteristic function for Ω_i. The system (1.8) provides a clear motivation for the micro-models, since formally we expect the following limits:

$$\begin{aligned} X_{\Omega_f} &\to m_1, && \text{the fissure volume ratio} \\ X_{\Omega_f} \tilde{A} &\to A, && \text{the homogenized coefficients} \\ \tilde{\Omega}_i &\to x, && \text{(for a suitable subsequence)} \\ \gamma_i &\to \gamma_x, && U_i(\cdot) \to U(x, \cdot) \\ b_i &\to b(x), && \frac{1}{\varepsilon_i^2} B_i \to B(x) \end{aligned}$$

and

$$q(x) \to \frac{1}{|\Omega_x|} \int_{\partial\Omega_x} B(x) \frac{\partial}{\partial n} U(x, \cdot).$$

Substituting the indicated limits into (1.8) gives the matched micro-structure model (1.5). The above "limits" suggest that for a given physical problem (with a large but finite number of blocks), an appropriate matched micro-model would have cells whose geometry was similar to that of the blocks, and the cell conductivity should be chosen as $B(x) \approx (1/\varepsilon_i^2) B_i$ where $\varepsilon_i^n = |\tilde{\Omega}_i|$ (assuming the cells have unit measure). The block conductivity B_i is expected to be small, as is the block size (and hence ε_i); the formula $B(x) \approx (1/\varepsilon_i^2) B_i$ shows how these effects balance. If, during the limiting process, $(1/\varepsilon_i^2) B_i \to \infty$, the block equation will give $U_i = \text{constant}$, so in the limit $u = U$ and we recover the classical model (1.1). See Corollary 2. If Eq. (1.8.c) is replaced by the mixed condition

$$\frac{1}{\varepsilon_i} B_i \frac{\partial U_i}{\partial \nu} - \frac{\varepsilon_i}{\delta} (\gamma_i U_i - u|_{\partial\tilde{\Omega}_i}) = 0, \tag{1.8.c'}$$

then the regularized model (1.6) is obtained with $B(x) \approx (1/\varepsilon_i^2) B_i$. If during the limiting process we assume that $(1/\varepsilon_i^2) B_i \to \infty$, then the first-order kinetic model (1.3) results. See Corollary 3 below.

The double-porosity concept developed slowly during the first half of this century, and one finds an excellent presentation of the application of the system (1.2) to diffusion through a heterogeneous medium in [5, 20]. The

special case of (1.3) is often used as a model for diffusion through a slowly sorbing porous medium. See [17, 12] for a discussion of such models and [9, 16, 24] for some nonlinear extensions. Various nonlinear extensions of Eq. (1.4) have likewise been studied in [8, 7, 21, 22]. Distributed micro-structure models (1.5) were being studied in the early fifties in the kinetics of chromatography [19] and they have also arisen independently in soil science [6, 11] and reservoir modeling [2]. See [4, 14, 13] for the derivation of (1.5) as a limit of the exact singular system (1.7) and [3] for a proof of the convergence. Likewise, the regularized system (1.6) was studied in [19] and more recently in similar nonlinear systems in [10, 25]. See [23] for additional references to past and forthcoming work on micro-structure models.

2. Variational Formulation of MSM Problems

Here we give the mathematical formulations of the micro-structure model problems (1.5) and (1.6) as evolution equations on Hilbert spaces of Sobolev type. The corresponding Cauchy problems will be shown to be well-posed in (Lebesgue) L^2 spaces in Section 3. When these problems are formulated directly in $L^2(\Omega)$, the resulting evolution equations are complicated substantially by delay or memory effects. We have chosen to formulate the problems on a large space, a continuous direct sum of Hilbert spaces, and thereby obtain a technically much simpler operator in the Cauchy problem.

Let Ω be a bounded domain in $\mathbb{R}^n$ with smooth boundary, $\Gamma = \partial\Omega$. Denote by $L^2(\Omega)$ the space of (equivalence classes of) Lebesgue square-integrable functions on Ω, and let $C_0^\infty(\Omega)$ denote the subspace of infinitely differentiable functions with compact support. $H^m(\Omega)$ is the Hilbert space of functions in $L^2(\Omega)$ for which each partial derivative up to order m belongs to $L^2(\Omega)$. We shall let $H_0^1(\Omega)$ be the subspace obtained as the closure in $H^1(\Omega)$ of $C_0^\infty(\Omega)$. See [1, 18] for information on these Sobolev spaces. In addition, suppose that for each $x \in \Omega$ we are given a bounded domain Ω_x which lies locally on one side of its smooth boundary Γ_x. Denote by $\gamma_x: H^1(\Omega_x) \to L^2(\Gamma_x)$ the *trace* map which assigns boundary values. Since Γ_x is smooth, there is a unit outward normal $\nu_x(s)$ at each $s \in \Gamma_x$. Finally, we define $H_x^1(\Omega_x)$ to be that closed subspace of $H^1(\Omega_x)$ consisting of those $\varphi \in H^1(\Omega_x)$ with $\gamma_x \varphi \in \mathbb{R}$; i.e., each $\gamma_x(\varphi)$ is *constant* a.e. on Γ_x. We shall denote by ∇_z the gradient on $H^1(\Omega_x)$ and by ∇_x the gradient on $H^1(\Omega)$.

The essential construction to be used below is an example of a *continuous direct sum* of Hilbert spaces. The special case that is adequate for our purposes can be described as follows. Let S be a measure space and consider

the product (measure) space $Q = \Omega \times S$, where Ω has Lebesgue measure. If $U \in L^2(Q)$ then from the Fubini theorem it follows that $U(x)(z) \equiv U(x, z)$, $x \in \Omega$, $z \in S$ defines $U(x) \in L^2(S)$ at a.e. $x \in \Omega$, and for each $\Phi \in L^2(Q)$

$$\int_\Omega (U(x), \Phi(x))_{L^2(S)}\, dx \equiv \int_\Omega \left\{ \int_S U(x, z)\, \Phi(x, z)\, dz \right\} dx = \iint_Q U\Phi.$$

Thus $L^2(Q)$ is naturally identified as a (closed) subspace of $L^2(\Omega, L^2(S))$, the Bochner square integrable (equivalence classes of) functions from Ω to $L^2(S)$. Suppose $\mathbf{X}: \Omega \to \mathbb{R}$ is the characteristic function of a measurable $\Omega_* \subset \Omega$ and $w \in L^2(S)$. For each $a > 0$ we see

$$\{(x, z) \in Q: \mathbf{X}(x)\, w(z) < a\} = \Omega_* \times \{z \in S: w(z) < a\} \cup (\Omega \sim \Omega_*) \times S$$

and for $a \leqslant 0$ we delete the second term. Thus, $\mathbf{X} \cdot w$ is measurable on Q. It follows that each measurable step function $u = \sum \mathbf{X}_j w_j$ from $L^2(\Omega, L^2(S))$ is measurable on Q and, hence, belongs to $L^2(Q)$. This shows $L^2(Q)$ is dense in and therefore equal to $L^2(\Omega, L^2(S))$.

In order to prescribe a measurable family of cells, $\{\Omega_x, x \in \Omega\}$, set $S = \mathbb{R}^n$, let $Q \subset \Omega \times \mathbb{R}^n$ be a given measurable set, and set $\Omega_x = \{z \in \mathbb{R}^n: (x, z) \in Q\}$. Each Ω_x is measurable in $\mathbb{R}^n$ and by zero-extension we identify $L^2(Q) \subsetneq L^2(\Omega \times \mathbb{R}^n)$ and each $L^2(\Omega_x) \subsetneq L^2(\mathbb{R}^n)$. Thus we obtain from above

$$L^2(Q) \cong \{U \in L^2(\Omega, L^2(\mathbb{R}^n)): U(x) \in L^2(\Omega_x), \text{ a.e. } x \in \Omega\}.$$

Hereafter we shall denote this Hilbert space with scalar-product

$$(U, \Phi)_{\mathscr{H}} = \int_\Omega \left\{ \int_{\Omega_x} U(x, z)\, \Phi(x, z)\, dz \right\} dx$$

by $\mathscr{H} = L^2(\Omega, L^2(\Omega_x))$, and we shall set $H_x = L^2(\Omega_x)$ for each $x \in \Omega$ and $H_0 = L^2(\Omega)$. The *state space* for our problems will be the product

$$H \equiv H_0 \times \mathscr{H} \cong L^2(\Omega) \times L^2(Q).$$

Suppose $\{W_x: x \in \Omega\}$ is the collection of Sobolev spaces $W_x = H^1(\Omega_x)$ so that each W_x is continuously imbedded in H_x, uniformly for $x \in \Omega$. It follows that the direct sum

$$\begin{aligned} \mathscr{W} &\equiv L^2(\Omega, W_x) \\ &\equiv \left\{ U \in \mathscr{H}: U(x) \in W_x, \text{ a.e. } x \in \Omega, \text{ and } \int_\Omega \|U(x)\|_{W_x}^2\, dx < \infty \right\} \end{aligned}$$

is a Hilbert space. We shall use a variety of such subspaces of $\mathscr{H}$ which can be constructed in this manner. Moreover we shall assume that each Ω_x is

a bounded domain in $\mathbb{R}^n$ which lies locally on one side of its boundary, Γ_x, and Γ_x is a C^2-manifold of dimension $n-1$. This permits us to use Green's Theorem and regularity theory for elliptic equations on each Ω_x. We shall also assume the trace maps γ_x: $W_x \to L^2(\Gamma_x)$ are *uniformly* bounded. Thus for each $U \in \mathscr{W}$ it follows that the *distributed trace* $\gamma(U)$ defined by $\gamma(U)(x, s) \equiv (\gamma_x U(x))(s)$, $s \in \Gamma_x$, $x \in \Omega$, belongs to $L^2(\Omega, L^2(\Gamma_x))$.

Next consider the collection $\{V_x : x \in \Omega\}$ of Sobolev spaces given above by $V_x = H_x^1(\Omega_x)$, $x \in \Omega$, and denote by $\mathscr{V}_1 \equiv L^2(\Omega, V_x)$ the corresponding direct sum. Thus for each $U \in \mathscr{V}_1$ it follows that the distributed trace $\gamma(U)$ belongs to $L^2(\Omega)$. We define $\mathscr{V}_0$ to be the subspace of those $U \in \mathscr{V}_1$ for which $\gamma(U) \in V_0 \equiv H_0^1(\Omega)$. Since γ: $\mathscr{V}_1 \to L^2(\Omega)$ is continuous, $\mathscr{V}_0$ is complete with the scalar-product

$$(U, \Phi)_{\mathscr{V}_0} \equiv \int_\Omega (U(x), \Phi(x))_{V_x} \, dx + (\gamma U, \gamma \Phi)_{V_0}.$$

This Hilbert space $\mathscr{V}_0$ will be the *energy space* for our problem (1.5). Note that this space is imbedded in the state space by the injection $\hat{\gamma}$: $\mathscr{V}_0 \to \mathscr{W}_1$ defined by $\hat{\gamma}(v) = [\gamma v, v]$ where $\mathscr{W}_1 \equiv H_0^1(\Omega) \times L^2(\Omega, H^1(\Omega_x))$ is a subspace of H. Moreover, this injection is continuous with range $\mathscr{W}_0 \equiv \hat{\gamma}(\mathscr{V}_0)$ dense in H, so we can identify H' with a subspace of $\mathscr{W}_0'$ by the adjoint $\hat{\gamma}^*$: if $f_0 \in H_0'$ and $\mathfrak{f} \in \mathscr{H}'$, then $\hat{\gamma}^*([f_0, \mathfrak{f}])$ is defined in $\mathscr{W}_0'$ by

$$\hat{\gamma}^*([f_0, \mathfrak{f}])(\Phi) \equiv [f_0, \mathfrak{f}] \, \hat{\gamma}(\Phi) = f_0(\gamma \Phi) + \mathfrak{f}(\Phi), \qquad \Phi \in \mathscr{W}_0.$$

The space $\mathscr{W}_1$ will be the energy space for the regularized problem (1.6).

In order to state the *Cauchy–Dirichlet Problem* for the matched micro-structure model (1.5), assume we are given the functions

$$a, A \in L^\infty(\Omega), \qquad u_0 \in L^2(\Omega),$$

$$b, B \in L^\infty(Q), \qquad U_0 \in L^2(Q),$$

and $f \in L^2(\Omega \times [0, T])$. A *solution* is a function U: $(0, T] \to \mathscr{V}_0$ for which

$$U \in C([0, T], L^2(Q)) \cap C^1((0, T], L^2(Q)),$$

$$u \equiv \gamma(U) \in C([0, T], L^2(\Omega)) \cap C^1((0, T], L^2(\Omega)),$$

for every $t \in (0, T]$ and a.e. $x \in \Omega$

$$\frac{\partial}{\partial t}\left(a(x)\, u(x, t) + \int_{\Omega_x} b(x, z)\, U(x, z, t)\, dz\right) - \nabla_x \cdot A(x)\, \nabla_x u(x, t) = f(x, t), \tag{2.1.a}$$

$$\frac{\partial}{\partial t}\,(b(x, z)\, U(x, z, t)) - \nabla_z \cdot (B(x, z)\, \nabla_z U(x, z, t)) = 0, \tag{2.1.b}$$

and

$$\lim_{t \to 0} u(x, t) = u_0(x), \qquad \lim_{t \to 0} U(x, z, t) = U_0(x, z), \tag{2.1.c}$$

for a.e. $z \in \Omega_x$. Specifically, each term in (2.1.a) and (2.1.b) is in $L^2(\Omega)$ and $L^2(Q)$, respectively, and the limits in (2.1.c) are taken in these spaces. Note that the prescribed u_0 is not necessarily the trace of U_0, and, moreover, the trace is not even meaningful for U_0 as given. Also, from the definition of γ and $\mathscr{V}_0$ it is implicit that the boundary conditions

$$u(x, t) = 0, \text{ a.e. } x \in \Gamma, \qquad U(x, z, t) = u(x, t), \text{ a.e. } x \in \Omega, z \in \Gamma_x, \tag{2.1.d}$$

hold at each $t > 0$. Finally, since each of the terms in (2.1.b) remains in $L^2(\Omega_x)$ at a.e. x and $t \in (0, T]$, it follows from regularity theory that $U(x, \cdot, t) \in H^2(\Omega_x)$ and then by Green's theorem that

$$\frac{\partial}{\partial t} \int_{\Omega_x} b(x, z)\, U(x, z, t)\, dz = \int_{\Gamma_x} B(x, s) \frac{\partial U(x, s, t)}{\partial \nu_x}\, ds \tag{2.2}$$

when $B(x, \cdot)$ is sufficiently smooth, or by a generalized Green's theorem otherwise. This is just the flux term by which (2.1.a) is coupled to (2.1.b); they are also coupled by the *matched* boundary conditions in (2.1.d).

The variational form of our problem follows quickly. Let U and $u = \gamma(U)$ constitute a solution of (2.1). Pick a $\Phi \in \mathscr{V}_0$ and $0 < t \leqslant T$. Multiply (2.1.b) by Φ and integrate over Ω_x, add this to the product of (2.1.a) and $\gamma_x(\Phi)$, and integrate this sum over Ω. Since $\gamma_x(\Phi)$ is constant on Γ_x we obtain with (2.2) the identity

$$\begin{aligned} &\int_\Omega \left\{ a(x) \frac{\partial u}{\partial t} - \nabla_x \cdot A(x) \nabla_x u \right\} \gamma_x(\Phi)\, dx \\ &\quad + \int_\Omega \left\{ \int_{\Omega_x} \left(b(x, z) \frac{\partial U}{\partial t} - \nabla_z \cdot B(x, z) \nabla_z U \right) \Phi(x, z)\, dz \right. \\ &\quad \left. + \int_{\Gamma_x} B(x, s) \frac{\partial U}{\partial \nu_x} \gamma_x \Phi\, ds \right\} dx \\ &= \int_\Omega f(x, t)\, \gamma_x(\Phi)\, dx. \end{aligned} \tag{2.3}$$

This leads to the pair of continuous linear operators $l: H \to H'$, $\mathscr{L}: \mathscr{W}_1 \to \mathscr{W}_1'$ given by

$$l([u, U], [\varphi, \Phi])$$

$$= \int_\Omega \left\{ a(x)\, u(x)\, \varphi(x) + \int_{\Omega_x} b(x, z)\, U(x, z)\, \Phi(x, z)\, dz \right\} dz,$$

$$u, \varphi \in L^2(\Omega), \quad U, \Phi \in L^2(Q), \tag{2.4.a}$$

$$\mathscr{L}([u, U], [\varphi, \Phi])$$

$$= \int_\Omega \left\{ A(x)\, \nabla_x u(x) \cdot \nabla_x \varphi(x) + \int_{\Omega_x} B(x, z)\, \nabla_z U(x, z) \cdot \nabla_z \Phi(x, z)\, dz \right\} dx,$$

$$u, \varphi \in H_0^1(\Omega), \quad U, \Phi \in L^2(\Omega, H^1(\Omega_x)). \tag{2.4.b}$$

Recall that $\hat{\gamma} = [\gamma, I]$ maps $\mathscr{V}_0$ onto $\mathscr{W}_0$ and $\mathscr{W}_0 \subset H$. It follows from (2.3) that a solution of (2.1) satisfies the evolution equation

$$\hat{\gamma}^* l\hat{\gamma}(U'(t)) + \hat{\gamma}^* \mathscr{L}\hat{\gamma}(U(t)) = F(t), \qquad 0 < t \leqslant T \tag{2.5}$$

in $\mathscr{W}_0'$ where the function $F\colon (0, T] \to \mathscr{W}_1'$ is given by

$$F(t)[\varphi, \Phi] = \int_\Omega f(x, t)\, \varphi(x)\, dx, \qquad [\varphi, \Phi] \in \mathscr{W}_1.$$

We shall show in the next section that there is a unique solution of this problem when the four coefficient functions a, b, A, B are uniformly lower-bounded by a positive number and the function $t \mapsto f(\cdot, t)$ from $[0, T]$ to $L^2(\Omega)$ is Hölder continuous. Moreover, we shall show that the flux term (2.2) is meaningful and can therefore be substituted in (2.3). Specifically we shall show that the dynamics are prescribed by a *holomorphic semigroup* on $H \cong H'$.

Next we state the Cauchy–Dirichlet Problem for the regularized micro-structure model (1.6). We have previously considered the measurable collection $\{W_x\colon x \in \Omega\}$ of Sobolev spaces given by $W_x = H^1(\Omega_x)$, $x \in \Omega$, and its corresponding direct sum $\mathscr{W} \equiv L^2(\Omega, W_x)$. The *energy space* for the regularized problem (1.6) is the "unconstrained" product space $\mathscr{W}_1 = H_0^1(\Omega) \times \mathscr{W}$. A *solution* of (1.6) is a pair of functions $u\colon (0, T] \to H_0^1(\Omega)$, $U\colon (0, T] \to \mathscr{W}$ for which

$$u \in C([0, T], L^2(\Omega)) \cap C^1((0, T], L^2(\Omega))$$

$$U \in C([0, T], L^2(Q)) \cap C^1((0, T], L^2(Q))$$

such that for every $t \in (0, T]$ and a.e. $x \in \Omega$

$$\frac{\partial}{\partial t}(a(x)\,u(x,t)) - \nabla_x \cdot A(x)\,\nabla_x u(x,t) + \frac{1}{\delta}\int_{\Gamma_x}(u(x,t) - \gamma_x U(x,s,t))\,ds = f(x,t), \tag{2.6.a}$$

$$\frac{\partial}{\partial t}(b(x,z)\,U(x,z,t)) - \nabla_z \cdot B(x,z)\,\nabla_z U(x,z,t) = 0, \qquad \text{a.e. } z \in \Omega_x \tag{2.6.b}$$

$$B(x,s)\frac{\partial U(x,s,t)}{\partial \nu_x} + \frac{1}{\delta}(\gamma_x U(s,t) - u(x,t)) = 0, \qquad \text{a.e. } s \in \Gamma_x \tag{2.6.c}$$

and

$$\lim_{t\to 0} u(x,t) = u_0(x), \qquad \lim_{t\to 0} U(x,z,t) = U_0(x,z) \tag{2.6.d}$$

in $L^2(\Omega)$ and $L^2(Q)$, respectively. Of course the boundary condition

$$u(x,t) = 0, \qquad \text{a.e. } x \in \Gamma,\ t \in (0,T] \tag{2.6.e}$$

is implicit above and we note that (2.6.c) is the flux on Γ_x by which (2.6.a) is coupled to (2.6.b) and is comparable to (2.1.a). Proceeding as before, we multiply (2.6.b) by $\Phi \in \mathscr{W}$ and integrate over Ω_x, add this to the product of (2.6.a) and $\varphi \in H_0^1(\Omega)$, and integrate this sum over Ω to obtain with (2.6.c)

$$\begin{aligned} &l([u'(t), U'(t)][\varphi, \Phi]) + \mathscr{L}([u(t), U(t)], [\varphi, \Phi]) \\ &\quad + \frac{1}{\delta}\int_\Omega \int_{\Gamma_x} (u(x) - \gamma_x U(s,t))(\varphi(x) - \gamma_x \Phi(s,t))\,ds\,dx \\ &\quad = F(t)[\varphi, \Phi], \qquad [\varphi, \Phi] \in \mathscr{W}_1. \end{aligned} \tag{2.7}$$

Recall that the trace operators are uniformly bounded, so γ is continuous from $\mathscr{W} = L^2(\Omega, H^1(\Omega_x))$ into $\mathscr{B} \equiv L^2(\Omega, L^2(\Gamma_x))$. If we denote by $\lambda: L^2(\Omega) \to \mathscr{B}$ the constant function in $L^2(\Gamma_x)$,

$$\lambda(u)(x,s) \equiv u(x), \qquad x \in \Omega, \quad s \in \Gamma_x,$$

and by $M: \mathscr{B} \times \mathscr{B} \to \mathscr{B}' \times \mathscr{B}'$ the boundary operator given by

$$\begin{aligned} &M([\varphi_0, \psi_0], [\varphi_1, \psi_1]) \\ &\quad = \int_\Omega \int_{\Gamma_x} (\varphi_0 - \psi_0)(\varphi_1 - \psi_1)\,ds\,dx, \qquad \varphi_j, \psi_j \in \mathscr{B}, \quad j = 0, 1, \end{aligned}$$

the third term in (2.7) is just $(1/\delta)[\lambda, \gamma]^* \circ M \circ [\lambda, \gamma]$ on $L^2(\Omega) \times \mathcal{W}$ into its dual. Thus, setting

$$\mathcal{L}_\delta \equiv \mathcal{L} + \frac{1}{\delta}[\lambda, \gamma]^* \circ M \circ [\lambda, \gamma], \qquad \delta > 0, \tag{2.8}$$

we have $\mathcal{L}_\delta \in \mathcal{L}(\mathcal{W}_1, \mathcal{W}_1')$ and (2.7) is equivalent to

$$l[u'(t), U'(t)] + \mathcal{L}_\delta[u(t), U(t)] = F(t), \qquad 0 < t \leqslant T. \tag{2.7'}$$

We shall show the Cauchy problem for (2.7') has a unique solution $[u_\delta, U_\delta]$ under the same hypotheses as above, and that as $\delta \to 0$ this solution converges to that of (2.5). The function M provides a *penalty function* which is used to approximate the problem (2.5) with *constraint*

$$M[\lambda(u), \gamma U] = 0 \qquad \text{in } \mathcal{B}'$$

as indicated above.

3. Cauchy Problems and Continuous Dependence

The objectives in this final section are to show that each of the micro-structure models (2.1) and (2.6) is a well-posed initial-boundary-value problem, that the solution of (2.6) depends continuously on the parameter $\delta > 0$ and converges to the solution of (2.1) as $\delta \to 0$, and that the first-order kinetic model (1.3) can be obtained as the limit of (2.6) when the conductivity B increases (uniformly) to $+\infty$. These results will be achieved by showing that the dynamics of each model is governed by a *holomorphic semigroup* in the Hilbert space H and then using the Trotter–Kato theory of convergence of semigroups to reduce the convergence claims to the stationary cases.

The Hilbert space $H = L^2(\Omega) \times L^2(Q)$ is a product space on which the operator l: $H \to H'$ given by (2.4.a) gives an equivalent scalar-product. We shall consider H to be endowed with $l(\cdot, \cdot)$ as scalar-product in the following. We constructed the Hilbert spaces $\mathcal{W}_1 = H_0^1(\Omega) \times L^2(\Omega, H^1(\Omega_x))$ and $\mathcal{W}_0 = \hat{\gamma}(\mathcal{V}_0) = \{[\gamma U, U]\colon\ U \in L^2(\Omega, H_x^1(\Omega_x)),\ \gamma(U) \in H_0^1(\Omega)\}$ for which we have $\mathcal{W}_0 \subset \mathcal{W}_1 \subset H$ and $\mathcal{W}_0$ is dense in H. Also the operators $\mathcal{L}$ and $\mathcal{L}_\delta$, given by (2.4.b) and by (2.8) for $\delta > 0$, are continuous and linear from $\mathcal{W}_1$ to its dual; each $\mathcal{L} + \varepsilon l$, $\mathcal{L}_\delta + \varepsilon l$ is $\mathcal{W}_1$-coercive for any $\varepsilon > 0$. First-order derivatives to model convection could be added with no substantial change.

First we consider the matched micro-structure model (2.1) in the form

(2.5). Given $\mathscr{L}: \mathscr{W}_0 \to \mathscr{W}_0'$ and $l: H \to H'$ with $\mathscr{W}_0 \subset H$ dense, so that $H' \subset \mathscr{W}_0'$ by restriction, we define

$$D(\mathbf{L}_0) \equiv \{[\gamma U, U] \in \mathscr{W}_0 : l[f, \mathfrak{f}] = \mathscr{L}[\gamma U, U] \text{ for some } [f, \mathfrak{f}] \in H\}$$
$$\mathbf{L}_0[\gamma U, U] = [f, \mathfrak{f}], \qquad [\gamma U, U] \in D(\mathbf{L}_0).$$

Since l is one-to-one, $\mathbf{L}_0$ is a function with domain $D(\mathbf{L}_0)$ and range in H. Moreover, $\mathbf{L}_0[\gamma U, U] = [f, \mathfrak{f}]$ is equivalent to

$$l([f, \mathfrak{f}], [\gamma \Phi, \Phi]) = \mathscr{L}([\gamma U, U], [\gamma \Phi, \Phi]), \qquad \Phi \in \mathscr{V}_0,$$

so setting $\Phi = U$ shows that

$$l(\mathbf{L}_0[\gamma U, U], [\gamma U, U]) = \mathscr{L}([\gamma U, U], [\gamma U, U])$$

and hence, $\mathbf{L}_0$ is *sectorial* on H. Likewise we obtain for $[\gamma U, U] \in D(\mathbf{L}_0)$

$$l((\varepsilon I + \mathbf{L}_0)[\gamma U, U]) = (\varepsilon l + \mathscr{L})[\gamma U, U] \qquad \text{in } \mathscr{W}_0';$$

since $\varepsilon l + \mathscr{L}$ is onto $\mathscr{W}_0'$ this shows that $\varepsilon I + \mathbf{L}_0$ maps $D(\mathbf{L}_0)$ onto H. That is, $\mathbf{L}_0$ is *m-sectorial* on H. From [15] it follows that $-\mathbf{L}_0$ is the generator of an holomorphic semigroup on H and this essentially establishes the following.

THEOREM 0. *Let the spaces H, $\mathscr{W}_0$ be given as in Section* 2: *we assume each Γ_x is a C^1 manifold of dimension $n-1$ with Ω_x locally on one side, and that the trace maps γ_x are uniformly bounded.* (a) *Assume the coefficient functions a, $A \in L^\infty(\Omega)$, b, $B \in L^\infty(Q)$, are uniformly lower-bounded by a positive number, and define l, $\mathscr{L}$ by* (2.4). *Then for each pair*

$$u_0 \in L^2(\Omega), \qquad U_0 \in L^2(Q)$$

and each Hölder continuous $f: [0, T] \to L^2(\Omega)$ there exists a unique $\hat{U} \in C([0, T], H) \cap C^1((0, T], H)$ which satisfies

$$\hat{U}'(t) + \mathbf{L}_0(\hat{U}(t)) = \left[\frac{1}{a} f(t), 0\right] \qquad \text{in } H, \quad 0 < t \leqslant T,$$
$$\hat{U}(0) = [u_0, U_0].$$

For $t > 0$, $\hat{U}(t) \in D(\mathbf{L}_0)$, so $\hat{U}(t) \equiv [\gamma U(t), U(t)] \in \mathscr{W}_0$. (b) *If each $B(x, \cdot) \in C^1(\bar{\Omega}_x)$ and Γ_x is a C^2 manifold, then $U: (0, T] \to \mathscr{V}_0$ is the unique solution to* (2.1).

Proof. It remains only to prove (b), i.e., $u \equiv \gamma U$ and U satisfy the system

(2.1), and for this it suffices to consider the stationary equation, $\mathbf{L}_0[\gamma U, U] = [f, \mathfrak{f}]$. That is,

$$U \in L^2(\Omega, H^1_x(\Omega_x)), \qquad u = \gamma U \in H^1_0(\Omega)$$

and (3.1)

$$\mathscr{L}([u, U], [\gamma\Phi, \Phi]) = l([f, \mathfrak{f}], [\gamma\Phi, \Phi]), \qquad \Phi \in \mathscr{V}_0.$$

The subspace of those $\Phi \in \mathscr{V}_0$ for which $\gamma\Phi = 0$ is dense in $\mathscr{H}$, so we obtain

$$-\nabla_z \cdot B(x, z)\, \nabla U(x, z) = b(x, z)\, \mathfrak{f}(x, z), \qquad \text{a.e. } x \in \Omega,\ z \in \Omega_x. \quad (3.2)$$

The regularity theory for these Dirichlet problems on Ω_x implies that a.e. $U(x, \cdot) \in H^2(\Omega_x)$. Integrating (3.2) over Ω_x by Gauss' Theorem shows that

$$\int_{\Gamma_x} B(x, s) \frac{\partial U(x, s)}{\partial \nu_x}\, ds = \int_{\Omega_x} b(x, z)\, \mathfrak{f}(x, z)\, dz, \qquad \text{a.e. } x \in \Omega, \quad (3.3)$$

and this is then in $L^2(\Omega)$. Likewise applying Gauss' Theorem after substituting (3.2) back into (3.1) yields

$$\int_\Omega \left\{ A(x)\, \nabla_x u(x)\, \nabla_x \varphi(x) + \int_{\Gamma_x} B(x, s) \frac{\partial U(x, s)}{\partial \nu_x}\, ds\, \varphi(x) \right\} dx$$

$$= \int_\Omega a(x)\, f(x)\, \varphi(x)\, dx, \qquad \varphi \in H^1_0(\Omega).$$

It follows from this variational equation that

$$-\nabla_x A(x)\, \nabla_x u(x) + \int_{\Gamma_x} B(x, s) \frac{\partial U(x, s)}{\partial \nu_x}\, ds = a(x)\, f(x), \qquad \text{a.e. } x \in \Omega, \quad (3.4)$$

and each of the three terms belongs to $L^2(\Omega)$.

Remark. If each $B(x, \cdot) \in C^1(\Omega_x)$ and Γ_x is a C^2 manifold with all constants independent of x, i.e., the regularity for (3.2) is *uniform* in $x \in \Omega$, then $U \in L^2(\Omega, H^2(\Omega_x))$. If $A \in C^1(\bar{\Omega})$ and $\partial\Omega$ is a C^2 manifold, then $u \in H^2(\Omega)$ follows from (3.4).

Next we treat the regularized micro-structure model similarly. Given $\mathscr{L}_\delta : \mathscr{W}_1 \to \mathscr{W}_1'$ and $l : H \to H'$ continuous and linear with $\mathscr{W}_1 \subset H$ dense and $H' \subset \mathscr{W}_1'$, define

$$D(\mathbf{L}_\delta) = \{[u, U] \in \mathscr{W}_1 : \mathscr{L}[u, U] \in H'\}.$$

Since l is an isomorphism onto H', it follows that $[u, U] \in D(\mathbf{L}_\delta)$ if and only if there is a unique $[f, \mathfrak{f}] \in H$ with $\mathscr{L}[u, U] = l[f, \mathfrak{f}]$, and then we define $\mathbf{L}_\delta[u, U] = [f, \mathfrak{f}]$. Thus $\mathbf{L}_\delta = l^{-1} \circ \mathscr{L}$ is a function with domain $D(\mathbf{L}_\delta)$ and range in H for which

$$l((\varepsilon I + \mathbf{L}_\delta)[u, U], [\varphi, \Phi]) = (\varepsilon l + \mathscr{L})([u, U], [\varphi, \Phi]), \qquad [\varphi, \Phi] \in \mathscr{W}_1,$$

for every $\varepsilon > 0$. It follows as before that $\mathbf{L}_\delta$ is an m-sectorial operator on the space H with respect to the scalar-product $l(\cdot, \cdot)$.

Let's characterize $\mathbf{L}_\delta[u, U] = [f, \mathfrak{f}]$, i.e.,

$$\begin{aligned} u \in H_0^1(\Omega),\ U \in L^2(\Omega, H^1(\Omega_x))\colon\ & \mathscr{L}([u, U], [\varphi, \Phi]) \\ & + \frac{1}{\delta} \int_\Omega \int_{\Gamma_x} (u(x) - \gamma_x U(x, s))(\varphi(x) - \gamma_x \Phi(x, s))\, ds\, dx \\ = \int_\Omega & \left\{ a(x) f(x) \varphi(x) + \int_{\Gamma_x} b(x, z)\, \mathfrak{f}(x, z)\, \Phi(x, z)\, dz \right\} dx, \\ & \varphi \in H_0^1(\Omega), \quad \Phi \in L^2(\Omega, H^1(\Omega_x)). \end{aligned} \tag{3.5}$$

Choosing $\Phi = 0$ shows the first component satisfies

$$\begin{aligned} u \in H_0^1(\Omega)\colon\ & -\nabla \cdot A \nabla u + \frac{1}{\delta} \int_{\Gamma_x} (u(x) - \gamma_x U(x, s))\, ds \\ & = a(x) f(x), \qquad \text{a.e. } x \in \Omega, \end{aligned} \tag{3.6}$$

with each of the three terms in $L^2(\Omega)$. Likewise setting $\varphi = 0$ yields the variational characterization of

$$\begin{aligned} U(x, \cdot) \in H^1(\Omega_x)\colon\ & -\nabla_z \cdot B(x, \cdot) \nabla_z U(x, \cdot) + \frac{1}{\delta} \int_{\Gamma_x} (\gamma U(x, s) - u(x))\, ds \\ & = b(x, \cdot)\, \mathfrak{f}(x, \cdot) \qquad \text{in } L^2(\Omega_x), \\ & \frac{\partial U(x, \cdot)}{\partial \nu_x} + \frac{1}{\delta} (\gamma U(x, \cdot) - u(x)) = 0 \qquad \text{in } L^2(\Omega_x), \quad \text{a.e. } x \in \Omega. \end{aligned} \tag{3.7}$$

These calculations finish the proof of the following.

THEOREM δ. *Let the spaces H, $\mathscr{W}_1$ be given as in Section* 2. *Part* (a) *of Theorem* 0 *holds for $\mathbf{L}_\delta$, $\mathscr{W}_1$ in place of $\mathbf{L}_0$, $\mathscr{W}_0$, and $\hat{U}(t) = [u(t), U(t)] \in \mathscr{W}_1$ for each $t > 0$.*

(a) *The first component satisfies $u(t) \in H_0^1(\Omega)$ and* (2.6.a) *in $L^2(\Omega)$, the second satisfies $U(t) \in L^2(\Omega, H^1(\Omega_x))$ and* (2.6.b) *for each $t > 0$, and the initial conditions* (2.6.d) *hold.*

(b) *If each* $B(x, \cdot) \in C^1(\bar{\Omega}_x)$ *and* Γ_x *is a* C^2 *manifold, then* (2.6.c) *holds at each* $t > 0$.

We consider the dependence of the solution of (2.6) on the coupling parameter, $0 \leqslant \delta \leqslant +\infty$. First we remark that the coupling operator in (2.8) is continuous on $L^2(\Omega) \times \mathscr{W}$ and that $\varepsilon I + \mathscr{L}$ is coercive over $\mathscr{W}_1 = H_0^1(\Omega) \times \mathscr{W}$, hence, over $L^2(\Omega) \times \mathscr{W}$. Thus the operator (2.8) is *relatively bounded* by $\varepsilon I + \mathscr{L}$ on H, and $\varepsilon I + \mathbf{L}_\delta$ is a *holomorphic family* [15] in the parameter $0 < \delta \leqslant +\infty$. Thus the solution of (2.6) depends analytically on the parameter $\delta > 0$.

Next we consider the behavior as $\delta \to O^+$. Let $\tilde{f} \in H$ and consider the unique solutions of the resolvent equations

$$(I + \mathbf{L}_\delta)\, \tilde{u}_\delta = \tilde{f}, \qquad (I + \mathbf{L}_0)\, \tilde{u}_0 = \tilde{f}.$$

These are characterized respectively by

$$\tilde{u}_\delta \in \mathscr{W}_1 \colon l(\tilde{u}_\delta) + \mathscr{L}(\tilde{u}_\delta) + \frac{1}{\delta}\mathscr{M}(\tilde{u}_\delta) = \tilde{f} \qquad \text{in } \mathscr{W}_1', \tag{3.8}$$

$$\tilde{u}_0 \in \mathscr{W}_0 \colon l(\tilde{u}_0) + \mathscr{L}(\tilde{u}_0) = \tilde{f} \qquad \text{in } \mathscr{W}_0', \tag{3.9}$$

where $\mathscr{M} \equiv [\lambda, \gamma]^* \circ M \circ [\lambda, \gamma]$. Note that $\mathscr{M}(\tilde{u}_0) = 0$. Thus we have

$$\begin{aligned} c\, \|\tilde{u}_\delta - \tilde{u}_0\|_{\mathscr{W}_1}^2 &\leqslant \left(l + \mathscr{L} + \frac{1}{\delta}\mathscr{M}\right)(\tilde{u}_\delta - \tilde{u}_0, \tilde{u}_\delta - \tilde{u}_0) \\ &= (f - (l + \mathscr{L})\, \tilde{u}_0, \tilde{u}_\delta - \tilde{u}_0), \qquad \delta > 0. \end{aligned} \tag{3.10}$$

This shows (a subsequence of) $\{\tilde{u}_\delta\}$ converges weakly to some $\tilde{u} \in \mathscr{W}_1$. Thus $\mathscr{L}\tilde{u}_\delta \rightharpoonup \mathscr{L}\tilde{u}$ and $\mathscr{M}\tilde{u}_\delta \to 0 = \mathscr{M}\tilde{u}$, so $\tilde{u} \in \mathscr{W}_0$. The restriction of $\mathscr{M}(\tilde{u}_\delta)$ to $\mathscr{W}_0$ is always zero, so by restricting (3.8) to $\mathscr{W}_0$ and letting $\delta \to 0$ we find that $\tilde{u}$ satisfies (3.9), hence, $\tilde{u} = u_0$. Finally, from (3.10) it follows that $\tilde{u}_\delta \to \tilde{u}_0$ in $\mathscr{W}_1$, hence, in H. This shows that $\tilde{u}_\delta$ converges *strongly* to $\tilde{u}_0$ in H as $\delta \to 0$. From Theorem IX.2.16 of [15] it follows that the semigroup generated by $-\mathbf{L}_\delta$ converges strongly to that generated by $-\mathbf{L}_0$, uniformly on bounded intervals, $0 \leqslant t \leqslant T$. This yields the following result.

COROLLARY 1. *In the situation of Theorem* 0, *denote the solution of* (2.1) *by* $[\gamma U, U]$ *and for* $\delta > 0$ *let* $[u_\delta, U_\delta]$ *be the solution of* (2.6). *Then we have strong convergence* $u_\delta(t) \to \gamma U(t)$ *in* $L^2(\Omega)$, $U_\delta(t) \to U(t)$ *in* $L^2(Q)$, *as* $\delta \to 0$, *uniformly for* $0 \leqslant t \leqslant T$.

Remark. This result holds also if $[u_\delta(0), U_\delta(0)] \to [u_0, U_0]$ in H and if the right side of (2.6.a) is replaced by a sequence $\{f_\delta\}$ of Hölder continuous functions which convergence in $L^2(0, T; L^2(\Omega))$ to the right side of

(2.1.a). Thus, the Cauchy problem (2.6) depends continuously on initial conditions, source terms, and the parameter $\delta > 0$ as indicated. The same argument also establishes continuous dependence on the coefficients.

Finally we consider the limiting behavior of the solutions to (2.1) and (2.6) as the cell permeability coefficient $B(x)$ increases without bound. To implement this we replace $B(x)$ by $(1/\varepsilon)\, B(x)$ in each of (2.1) and (2.6) for $0 < \varepsilon \leqslant 1$, and establish the convergence of the solutions as $\varepsilon \to 0$.

COROLLARY 2. *In the situation of Theorem* 0, *for each* $\varepsilon > 0$, *let* $[\gamma U^\varepsilon, U^\varepsilon]$ *be the solution of* (2.1) *in which* $B(x)$ *is replaced by* $(1/\varepsilon)\, B(x)$. *Then we have strong convergence*

$$\gamma U^\varepsilon(t) \to u(t) \text{ in } L^2(\Omega), \qquad U^\varepsilon(t) \to u(t) \text{ in } L^2(Q)$$

as $\varepsilon \to 0$, *uniformly for* $0 \leqslant t \leqslant T$, *where* $u \in C([0, T], L^2(\Omega)) \cap C^1((0, T], L^2(\Omega))$ *is the unique solution of the classical diffusion equation*

$$\frac{\partial}{\partial t}\left(a(x) + \int_{\Omega_x} b(x, z)\, dz\right) u(x, t) - \nabla_x A(x)\, \nabla_x u(x, t)$$

$$= f(x, t), \qquad \text{a.e. } x \in \Omega, \tag{3.11.a}$$

$$u(x, t) = 0, \qquad \text{a.e. } x \in \Gamma,\ t > 0, \tag{3.11.b}$$

$$\lim_{t \to 0} u(x, t) = u_0(x) + \int_{\Omega_x} U_0(x, z)\, dz, \qquad \text{a.e. } x \in \Omega. \tag{3.11.c}$$

Specifically, $u(t) \in H_0^1(\Omega)$, *so* (3.11.b) *is meaningful, and* (3.11.c) *holds in* $L^2(\Omega)$.

Proof. As before the result follows from the Trotter–Kato Theorem and the corresponding result for the stationary problem. For the stationary problem for $u^\varepsilon = \gamma U^\varepsilon$ we have

$$[u^\varepsilon, U^\varepsilon] \in \mathcal{W}_0 \colon l([u^\varepsilon, U^\varepsilon], [\varphi, \Phi])$$

$$+ \int_\Omega A\, \nabla u^\varepsilon \cdot \nabla \varphi\, dx + \frac{1}{\varepsilon} \iint_Q B\, \nabla_z U^\varepsilon \cdot \nabla_z \Phi\, dz\, dx$$

$$= \int_\Omega f\varphi\, dx + \iint_Q f\Phi\, dz\, dx, \qquad [\varphi, \Phi] \in \mathcal{W}_0.$$

It follows by calculations similar to those leading to (3.10) that $\{[u^\varepsilon, U^\varepsilon]\}$ converges strongly in $\mathcal{W}_0$ to $[u, U] \in \mathcal{W}_0$ which satisfies the *constraint*

$$\iint_Q B(x, z)\, \nabla_z U(x, z) \cdot \nabla_z U(x, z)\, dz\, dx = 0.$$

That is, $U(x, z) = u(x)$ for a.e. $z \in \Omega_x$, so $u \in H_0^1(\Omega) \subsetneq L^2(\Omega, H_x^1(\Omega_x))$ in the sense of functions constant on Ω_x, and this limit satisfies

$$\int_\Omega \left(a(x) + \int_{\Omega_x} b(x, z)\, dz \right) u(x)\, \varphi(x)\, dx + \int_\Omega A(x)\, \nabla u(x)\, \nabla \varphi(x)\, dx$$

$$= \int_\Omega \left(f(x) + \int_{\Omega_x} \mathfrak{f}(x, z)\, dz \right) \varphi(x)\, dx, \qquad \varphi \in H_0^1(\Omega). \tag{3.12}$$

Here we identify φ with $[\varphi, \Phi] \in \mathscr{W}_0$, where $\Phi(x, z) \equiv \varphi(x)$ defines $\Phi \in L^2(\Omega, H_x^1(\Omega_x))$ as above. Finally note that (3.12) is just the variational form of the resolvent equation for (3.11) when $\mathfrak{f} = 0$, so the proof is complete.

By the similar but somewhat easier calculations we obtain the following.

COROLLARY 3. *For each $\varepsilon > 0$, let $[u^\varepsilon, U^\varepsilon]$ be the solution of* (2.6) *in which $B(x)$ is replaced by $(1/\varepsilon)\, B(x)$. Then we have the strong convergence*

$$u^\varepsilon(t) \to u(t) \text{ in } L^2(\Omega), \qquad U^\varepsilon(t) \to U(t) \text{ in } L^2(Q)$$

as $\varepsilon \to 0$, uniformly for $0 \leqslant t \leqslant T$, where

$$u, U \in C([0, T], L^2(\Omega)) \cap C^1((0, T]; L^2(\Omega))$$

is the unique solution of the first-order kinetic model

$$\frac{\partial}{\partial t}(a(x)\, u(x, t)) - \nabla_x A(x)\, \nabla_x u(x, t) + \frac{|\Gamma_x|}{\delta}(u(x, t) - U(x, t))$$

$$= f(x, t), \qquad \text{a.e. } x \in \Omega, \tag{3.13.a}$$

$$\frac{\partial}{\partial t}\left(\int_{\Omega_x} b(x, z)\, dz\, U(x, t) \right) + \frac{|\Gamma_x|}{\delta}(U(x, t) - u(x, t)) = 0, \qquad \text{a.e. } x \in \Omega, \tag{3.13.b}$$

$$u(x, t) = 0, \qquad \text{a.e. } x \in \Gamma,\ t > 0, \tag{3.13.c}$$

and

$$\lim_{t \to 0} u(x, t) = u_0(x), \qquad \lim_{t \to 0} U(x, t) = \int_{\Omega_x} U_0(x, z)\, dz, \text{ a.e. } x \in \Omega. \tag{3.13.d}$$

The boundary condition (3.13.c) *holds in the sense of trace, i.e., $u(t) \in H_0^1(\Omega)$, and the initial conditions* (3.13.d) *are strong limits in $L^2(\Omega)$.*

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