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Diffusion Models for Fractured Media*

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Two models for diffusion in fractured media are described; the compartment model as an example of a double-porosity system, and the micro-structure model as the limit by homogenization of local flux-coupled classical diffusion models which depend on the geometry. These two models are shown to be examples of a single evolution equation for which the appropriate initial–boundary-value problems are well-posed. This gives a unified theoretical basis for these two (as well as classical diffusion) models in which they can be compared and studied.

1. INTRODUCTION—DIFFUSION MODELS

We begin with a review of certain models for diffusion in fractured media. In Section 2 we present a more recent micro-structure model and describe its derivation by homogenization. The objective is to obtain a model which more accurately portrays the exchange of fluid between the blocks and fractures of the structure and therefore must take account of the geometry of the local structure. In Section 3 initial–boundary-value problems are shown to be well-posed for a single implicit evolution equation which contains all of these diffusion models and thereby establishes a unified theoretical basis for all of them.

The classical equation for miscible displacement in a homogeneous porous medium takes the form

$$\partial_t(\theta v) = \nabla \cdot (D\nabla v - \mathbf{u}v), \tag{1.1}$$

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where

θ = saturation,

v = concentration of the solute,

D = diffusion/dispersion tensor,

$\mathbf{u}$ = Darcy velocity of the fluid.

Formally, the same equation also describes heat transport in porous media. Dropping the convective term and keeping the saturation fixed, one is led to the classical diffusion equation

$$\theta\, \partial_t v = \nabla \cdot (D \nabla v). \tag{1.2}$$

On the other hand, fluid flow through porous media is described by the seepage equation

$$\partial_t(\theta \rho) = \nabla \cdot (\rho K(\nabla p + g\rho e)), \tag{1.3}$$

where

ρ = density of fluid,

K = conductivity of the fluid,

p = pressure in the fluid,

g = gravitational acceleration,

$e = (0, 0, 1)$ = unit vector in vertical direction.

Neglecting the influence of gravity and assuming both fixed saturation and constant compressibility, i.e.,

$$\rho \frac{dp}{d\rho} = c,$$

one arrives again at

$$\theta\, \partial_t \rho = \nabla \cdot (cK \nabla \rho), \tag{1.4}$$

the same form as (1.2). Thus, Eq. (1.2), here called the *classical model*, describes a general diffusion process in homogeneous porous media.

In order to describe fluid flow through a heterogeneous medium consisting of two components, Barenblatt *et al.* [3] introduced a system of two such equations coupled in the form

$$\theta_{11}\, \partial_t v - \theta_{12}\, \partial_t w + \alpha(v - w) = \nabla \cdot (D_1 \nabla v) \tag{1.5a}$$

$$-\theta_{21}\, \partial_t v + \theta_{22}\, \partial_t w - \alpha(v - w) = \nabla \cdot (D_2 \nabla w). \tag{1.5b}$$

Models of this type and modifications thereof are usually called *double porosity models.* One important special case is the *compartment model*

$$\theta_1\, \partial_t v + \theta_0\, \partial_t w = \nabla \cdot (D \nabla v) \tag{1.6a}$$

$$\partial_t w = \alpha(v - w), \tag{1.6b}$$

which is commonly called the *kinetic model* or *first-order rate model.* This results from the essential assumption that the first component is a fracture system sufficiently well-developed that no flow occurs directly between the blocks. the second component, so $D_2 = 0$. (Also, one ignores as usual the effect of fissure pressure on the block porosity, so $\theta_{21} = 0$.) Thus the individual blocks are involved only by way of exchange of fluid with the surrounding fracture system by which they are locally isolated.

A similar set of equations was considered by Deans [9], Coats and Smith [8], Warren and Root [21], and many others in the context of miscible displacements in the following years. A discussion of exact solutions of this compartment model was given in Lindstrom and Narasimham [17]. Van Genuchten and Wierenga [12] studied in detail the sensitivity of parameters and performed careful laboratory experiments and comparisons. Charlaix *et al.* [7] used (1.6) to describe dispersion in glass beads. A special case of (1.6), the *fissured medium model,* was investigated by Böhm and Showalter [6] as a nonlinear form of the system

$$\theta_0\, \partial_t w = \nabla \cdot (D \nabla v), \tag{1.7a}$$

$$\partial_t w = \alpha(v - w). \tag{1.7b}$$

This system results whenever $\theta_1 = 0$ in (1.6), that is, the relative volume of the fissure system is assumed to be so small that one can ignore the storage of fluid in the fissures compared to that in the blocks.

Whereas the coupling in (1.6) models a simple distributed exchange due to pressure difference, Barker [4] introduced a concept that takes into account in more detail the dynamics of the flux exchange on the micro-scale of the individual blocks. Studying miscible displacement, he used what he calls *block-geometry functions.* In their survey paper van Genuchten and Dalton [11] developed the same ideas. On similar lines, Arbogast [1] and Douglas *et al.* [10] studied models for fluid flow with constant compressibility in which they couple the fractures to the blocks via the pressures and the fluxes across the interfaces. Models of this type will here be called *micro-structure models* of diffusion in fractured media. Vogt [20] used the method of homogenization to derive a model for chromotography. Hornung and Jäger [15, 16] used similar techniques for heterogeneous catalysis. In Hornung [14] a mathematical justification and discussion of the micro-structure model for miscible displacement is given. Arbogast *et al.* [2] apply this technique to oil reservoir simulation.

In the following section it will be shown that the micro-structure model can be written in the form

$$\theta_1\,\partial_t v + \theta_0\,\partial_t w = \nabla\cdot(D\nabla v) \tag{1.8a}$$

$$\partial_t w = \tau * \partial_t v, \tag{1.8b}$$

where "$*$" means *convolution* with respect to time, i.e.,

$$(f * g)(t) = \int_0^t f(t-s)\,g(s)\,ds.$$

The *history function* τ: $\mathbb{R}^+ \to \mathbb{R}^+$ is convex, monotone decreasing, and has a singularity at $t=0$; one typical example is

$$\tau(t) = 6\alpha \sum_{k=1}^{\infty} \exp(-\pi^2 k^2 \alpha t). \tag{1.9}$$

Interestingly enough, the compartment model is a special case of (1.8) obtained by setting

$$\tau(t) = \alpha \exp(-\alpha t) \tag{1.10}$$

instead of (1.9). Also one recovers the classical model by formally setting

$$\tau = \text{Dirac measure at } t = 0. \tag{1.11}$$

From this standpoint, all three models are of the same form, namely,

$$\theta_1\,\partial_t v + \theta_0 \tau * \partial_t v = \nabla\cdot(D\nabla v), \tag{1.12}$$

the only difference being the special choice of the history function τ. Section 3 presents a general elementary proof for well-posedness of initial–boundary value problems for the integro-differential equation (1.12) with the additional appropriate initial and boundary conditions. Thus, equations of the form (1.12) include at least all of the diffusion models presented above, each such model corresponding to a choice of the storage history function τ, and so this equation will provide a basis on which to compare these various models.

2. The Micro-Structure Model

We shall briefly describe how the micro-structure model can be obtained by homogenization. In this technique one starts from a *micro-model* and passes to a certain limit which is the *macro-model*. For a discussion of homogenization techniques in general, we refer to Bensoussan, Lions and Papanicolaou [5].

To describe the micro-model we use the following notation in $\mathbb{R}^3$. Denote by $\mathbf{e}_j$ the jth unit vector and the *unit cell* by $U = \{\sum_{j=1}^{3} \lambda_j \mathbf{e}_j : 0 \leqslant \lambda_j \leqslant 1\}$. For any $X \subset U$ we let $X^m = m_j \mathbf{e}_j + X$ be a corresponding integer translate of X by $m = (m_1, m_2, m_3) \in \mathbb{Z}^3$. The geometry of the micro-model is given by specifying a representative block Y_0, a domain whose closure is in the interior of U, and the corresponding fracture $Y_1 \equiv U \sim Y_0$, interface $\Gamma = \partial Y_0$, and unit normal ν on Γ. The porous medium is a bounded domain Ω in $\mathbb{R}^3$. Let the scale parameter $\varepsilon > 0$ be given. Then the geometry of the micro-structure is given by

$$\Omega_1^\varepsilon = \Omega \cap \bigcup \{\varepsilon Y_1^m : m \in \mathbb{Z}^3\},$$

$$\Omega_0^\varepsilon = \Omega \cap \bigcup \{\varepsilon Y_0^m : m \in \mathbb{Z}^3\},$$

which represent the blocks and fractures, the interface $\Gamma^\varepsilon = \Omega \cap \bigcup \{\varepsilon \Gamma^m : m \in \mathbb{Z}^3\}$, and the corresponding inner normal ν^ε on Γ^ε.

The variables in this model will be the density of fluid in the fracture system, $v^\varepsilon : [0, T] \times \Omega_1^\varepsilon \to \mathbb{R}$, and the density in the blocks, $w^\varepsilon : [0, T] \times \Omega_0^\varepsilon \to \mathbb{R}$. The flow characteristics will be described by constants θ_1 and θ_0, the relative pore volume in the fractures and blocks, respectively, and numbers d, $\varepsilon^2 \alpha$, the diffusivity in the fractures and blocks, respectively. With the preceding notation, the *micro-model* is described by the system

$$\theta_1 \, \partial_t v^\varepsilon(t, x) = d \Delta v^\varepsilon(t, x), \qquad x \in \Omega_1^\varepsilon \tag{2.1a}$$

$$w^\varepsilon(t, s) = v^\varepsilon(t, s), \qquad s \in \Gamma^\varepsilon \tag{2.1b}$$

$$\varepsilon^2 \alpha \nu^\varepsilon \cdot \nabla w^\varepsilon(t, s) = d \nu^\varepsilon \cdot \nabla v^\varepsilon(t, s), \qquad s \in \Gamma^\varepsilon \tag{2.1c}$$

$$\theta_0 \, \partial_t w^\varepsilon(t, x) = \varepsilon^2 \alpha \, \Delta w^\varepsilon(t, x), \qquad x \in \Omega_0^\varepsilon \tag{2.1d}$$

for each $t > 0$. Thus the flows in each of the blocks as well as the fracture system are described locally by the classical model of diffusion, and they are connected by the requirements that pressure and flux be matched along the interface.

The direct numerical solution of (2.1) when ε is small is a complicated and ill-conditioned problem. By the homogenization method one obtains a much simpler problem, not necessarily of the same form, which frequently is a simpler and numerically well-conditioned problem. The coefficients are called the *effective parameters*, and this simpler homogenized problem will be a good approximation to the original (2.1). The most important aspect of this method is the explicit analytical construction of the limiting problem.

We shall give the form of this limit by homogenization, or macro-model, so obtained from (2.1). Its coefficients will be constructed from averages of solutions of a pair of *cell problems*. Thus, for each $j = 1, 2, 3$, let $\sigma_j : \mathbb{R}^3 \to \mathbb{R}$ be a U-periodic solution of the problem

$$\begin{aligned} \Delta_y \sigma_j(y) &= 0, && y \in Y_1 \\ \nu \cdot \nabla_y \sigma_j(y) &= -\nu \cdot \mathbf{e}_j, && y \in \Gamma \end{aligned}$$

and define S to be the tensor with coefficients

$$s_{ij} = |Y_1|\, \delta_{ij} + \int_{Y_1} \partial_i \sigma_j(y)\, dy.$$

Here $|Y_1|$ is the measure of Y_1. The tensor D is then defined by $D = dS$. It is easily seen that the tensor S, and thus also D, are symmetric and positive definite. Similarly, let r: $[0, \infty) \times Y_0 \to \mathbb{R}$ be the solution of the cell problem

$$\begin{aligned} \theta_0\, \partial_t r(t, y) &= \alpha\, \Delta_y r(t, y), && y \in Y_0, \\ r(t, s) &= 0, && s \in \Gamma,\ t > 0, \\ r(0, y) &= 1, && y \in Y_0, \end{aligned}$$

and denote its average by

$$\rho(t) \equiv \frac{1}{|Y_0|} \int_{Y_0} r(t, y)\, dy, \qquad t > 0.$$

From this we compute the *history function*

$$\tau(t) \equiv -\frac{d}{dt} \rho(t),$$

which plays a primary role below. For example, if Y_0 is a ball of radius R at the origin, then

$$r(t, y) = \frac{2}{\pi} \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k}\, e^{-k^2\pi^2\alpha t} \sin\left(\frac{k\pi |y|}{R}\right)$$

and so τ is given as in Section 1.

From the preceding we can write out the *macro-model* as follows. If v and w denote the density of fluid in the fracture system and in the block system, respectively, then they are to satisfy

$$\Theta_1\, \partial_t v(t, x) + \Theta_0\, \partial_t w(t, x) = \nabla \cdot (D \nabla v(t, x)), \tag{2.2a}$$

$$\partial_t w(t, x) = (\tau * \partial_t v(\cdot, x))(t), \qquad x \in \Omega,\ t > 0, \tag{2.2b}$$

where $\Theta_j = |Y_j|\, \theta_j$ for $j = 1, 2$, and the indicated convolution is in t only. When both of (2.1) and (2.2) are supplemented with appropriate boundary and initial conditions, one can show the following result.

THEOREM. *The sequence v^ε converges strongly in $L^2(0, T; L^2(\Omega))$ to a solution v of* (2.2).

Here we mean by v^ε the extension onto all Ω which is harmonic in Ω_0^ε. A formal derivation is given in Hornung [14]. For our purposes the following observation is useful:

PROPOSITION. *Let $\omega_k\colon Y_0 \to \mathbb{R}$, $k = 1, 2, \ldots$, be the orthonormal system of eigenfunctions of $-(1/\theta_0)\, \Delta_y$ with eigenvalues μ_k, i.e.,*

$$\begin{cases} -\Delta_y \omega_k(y) = \theta_0 \mu_k \omega_k(y), & y \in Y_0 \\ \omega_k = 0, & y \in \Gamma. \end{cases}$$

Then one has

$$\rho(t) = \sum_k \left(\int_{Y_0} \omega_k(y)\, dy \right)^2 \exp(-\mu_k \alpha t).$$

Proof. One derives easily the representation

$$r(t, y) = \sum_k \int_{Y_0} \omega_k(y)\, dy \exp(-\alpha \mu_k t)\, \omega_k(y)$$

from which the result follows by averaging. ∎

Remark. It follows from the Proposition that the corresponding history function τ, obtained from ρ as above, always has the properties mentioned above for the example (1.9), and we shall exploit these properties in the next section.

3. THE CAUCHY PROBLEM

We shall prove that initial–boundary-value problems are well-posed for the linear functional partial differential equation

$$\frac{\partial v}{\partial t} + \tau * \frac{\partial v}{\partial t} - \nabla \cdot (D \nabla v) = f \quad \text{in} \quad \Omega \times (0, T). \tag{3.1}$$

These will be formulated in Hilbert space as a linear implicit evolution equation for which we shall show that the Cauchy problem is well-posed.

Then (3.1) is recovered by choosing appropriate elliptic operators in Sobolev space.

The following abstract setting suffices for our problem. Let V and H be Hilbert space with V dense in H and assume the embedding $V \hookrightarrow H$ is compact. Let $a(\cdot, \cdot)$ be a continuous symmetric non-negative bilinear form on V such that for each $\varepsilon > 0$ there is a $c_\varepsilon > 0$ for which

$$a(v, v) + \varepsilon \|v\|_H^2 \geq c_\varepsilon \|v\|_V^2, \qquad v \in V.$$

Identify H with its dual space H' by means of the Riesz isomorphism, so we have $V \subset H \subset V'$ by duality and $f(v) = (f, v)_H$ for $v, f \in H$. Define $D(A)$ to be the set of $u \in V$ such that $a(u, \cdot)$ is continuous on V *with the norm of* H; equivalently, there is a unique $Au \in H$ such that

$$a(u, v) = (Au, v)_H, \qquad u \in D(A),\ v \in V.$$

This defines A: $D(A) \to H$ with the following structure: there is an orthonormal basis $\{\varphi_j\}$ of H such that $A\varphi_j = \lambda_j \varphi_j$, $j \geq 1$, and $0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \to \infty$ as $n \to +\infty$. Such structure is standard for elliptic boundary value problems on a bounded domain Ω. The fractional powers A^α are easily defined by this spectral resolution and it follows that $V = D(A^{1/2})$.

Consider the convolution operator

$$(Lu)(t) = (\tau * u)(t) = \int_0^t \tau(t-s)\, u(s)\, ds, \qquad 0 < t < T,\ u \in L^2(0, T).$$

We shall assume $\tau \in L^1(0, T) \cap C^1(0, T)$, $\tau \geq 0$, $\tau' \leq 0$, and τ' is non-decreasing and not a constant; these hypotheses are consistent with the preceding Proposition. It follows from results of MacCamy and Wong [18] that L is a *monotone* continuous linear operator on $L^2(0, T)$, i.e.,

$$\int_0^T (Lu)(t)\, u(t)\, dt \geq 0, \qquad u \in L^2(0, T).$$

Since L is maximal monotone operator it follows that $(I + L)^{-1}$ is a contraction defined on all of $L^2(0, T)$. Finally we note that L is a forward evolution operator in the sense that for each $t \in (0, T)$, the value $u(t)$ is determined by $(I + L)u = v$ and $\{v(s) \colon 0 \leq s \leq t\}$.

In the following we denote by $L^2(0, T; H)$ the space of (equivalence classes of) Bochner square-integrable functions from the real interval $[0, T]$ to the Hilbert space H. Also we let $H^1(0, T; H)$ be the Hilbert space of those absolutely continuous H-valued functions on $[0, T]$ whose derivatives belong to $L^2(0, T; H)$. We shall prove that the *Cauchy problem*

$$(I+L)\frac{dv}{dt}+Av=f \qquad \text{in } L^2(0, T; H) \tag{3.2a}$$

$$v(0)=v_0 \qquad \text{in } H \tag{3.2b}$$

has a unique solution $v \in H^1(0, T; H)$ with data f, v_0 as given. This will be achieved by a standard Galerkin scheme. Thus, we consider the projection of (3.2) onto $\langle \varphi_j \rangle$, namely,

$$(I+L)\,\dot{v}_j+\lambda_j v_j=f_j \qquad \text{in } L^2(0, T) \tag{3.3a}$$

$$v_j(0)=v_0^j, \tag{3.3b}$$

where $f_j(t)=(f(t), \varphi_j)_H$ and $v_0^j=(v_0, \varphi_j)_H$ for $j \geqslant 1$. Then we shall obtain the solution to (3.2) in the form

$$v(t)=\sum_{j=1}^{\infty} v_j(t)\,\varphi_j \text{ in } H, \qquad 0 \leqslant t \leqslant T. \tag{3.4}$$

Note by our earlier remark on the forward evolution property of L that the problem (3.3) can be written in the equivalent form

$$v_j(t)=v_0^j+\int_0^t (I+L)^{-1}\,(f_j-\lambda_j v_j)(s)\,ds, \qquad 0 \leqslant t \leqslant T,$$

even though L is a *non-local* operator.

A fixed-point argument shows that (3.3) has a solution. Suppress the j, let $f \in L^2(0, T)$ and $v_0 \in \mathbb{R}$ be given, and define $\mathscr{F}: L^2(0, T) \to L^2(0, T)$ by

$$\mathscr{F}v(t) \equiv v_0+\int_0^t (I+L)^{-1}\,(f-\lambda v)(s)\,ds, \qquad 0 \leqslant t \leqslant T.$$

Then by Cauchy–Schwartz follows

$$|(\mathscr{F}u-\mathscr{F}v)(t)|^2 \leqslant \lambda^2 t \|u-v\|^2_{L^2(0,t)}, \qquad 0 \leqslant t \leqslant T, \tag{$*$}$$

since $(I+L)^{-1}$ is a contraction on $L^2(0, t)$. Integration of $(*)$ gives

$$\|\mathscr{F}u-\mathscr{F}v\|_{L^2(0,t)} \leqslant \frac{\lambda t}{\sqrt{2}}\,\|u-v\|_{L^2(0,t)}, \qquad 0 \leqslant t \leqslant T,\ u, v \in L^2(0, T).$$

By an induction argument which uses $(*)$, there follows by standard methods the estimate

$$\|\mathscr{F}^n u-\mathscr{F}^n v\|_{L^2(0,t)} \leqslant \left(\frac{\lambda t}{\sqrt{2}}\right)^n \frac{1}{\sqrt{n!}}\,\|u-v\|_{L^2(0,t)},$$
$$0 \leqslant t \leqslant T,\ u, v \in L^2(0, T),$$

for integer $n \geqslant 1$. This shows that for sufficiently large n, $\mathscr{F}^n$ is a strict contraction and, hence, $\mathscr{F}$ has a unique fixed point in $L^2(0, T)$. This fixed point is the desired solution of (3.3).

Let v_j be the solution of (3.3); since $(I+L)^{-1}$ and L are continuous on $L^2(0, T)$ it follows that $v_j \in H^1(0, T)$ and $L\dot{v}_j \in L^2(0, T)$. Since L is monotone on every $L^2(0, t)$, $0 \leqslant t \leqslant T$, it follows by multiplying (3.3) by $\dot{v}_j$ and integrating that

$$\int_0^t |\dot{v}_j|^2 + \frac{\lambda_j}{2} |v_j(t)|^2 \leqslant \frac{\lambda_j}{2} |v_j^0|^2 + \|f_j\|_{L^2(0,t)} \|\dot{v}_j\|_{L^2(0,t)}.$$

This yields the fundamental estimate

$$\|\dot{v}_j\|^2_{L^2(0,t)} + \lambda_j |v_j(t)|^2 \leqslant \lambda_j |v_j^0|^2 + \|f_j\|^2_{L^2(0,t)}, \qquad 0 \leqslant t \leqslant T, \tag{3.5}$$

for any solution of (3.3). If $f \in L^2(0, T; H)$ and $v_0 \in D(A^{1/2})$, that is,

$$\|f\|^2_{L^2(0,T;H)} = \sum_{j=1}^{\infty} \|f_j\|^2_{L^2(0,T)} < \infty, \qquad \|A^{1/2} v_0\|^2_H = \sum_{j=1}^{\infty} \lambda_j |v_j^0|^2 < \infty,$$

then the sequence of partial sums in (3.4) is Cauchy, hence, convergent to $v \in H^1(0, T; H)$ with $A^{1/2} v(\cdot) \in C(0, T; H)$.

THEOREM. *Let the operators A and L be given as above. For each $v_0 \in D(A^{1/2})$ and $f \in L^2(0, T; H)$, there is a unique solution $v \in H^1(0, T; H)$ with $A^{1/2} v(\cdot) \in C(0, T; H)$ of the Cauchy problem* (3.2), *and it satisfies*

$$\left\| \frac{dv}{dt} \right\|_{L^2(0,t;H)} + \|A^{1/2} v(t)\|^2_H \leqslant \|A^{1/2} v_0\|^2_H + \|f\|^2_{L^2(0,T;H)}, \qquad 0 \leqslant t \leqslant T. \tag{3.6}$$

Proof. The existence of such a solution has been shown already. Note that for any solution $v \in H^1(0, T; H)$ of (3.2) it follows that $\dot{v}$ and $Av(\cdot)$ are in $L^2(0, T; H)$, and this implies $A^{1/2} v(\cdot)$ belongs to $C(0, T; H)$. The proof of (3.6) is the same as that of (3.5). From this follow the uniqueness and continuous dependence on data f, v_0 for the Cauchy problem.

Remark. Using the fractional powers A^α, one can show that if $v_0 \in D(A^\alpha)$ and $f \in L^2(0, T; D(A^{\alpha - 1/2}))$, there is a unique solution of (3.2) in $L^2(0, T; D(A^{\alpha - 1/2}))$ with $A^\alpha v(\cdot) \in C(0, T; H)$.

EXAMPLE. Let Ω be a bounded domain in $\mathbb{R}^n$ and $T > 0$. The Sobolev space $H^1(\Omega)$ is the Hilbert space of (equivalence classes of) functions u in $L^2(\Omega)$ for which all generalized derivatives, $\partial_j u$, $1 \leqslant j \leqslant n$, belong to $L^2(\Omega)$. Choose V to be the closed subspace of those $u \in H^1(\Omega)$ whose trace or boundary values are zero. Also we set $H = L^2(\Omega)$; the imbedding $V \hookrightarrow H$ is compact by a classical result of Rellich. Let the matrix $\{d_{ij}: 1 \leqslant i, j \leqslant n\}$ be symmetric and satisfy the *ellipticity* condition

$$\sum_{i,j=1}^{n} d_{ij} \xi_i \xi_j \geqslant d_0 \sum_{j=1}^{n} |\xi_j|^2, \qquad \xi \in \mathbb{R}^n,$$

with $d_0 > 0$, and define

$$a(u, v) = \sum_{i,j=1}^{n} \int_{\Omega} d_{ij}\, \partial_i u(x)\, \partial_j v(x)\, dx, \qquad u, v \in V.$$

This bilinear form on V satisfies the conditions required in the Theorem, and the operator A is determined by

$$Au = -\sum_{i,j=1}^{n} \partial_j(d_{ij}\, \partial_i u), \qquad D(A) = \{u \in V: Au \in L^2(\Omega)\}.$$

Note that such a $u \in D(A)$ necessarily satisfies the generalized homogeneous Dirichlet boundary condition in the sense of trace, and that the value of Au is computed in the sense of distributions on Ω. See Showalter [19] for details on such now standard constructions. Let the function τ be given as above; cf. the example at the end of Section 1. From the Theorem it follows that the initial–boundary-value problem

$$\frac{\partial v(x, t)}{\partial t} + \int_0^t \tau(t-s) \frac{\partial v(x, s)}{\partial t}\, ds - \sum_{i,j=1}^{n} \partial_j(d_{ij}\, \partial_i v(x, t)) = f(x, t), \qquad x \in \Omega, \tag{3.7a}$$

$$v(s, t) = 0, \qquad s \in \partial\Omega, \qquad 0 < t < T, \tag{3.7b}$$

$$v(x, 0) = v_0(x), \qquad x \in \Omega, \tag{3.7c}$$

is well-posed whenever $f \in L^2(\Omega \times (0, T))$ and $v_0 \in V$ are given. In particular, each term in (3.7a) belongs to $L^2(\Omega \times (0, T))$, so at a.e. $t \in (0, T)$ we find $v(\cdot, t) \in D(A)$; when $\partial\Omega$ is smooth this establishes spatial-regularity of the solution.

Remark. The recent very deep results of [13] include Lipschitz perturbations of (3.1) in which A is an *m-accretive* operator in a general Banach space. These are obtained by rather technical and interesting non-standard approximations; additional related references can be found in [13].

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