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An Approximate Scalar Conservation Law from Dynamics of Gas Absorption

R. E. SHOWALTER* AND XIANGSHENG XU

Department of Mathematics, The University of Texas at Austin, Austin, Texas 78712

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The Cauchy problem for

$$\varepsilon u_{xt} + u_t + \alpha(u)_x \ni 0 \qquad (*)$$

is shown to be well-posed in L^1 by semi-group methods. The solution $u^{\varepsilon,\alpha}$ depends continuously on the maximal monotone graph α and on $\varepsilon \geqslant 0$.

1. INTRODUCTION

Our interest in Eq. $(*)$ arose originally from a classical problem in the dynamics of gas absorption. This problem provides a useful conduit through which to introduce our results and to suggest some physical relevance, so we shall briefly recall this problem before describing the results obtained. Consider a cylinder of constant cross section, whose axis is parametrized by $x \in \mathbb{R}$, and which contains an absorbing material. A gas–air mixture is passed at a uniform velocity through the cylinder; we want to describe the exchange of gas between this flow field in the pores and the absorbing material fixed in the cylinder. Let $u(x, t)$ denote the concentration of the gas in the absorbent at position x and time $t > 0$, and let $w(x, t)$ be the concentration of gas moving past x in the system of pores. The essential properties of the absorbent are defined by a relation or function α called the *absorption isotherm.* Thus, $v \in \alpha(u)$ or $[u, v] \in \alpha$ if v is a concentration of gas on the surface of the absorbent which is in equilibrium with the concentration u in its interior. The rate of exchange of gas between the absorbent and its pores is then proportional to $w - v$. The dynamical

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problem that directly results from mass-conservation is of the form

$$\begin{cases} u_t + \frac{1}{\varepsilon}(v - w) = 0, & w_x + \frac{1}{\varepsilon}(w - v) = 0, \\ v \in \alpha(u), & -\infty < x < \infty, \quad t > 0, \end{cases} \tag{1.1}$$

where u_t is the rate at which gas is stored in the absorbent, w_x is the flux due to transport with unit rightward velocity, and the *kinetic coefficient* $1/\varepsilon$ is a measure of the surface area common to pores and absorbent. By eliminating w from (1.1) we formally obtain (*).

We shall show that the Cauchy problem for (1.1) is well-posed in the Banach space $L^1(\mathbb{R})$ by standard methods of nonlinear semigroup theory. This system is essentially equivalent to

$$u_t + \frac{1}{\varepsilon}(I - (I + \varepsilon\,\partial)^{-1})v = 0, \qquad v \in \alpha(u), \tag{1.2}$$

wherein α is a rather general maximal monotone graph and ∂ is an extension of d/dx on an appropriate domain in $L^1(\mathbb{R})$. In (1.2) we recognize $\partial_\varepsilon \equiv (1/\varepsilon)(I - (I + \varepsilon\,\partial)^{-1})$ as the *Yoshida approximation* of ∂; ∂_ε is a bounded operator for which $\partial_\varepsilon \to \partial$ as $\varepsilon \to 0$ in an appropriate sense. This suggests that the solution u^ε of (1.1) should converge to that of the *scalar conservation law*

$$u_t + \partial v = 0, \qquad v \in \alpha(u), \tag{1.3}$$

with the same initial data. The proof of this convergence is our primary goal and it is attained in Section 4. That the Cauchy problem for (1.3) is well-posed in L^1 will be established as a useful preliminary result of incidental interest; the corresponding result for (1.1) is surprisingly somewhat more delicate. For the absorption dynamics problem our convergence result means that the conservation law (1.3) is the limiting form of the model (1.1) as $\varepsilon \to 0$. The parameter $\varepsilon > 0$ is the mean radius of the pore paths with a fixed total cross section area, so (1.3) is the limiting homogeneous model with a continuum of pore paths.

The semigroup treatment of the conservation law is certainly not new; see [2, 10] where α is continuous, strictly monotone, and surjective, but any spatial dimension is permitted, and the relation with Kruzkov's entropy solution is established. Here we treat the case $n = 1$; the simplicity of the method is a measure of the power of the semigroup method. Also we obtain some convergence results when the mass-conservation holds, and this will here be related to the single-valuedness of $\alpha(u)$ at $u = 0$.

Equation (1.2) is easily resolved in any L^p by the Cauchy–Picard

Theorem whenever α is Lipschitz, and in that case it is equivalent to (1.1). This equation has the interesting property of preserving spatial regularity and discontinuities at a point, but these estimates are lost, of course, as $\varepsilon \to 0$. See [14] where this is done in higher dimension and with locally Lipschitz α. If the operator ∂ is replaced by the Laplacian, $-\Delta$, then (1.2) and (1.3) formally become, respectively, the *fissured medium equation* [5] and the *porous medium equation* [3, 4, 16]. See the cited references for analogous existence and continuous dependence results, and [2] for the corresponding convergence with ε. See [15] for the addition of a linear convective term.

The convergence of u^ε to u exhibits the scalar conservation law as a limit of the first-order hyperbolic partial differential equation (1.1) instead of the classical parabolic viscosity approximation. This nonstandard approximation of (1.3) is the basis for some numerical regularization schemes which display substantial promise [8]. Note that (1.2) is *not* the Yoshida approximation of (1.3), since only the linear factor has been regularized. For a discussion of the approaches to (1.3) by calculus of variations and Hamilton–Jacobi theory, finite difference schemes, viscosity methods, and the method of characteristics, see [17].

The semigroup method is to realize the given problem as an abstract Cauchy problem

$$u'(t) + A(u(t)) \ni 0, \qquad t > 0, \quad u(0) = u_0. \tag{1.4}$$

The semigroup theory gives a (generalized) solution to (1.4) from appropriate conditions on the multi-valued operator A in the Banach space X. Sufficient conditions are that A be *accretive*, i.e., if $w_j \in A(u_j)$ for $j = 1, 2$, then for each $\mu > 0$

$$\|u_1 + \mu w_1 - (u_2 + \mu w_2)\| \geqslant \|u_1 - u_2\|,$$

and that A satisfy the range condition, $Rg(I + \mu A) = X$. Then A is called *m-accretive*. We shall construct m-accretive operators A and A^ε on the appropriate domains in $L^1(\mathbb{R})$ which are realizations of $\partial \circ \alpha$ and $\partial_\varepsilon \circ \alpha$, respectively, in Section 2 and Section 4. We refer to [9, 12] for expositions of the semigroup theory and various applications to initial-boundary-value problems for partial differential equations.

Standard notation L^p, $W^{m,p}$, will be used for the Lebesgue and Sobolev spaces; each is a Banach space with its usual norm. For $-\infty \leqslant a < b \leqslant +\infty$ we denote by $C[a, b]$ the space of bounded uniformly continuous real-valued functions on the interval $[a, b]$, and by $C_0[a, b]$ those which vanish at a. For the case $a = -\infty$ which occurs frequently below we use the notations $L^p(b) = L^p(-\infty, b)$, $C_0(b) = C_0[-\infty, b]$, and so on. The pre-compact subsets of each of the above spaces are characterized by the classical theorems of Ascoli–Arzela and Kondrachov and their variants [1].

Whether an operator A in $L^1(a, b)$ is accretive can be characterized by means of the $L^1 - L^\infty$ duality map involving the graph *sgn*, defined by $\operatorname{sgn}(x) = \{x/|x|\}$ for $x \neq 0$ and $\operatorname{sgn}(0) = [-1, 1]$. Thus, A is accretive in $L^1(a, b)$ if and only if for each pair $f_j \in A(u_j)$ for $j = 1, 2$ there is a measurable selection $\sigma \in \operatorname{sgn}(u_1 - u_2)$ such that $\int_a^b (f_1 - f_2)\sigma \geqslant 0$. Similarly, we shall use the graphs $\operatorname{sgn}^+(x) = \frac{1}{2}(1 + \operatorname{sgn}(x))$ and $\operatorname{sgn}^-(x) = \frac{1}{2}(\operatorname{sgn}(x) - 1)$, and we shall denote by $x^+ = \operatorname{sgn}^+(x) \cdot x$, $x^- = \operatorname{sgn}^-(x) \cdot x$ the positive and negative parts of $x \in \mathbb{R}$. Finally, the *minimal section* of a maximal monotone graph α is denoted by α_0, so we denote by sgn_0 and sgn_0^+ those functions which agree with sgn and sgn^+, respectively, at each $x \neq 0$ and for which $\operatorname{sgn}_0(0) = \operatorname{sgn}_0^+(0) = \{0\}$. See [6] for information on maximal monotone graphs in $\mathbb{R}$.

Standard results on the convergence of m-accretive operators A_n to another such A_∞ will be used. Specifically, we say $A_n \to A_\infty$ if $\lim_{n \to \infty}(I + \mu A_n)^{-1} f = (I + \mu A_\infty)^{-1} f$ for each $f \in X$ and $\mu > 0$. This is easily seen to be equivalent to the property that for each $[u, f] \in A_\infty$ there is a sequence $[u_n, f_n] \in A_n$ such that $u_n \to u$ and $f_n \to f$. That is, *resolvent convergence* and *graph convergence* are equivalent for m-accretive operators. A useful consequence is the following.

LEMMA 0. *Let α, α_n be maximal monotone graphs in $\mathbb{R}$ and assume $\alpha_n \to \alpha$. If $u_n \to u$, $v_n \to v$, and $v_n \in \alpha_n(u_n)$ in $L^1(a, b)$, then $v \in \alpha(u)$.*

Proof. Let $v^* \in \alpha(u^*)$ in $L^1(a, b)$; then there is a sequence $v_n^* \in \alpha_n(u_n^*)$ with $v_n^* \to v^*$ and $u_n^* \to u^*$ in $L^1(a, b)$. Since $\int_a^b (v_n - v_n^*) \operatorname{sgn}_\varepsilon(u_n - u_n^*) \geqslant 0$ by the monotonicity of α_n and the Yoshida approximation $\operatorname{sgn}_\varepsilon$, $\varepsilon > 0$, by letting $n \to \infty$ and then $\varepsilon \to 0$ we obtain $\int_a^b (v - v^*) \operatorname{sgn}_0(u - u^*) \geqslant 0$ for all such $[u^*, v^*] \in \alpha$. This implies $[u, v] \in \alpha$ by the maximality of α in $L^1(a, b)$.

2. $d/dx \circ \alpha$ IN L^1

Assume α is a maximal monotone graph in $\mathbb{R}$ with $0 \in \alpha(0)$. We consider the following realization of $d/dx \circ \alpha$ in $L^1(b)$ for each $b \leqslant +\infty$.

DEFINITION. $w \in A(u)$ if $w, u \in L^1(b)$ and there exists $v \in L^1_{\text{loc}}(b)$ with $v_x = w$ a.e. on $(-\infty, b)$, $v(x) \in \alpha(u(x))$ a.e. $x < b$, and $v(-\infty) = 0$.

Note that any distribution on $\mathbb{R}$ whose derivative is in L^1_{loc} is necessarily an absolutely continuous function. Thus, it is clear that $w \in A(u)$ if and only if $u, w \in L^1(b)$ and $v(x) \equiv \int_{-\infty}^x w \in \alpha(u(x))$, a.e. $x < b$. In particular, $v \in C_0(b)$. We shall prove the following.

THEOREM 1. *The operator A is m-accretive on $L^1(b)$ for each $b \leqslant +\infty$. If $\alpha(0) = \{0\}$ then $w \in A(u)$ if and only if there exists $v \in L^1_{\mathrm{loc}}(b)$ with $v_x = w$ a.e. on $(-\infty, b)$ and $v(x) \in \alpha(u(x))$, a.e. $x < b$. If, in addition, $b = +\infty$, then $v(+\infty) = \int_{-\infty}^{+\infty} w = 0$.*

For the proof we begin with the following calculus result. Let $D = d/dx$ on distributions.

LEMMA 1. *If $v \in L^\infty(a, b)$, $Dv \in L^1(a, b)$, and the measurable $\sigma \in \operatorname{sgn}^+(v)$ a.e., then*

$$Dv^+ = \sigma \cdot Dv \qquad \text{a.e.} \quad x \in (a, b).$$

Proof. A standard result is that $Dv^+ = \operatorname{sgn}_0^+(v) \cdot Dv$ a.e.; see [11, 13] for example. If $x \in \Omega = \{x \in (a, b) : v(x) \neq 0\}$, then $\sigma(x) = \operatorname{sgn}_0^+(v(x))$. The complement of Ω consists of its accumulation points x, at which $v(x) = v^+(x) = 0$ and $Dv(x) = Dv^+(x) = 0$, and the isolated points which are countable.

PROPOSITION 1. *A is accretive on $L^1(b)$.*

Proof. Let $\mu > 0$ and $f_j \in (I + \mu A)(u_j)$ with $v_j \in \alpha(u_j)$ as above for $j = 1, 2$. Choose $\sigma(x) = \operatorname{sgn}_0^+(u_1(x) - u_2(x) + v_1(x) - v_2(x))$, $x < b$. Since $u_j + \mu\, Dv_j = f_j$, $j = 1, 2$, and $0 \leqslant \sigma \leqslant 1$, it follows that

$$\sigma(u_1 - u_2) + \mu\sigma\, D(v_1 - v_2) \leqslant (f_1 - f_2)^+, \qquad \text{a.e.} \tag{2.1}$$

The graph α is monotone, $\sigma \in \operatorname{sgn}^+(v_1 - v_2)$ a.e., so Lemma 1 shows $\sigma\, D(v_1 - v_2) = D(v_1 - v_2)^+$ a.e. Likewise, $\sigma \in \operatorname{sgn}^+(u_1 - u_2)$, so $\sigma(u_1 - u_2) = (u_1 - u_2)^+$. Finally, $v_1(-\infty) = v_2(-\infty) = 0$ by the definition of A, so an integration of (2.1) yields

$$\int_{-\infty}^{x} (u_1 - u_2)^+ + \mu(v_1(x) - v_2(x))^+ \leqslant \int_{-\infty}^{x} (f_1 - f_2)^+, \qquad x \leqslant b. \tag{2.2}$$

The corresponding estimate holds for the negative part of the differences, so by addition of these we obtain

$$\int_{-\infty}^{x} |u_1 - u_2| + \mu\, |v_1(x) - v_2(x)| \leqslant \int_{-\infty}^{x} |f_1 - f_2|, \qquad x \leqslant b. \tag{2.3}$$

This shows A is accretive; in fact we have

$$\|u_1 - u_2\|_{L^1(b)}, \qquad \mu\, \|v_1 - v_2\|_{C_0(b)} \leqslant \|f_1 - f_2\|_{L^1(b)} \tag{2.4}$$

whenever $\mu > 0$ and $f_j \in (I + \mu A)(u_j)$, $j = 1, 2$.

COROLLARY 1. *The operator A is closed, $Rg(I+\mu A)$ is closed for $\mu>0$, and whenever*

$$f=u+\mu w, \qquad w\in A(u)$$

we have the estimates

$$\|u\|_{L^1(b)}, \qquad \mu\|v\|_{C_0(b)}\leqslant \|f\|_{L^1(b)},$$
$$\|u^+\|_{L^p(b)}\leqslant \|f^+\|_{L^p(b)}, \qquad 1\leqslant p\leqslant +\infty,$$
$$\|v^+\|_{C_0(b)}\leqslant \alpha_0(\|f^+\|_{L^\infty(b)}).$$

Proof. Let $w_n\in A(u_n)$ and $w_n\to w$, $u_n\to u$ in $L^1(b)$. Then $v_n(x)=\int_{-\infty}^x w_n$ converges in $C_0(b)$ to $v(x)=\int_{-\infty}^x w$. Since α is maximal monotone and, for some subsequence, $u_{n'}(x)\to u(x)$ a.e., it follows $v(x)\in\alpha(u(x))$ a.e., so $w\in A(u)$. Thus A is closed, and the closedness of the range of $I+\mu A$ now follows from (2.4).

The first desired estimate follows from (2.4), since $0\in\alpha(0)$, and the second with $p=1$ likewise from (2.2). For $1<p<\infty$, note first that

$$u^p \operatorname{sgn}^+ u+v_x u^{p-1}\operatorname{sgn}^+ u\leqslant f^+u^{p-1}\operatorname{sgn}^+ u, \qquad \text{a.e.}$$

and that

$$\int_{-\infty}^b u^p\operatorname{sgn}^+ u=\|u^+\|^p_{L^p(b)}, \qquad \int_{-\infty}^b f^+u^{p-1}\operatorname{sgn}^+ u\leqslant \|f^+\|_{L^p(b)}\,\|u^+\|^{p-1}_{L^p(b)}.$$

Since $\sigma(r)=r^{p-1}\operatorname{sgn}^+ r$ is single-valued, the composition $\sigma\circ\alpha^{-1}$ is maximal monotone, and so there is a convex lower-semi-continuous primitive $j\colon\mathbb{R}\to\mathbb{R}^+$. That is, $\partial j=\sigma\circ\alpha^{-1}$ with $j(0)=0$. Then

$$v_x u^{p-1}\operatorname{sgn}^+ u=v_x\,\partial j(v(x))$$

so we obtain

$$\int_a^b v_x u^{p-1}\operatorname{sgn}^+ u=j(v(b))-j(v(a))\geqslant 0-\int_0^{v(a)}(\sigma\circ\alpha^{-1})_0, \qquad 0<a<b.$$

Letting $a\to-\infty$ we obtain from $v\in C_0(b)$

$$\int_{-\infty}^b v_x u^{p-1}\operatorname{sgn}^+ u\geqslant 0,$$

so the case $1<p<+\infty$ is established. The case $p=+\infty$ follows from this, but we obtain it directly in the following.

Suppose now that $B\equiv\|f^+\|_{L^\infty(b)}<\infty$. If $u(x)<B$, a.e. $x<b$, then we are

done. Otherwise, $B \in \text{dom}(\alpha)$ and

$$(u(x)-B)\,\sigma(x)+\mu(v(x)-\alpha_0(B))_x\,\sigma(x)\leqslant (f^+(x)-B)\,\sigma(x)\leqslant 0,$$

where $\sigma(x)\equiv \text{sgn}_0^+(u(x)-B+v(x)-\alpha_0(B))$. Since α is monotone we have $\sigma(x)$ belonging to $\text{sgn}^+(u(x)-B)$ and to $\text{sgn}^+(v(x)-\alpha_0(B))$ at a.e. x, so

$$(u(x)-B)^+ +\mu\, D(v(x)-\alpha_0(B))^+ \leqslant 0, \qquad \text{a.e.}$$

From this follows

$$\int_{-\infty}^{x} (u-B)^+ +\mu(v(x)-\alpha_0(B))^+ \leqslant \mu(-\alpha_0(B))^+ = 0.$$

so we obtain

$$\begin{aligned} u(x)&\leqslant B, &&\text{a.e.}\quad x\leqslant b,\\ v(x)&\leqslant \alpha_0(B), &&\text{all}\quad x\leqslant b. \end{aligned}$$

Additional information on the domain and range of A follows from the next result.

LEMMA 2. *Suppose* $\alpha(0)=[r,s]\ni 0$. *If* $u\in L^1(b)$, $v\in L^1_{\text{loc}}(b)$, $v(x)\in \alpha(u(x))$ *a.e.* $x<b$, *and* $\lim_{x\to-\infty} v(x)=R$, *then* $R\in[r,s]$.

Proof. Assume $R<r$. Since R belongs to the closure of $Rg(\alpha)$, $(R,r)\subset Rg(\alpha)$ and so there is an $\xi\in(R,r)$ and an $\eta<0$ for which $\xi\in\alpha(\eta)$. Then for sufficiently negative x, $v(x)<\xi$ and so $u(x)\leqslant\eta<0$. This contradicts $u\in L^1(b)$, so we have $R\geqslant r$. Similarly we obtain $R\leqslant s$.

COROLLARY 2. *Assume* $\alpha(0)=\{0\}$. *Then* $w\in A(u)$ *if and only if* $u, w\in L^1(b)$ *and there exists* $v\in L^1_{\text{loc}}(b)$ *with* $Dv=w$ *and* $v\in\alpha(u)$ *a.e. on* $(-\infty,b)$. *In this case,* $v(x)=\int_{-\infty}^{x} w$, $x\leqslant b$. *If, in addition,* $b=+\infty$, *then* $v(\infty)=\int_{-\infty}^{\infty} w=0$.

The point of Corollary 2 is that the characterization of the domain can be simplified and the range is similarly delimited when α is single-valued at the origin. This condition on α is appropriate.

EXAMPLE. Let $\alpha=\text{sgn}^+$ and consider $f\in(I+A)(u)$. That is, $u, f\in L^1(\mathbb{R})$, $u+Dv=f$ and $v\in\text{sgn}^+(u)$ a.e., $v(-\infty)=0$. If $f\geqslant 0$ and $\int_{-\infty}^{\infty} f\leqslant 1$ then $u=0$, $v(x)=\int_{-\infty}^{x} f$, gives the solution. It is *not* necessary for $\int_{-\infty}^{\infty} f=0$. If we delete the condition "$v(-\infty)=0$," then a second solution, $u=f$, $v=1$, is obtained, and this nonuniqueness shows the corresponding operator is not accretive. In fact, both u and v are nonunique.

It remains to prove *existence* of

$$w \in A(u), \qquad u + w = f \tag{2.5}$$

for $f \in L^1(b)$. Since A is closed it suffices to consider $f \in L^1$ with bounded support. We shall choose an approximation α^λ of α and solve

$$u_\lambda + D\alpha^\lambda(u_\lambda) = f \quad \text{in} \quad L^1(b) \tag{2.6}$$

for u_λ, $\lambda > 0$, then show there is a limit, $u = \lim_{\lambda \to 0} u_\lambda$, which is the solution of (2.5).

The Yoshida approximation $\alpha_\lambda = (1/\lambda)(I - (I + \lambda\alpha)^{-1})$, characterized by $y = \alpha_\lambda(x)$ iff $y \in \alpha(x - \lambda y)$, is monotone and Lipschitz on $\mathbb{R}$. We shall instead use $\alpha^\lambda \equiv \lambda I + \alpha_\lambda$ which is additionally strictly monotone and surjective. It is characterized by $y^* = \alpha^\lambda(x) = \lambda x + y$ iff $y^* - \lambda x \in \alpha(x - \lambda(y^* - \lambda x))$.

LEMMA. *$\alpha^\lambda \to \alpha$ in $\mathbb{R}$.*

Proof. Set $y_\lambda \equiv (I + \alpha^\lambda)^{-1} x$ so we have

$$x - y_\lambda - \lambda y_\lambda \in \alpha(y_\lambda - \lambda(x - y_\lambda)), \qquad \lambda > 0.$$

Then, $|y_\lambda| \leqslant |x|$, so a subsequence $y_{\lambda'}$ converges to $y \in \mathbb{R}$, and this satisfies $x - y \in \alpha(y)$, hence $y = (I + \alpha)^{-1} x$. But such a y is unique, so $\lim_{\lambda \to 0} y_\lambda = y$.

For each $\lambda > 0$, the ordinary differential equation (2.6) clearly has a solution $u_\lambda \in L^1(b)$; note that α^λ is Lipschitz, f vanishes near $-\infty$, and we have the estimates of Corollary 1. Denote by τ_h the *translation* operator, $\tau_h v(x) = v(x - h)$, $h > 0$. Since $\tau_h u_\lambda$ satisfies (2.6) with f replaced by $\tau_h f$, we obtain with $w_\lambda = \alpha^\lambda(u_\lambda)$

$$\|u_\lambda\|_{L^1(a)},\ \|w_\lambda\|_{C_0(a)} \leqslant \|f\|_{L^1(a)}, \qquad -\infty < a \leqslant b,$$

$$\|\tau_h u_\lambda - u_\lambda\|_{L^1(b)}, \qquad \|\tau_h w_\lambda - \mathrm{w}_\lambda\|_{C_0(b)} \leqslant \|\tau_h f - f\|_{L^1(b)}.$$

Suppose that $b < +\infty$. These estimates imply that $\{u_\lambda\}$ is pre-compact in $L^1(b)$ and that $\{w_\lambda\}$ is pre-compact in $C_0(b)$, hence, there is a subsequence, which we denote again by $\{u_\lambda\}$, which converges to some u in $L^1(b)$. The corresponding sequence $\{w_\lambda\}$ satisfies

$$w_\lambda(x) = \int_{-\infty}^{x} (f - u_\lambda), \qquad -\infty < x \leqslant b$$

and converges to some w in $C_0(b)$. From Lemma 0 it follows that $w \in \alpha(u)$ a.e. on $(-\infty, b)$ and we clearly have $w(x) = \int_{-\infty}^{x} (f-u)$, so $u + A(u) \ni f$. In the case $b = +\infty$ we obtain a solution $u \in L^1(\mathbb{R})$ from above, but the convergence is in $L^1(b)$ for every $b < +\infty$.

We summarize our results for later reference. If α is maximal monotone on $\mathbb{R}$, $0 \in \alpha(0)$, and $b \leqslant +\infty$, then the operator A is m-accretive in $L^1(b)$ and the *resolvents*, $J_\mu = (I + \mu A)^{-1}$, $\mu > 0$, satisfy the following:

$$J_\mu \text{ commutes with translation } \tau_h,\ h > 0, \text{ and}$$

$$\|(J_\mu f_1 - J_\mu f_2)^+\|_{L^1(b)} \leqslant \|(f_1 - f_2)^+\|_{L^1(b)}, \qquad f_1, f_2 \in L^1(b); \tag{2.7a}$$

$$\|J_\mu f\|_{L^p} \leqslant \|f\|_{L^p}, \qquad f \in L^p(b), \qquad 1 \leqslant p \leqslant +\infty,$$

$$-\|f^-\|_{L^\infty(b)} \leqslant J_\mu f(x) \leqslant \|f^+\|_{L^\infty(b)}, \qquad \text{a.e. } x; \tag{2.7b}$$

if also $\alpha(0) = \{0\}$ and $b = +\infty$, then

$$\int_{\mathbb{R}} J_\mu f = \int_{\mathbb{R}} f, \qquad f \in L^1(\mathbb{R}).$$

3. Continuous Dependence and the Evolution

We recall certain facts from the nonlinear semigroup theory which formally provides a solution to the abstract Cauchy problem (1.4).

THEOREM A. *Let A be m-accretive in the Banach space X. Thus, for each $\varepsilon > 0$ and $u_0 \in X$ there is a unique u_ε: $[0, \infty) \to X$ for which*

$$\frac{1}{\varepsilon}(u_\varepsilon(t) - u_\varepsilon(t-\varepsilon)) + A(u_\varepsilon(t)) \ni 0, \qquad t > 0$$

$$u_\varepsilon(t) = u_0, \qquad t \leqslant 0.$$

If $u_0 \in \overline{D(A)}$ then $u(t) = \lim_{\varepsilon \to 0} u_\varepsilon(t)$ exists, with uniform convergence for t bounded.

This limit provides a generalized notion of a solution of (1.4) known as the *integral solution* [2]. This integral solution of (1.4) is unique; the following perturbation theorem shows that it depends continuously on the operator A.

THEOREM B. *Let A_n be m-accretive on X and $u_0^n \in \overline{D(A_n)}$ for $n = 1, 2, \ldots, +\infty$. Assume $u_0^n \to u_0^\infty$ in X and that $A_n \to A_\infty$ as graphs, (cf.*

Section 1). *Then the corresponding integral solutions* u_n *of*

$$u_n'(t) + A_n(u_n(t)) \ni 0, \qquad t > 0, \quad u_n(0) = u_0^n \tag{3.1}$$

satisfy $\lim_{n\to\infty} u_n(t) = u_\infty(t)$, *with uniform convergence for* t *bounded.*

From the preceding results it is natural to define a *generalized solution* of (1.3) with $u(\cdot, 0) = u_0$ in $L^1(b)$ to be the integral solution of (1.4) in the situation of Theorem 1. When α is smooth this agrees with the development in [2, 10]; in the general case we are interested in the continuous dependence on α.

THEOREM 2. *For each* $n = 1, 2, ..., +\infty$, *assume* α_n *is a maximal monotone graph in* $\mathbb{R}$ *with* $0 \in \alpha_n(0)$, $n = 1, 2, ..., \infty$, *and assume* $\alpha_\infty(0) = \{0\}$ *or* $b < +\infty$. *Let* A_n *be the m-accretive operator on* $L^1(b)$ *constructed from* α_n *as in Section 2. If* $\alpha_n \to \alpha_\infty$ *in* $\mathbb{R}$, *then* $A_n \to A_\infty$ *in* $L^1(b)$.

Proof. If the operator A is constructed from the graph α as above, then μA is the operator obtained from $\mu\alpha$ for each $\mu > 0$. Thus, to show resolvent convergence of A_n to A_∞ it suffices to show $\lim_{n\to\infty}(I + A_n)^{-1} f = (I + A_\infty)^{-1} f$ for each $f \in L^1(b)$. For each $n = 1, 2, ..., \infty$, we define $u_n = (I + A_n)^{-1} f$. Then from (2.7a) we obtain

$$\int_{-\infty}^{R} |u_n| \leqslant \int_{-\infty}^{R} |f|, \qquad -\infty < R \leqslant b, \tag{3.2a}$$

$$\|\tau_h u_n - u_n\|_{L^1(b)} \leqslant \|\tau_h f - f\|_{L^1(b)}, \qquad h > 0. \tag{3.2b}$$

Suppose $b < +\infty$, and let $[u_n, v_n, w_n]$ be the solution: $u_n + w_n = f$, $v_n(x) = \int_{-\infty}^{x} w_n \in \alpha_n(u_n(x))$, a.e. $x < b$. Since $\{u_n\}$ is pre-compact in $L^1(b)$, so also is $\{w_n\}$, so for some subsequence, $u_{n'} \to u$, $w_{n'} \to w$ in $L^1(b)$ and $v_{n'} \to v$ in $C_0(b)$. From Lemma 0 we obtain $[u, v, w] = [u_\infty, v_\infty, w_\infty]$; by uniqueness of this limit it follows that $\lim_{n\to\infty} u_n = u_\infty$ in $L^1(b)$.

Consider the case $b = +\infty$. In order to establish that $\{u_n\}$ is convergent, it suffices to show that it is pre-compact in $L^1(\mathbb{R})$, and so in addition to (3.2) we need only to establish

$$\lim_{R\to\infty} \int_{R}^{\infty} |u_n| = 0, \qquad \text{uniformly in } n.$$

From (2.7a) we obtain the pointwise estimates

$$0 \leqslant u_n^+ \leqslant (I + A_n)^{-1}(f^+), \qquad (I + A_n)^{-1}(f^-) \leqslant u_n^- \leqslant 0.$$

These yield

$$0 \leqslant \int_R^\infty u_n^+ \leqslant \int_R^\infty (I+A_n)^{-1}(f^+), \qquad \int_R^\infty (I+A_n)^{-1}(f^-) \leqslant \int_R^\infty u_n^- \leqslant 0.$$

Since we have $\int_R^\infty |u_n| = \int_R^\infty (u_n^+ - u_n^-)$, it suffices to consider the case $f \geqslant 0$. But then $u_n \geqslant 0$ by (2.7a), and so $u_n + u_\infty - |u_n - u_\infty| \geqslant 0$. Since $u_n(x) \to u_\infty(x)$, a.e. $x \in \mathbb{R}$, by the case $b < \infty$ above, we obtain from Fatou's Theorem and (2.7b)

$$\begin{aligned} 2\int_{-\infty}^\infty u_\infty &\leqslant \liminf_{n\to\infty} \int_{-\infty}^\infty (u_n + u_\infty - |u_n - u_\infty|) \\ &\leqslant \int_{-\infty}^\infty f + \int_{-\infty}^\infty u_\infty - \limsup_{n\to\infty} \int_{-\infty}^\infty |u_n - u_\infty|. \end{aligned}$$

Thus

$$\limsup_{n\to\infty} \|u_n - u_\infty\|_{L^1(\mathbb{R})} \leqslant \int_{-\infty}^\infty f - \int_{-\infty}^\infty u_\infty,$$

and the right side vanishes by (2.7c), so we are done.

The condition that α need be single-valued at the origin cannot be deleted from Theorem 2.

EXAMPLE. Let $\alpha_\infty = \mathrm{sgn}^+$ and denote its Yoshida approximations by $\alpha_n = n(I - (I + (1/n)\alpha)^{-1})$. Let $f \in L^1(\mathbb{R})$ with $0 \leqslant f$, $0 < \int f \leqslant 1$, and consider the problems in $L^1(\mathbb{R})$.

$$u_n + D\alpha_n(u_n) = f, \qquad n \geqslant 1, \tag{3.3}$$

$$u + Dv = f, \qquad v \in \alpha_\infty(u). \tag{3.4}$$

The solution of (3.4) is $u = 0$, $v(x) = \int_{-\infty}^x f$. If the solution u_n of (3.3) converges to 0 in $L^1(\mathbb{R})$, then

$$\lim_{R\to+\infty} \alpha_n(u_n)(R) = \lim_{R\to+\infty} \int_R^{+\infty} u_n = 0, \qquad \text{uniformly in} \quad n \geqslant 1,$$

and, hence, $\lim_{R\to+\infty} v(R) = \lim_{R\to+\infty} \int_{-\infty}^R f = 0$, a contradiction. Note, however, that $u_n \to 0$ in $L^1(b)$ for every $b < \infty$. It is easy and instructive to compute u_n explicitly in this example.

Finally, we collect and summarize various properties of the generalized solution. These are a direct consequence of corresponding properties of the resolvents J_ε of the operator A as summarized in (2.7), and the generation

Theorem A in which

$$u_\varepsilon(t)=J_\varepsilon^{[t/\varepsilon]+1}u_0, \qquad \varepsilon>0,$$

where $[t/\varepsilon]$ is the greatest integer in t/ε.

PROPOSITION 2. *Assume α is maximal monotone on $\mathbb{R}$, $0\in\alpha(0)$, and $b\leqslant+\infty$; let A be the operator constructed in Section 2. For $u_0, v_0\in\overline{D(A)}$, denote by $u, v\in C([0,\infty), L^1(b))$ the respective generalized solutions of* (1.3) *with $u(\cdot,0)=u_0$ and $v(\cdot,0)=v_0$. Then we have the following*:

(a) $\|(u(t)-v(t))^+\|_{L^1(b)}\leqslant\|(u_0-v_0)^+\|_{L^1(b)}$, $t\geqslant 0$

and $v(t)=\tau_h u(t)$ if $v_0=\tau_h u_0$ for $h>0$;

(b) $\|u(t)\|_{L^p(b)}\leqslant\|u_0\|_{L^p(b)}$, $t\geqslant 0$ *if* $1\leqslant p\leqslant+\infty$,

and a.e. on $(-\infty,b)\times(0,\infty)$ we have

$$-\|u_0^-\|_{L^\infty(b)}\leqslant u(x,t)\leqslant\|u_0^+\|_{L^\infty(b)};$$

(c) *If, in addition, $\alpha(0)=\{0\}$ and $b=+\infty$, then*

$$\int_{\mathbb{R}}u(t)=\int_{\mathbb{R}}u_0, \qquad t\geqslant 0.$$

(d) *Assume the situation of Theorem 2 holds, and for each $n=1,2,...,+\infty$, let u^n denote the generalized solution of*

$$u_t^n+Dv^n=0, \qquad v^n\in\alpha_n(u^n), \qquad \textit{in } L^1(b), \qquad t>0$$

with $u^n(0)=u_0^n\in\overline{D(A_n)}$, and assume $u_0^n\to u_0^\infty$ as $n\to\infty$. Then $u^n\to u^\infty$ in $C([0,T],L^1(b))$ for every $0<T<+\infty$.

Remark. Let α be a continuous function on $\mathbb{R}$ with $\alpha(0)=0$. Define the operator B_0 in $L^1(\mathbb{R})$ by $w\in B_0(v)$ if $v, w\in L^1(\mathbb{R})$, $\alpha(v)\in L^1_{\text{loc}}$ and

$$\int\operatorname{sgn}_0(v-k)\{(\alpha(v(x))-\alpha(k))f_x(x)+w(x)f(x)\}\,dx\geqslant 0$$

for every $f\in C_0^\infty(\mathbb{R})$ for which $f\geqslant 0$ and every $k\in\mathbb{R}$. Let B be the closure of B_0 in $L^1(\mathbb{R})$. That B is an m-accretive operator in $L^1(\mathbb{R})$ is an important contribution of [10] where it is also shown that

$$-\|h^-\|_{L^\infty}\leqslant(I+\varepsilon B)^{-1}h(x)\leqslant\|h^+\|_{L^\infty} \qquad \text{a.e.}$$

for $h\in L^1\cap L^\infty$. By employing the same ideas of the proof of Theorem 2, one can show that such results hold for B and, hence, $\overline{D(B)}=L^1(R)$ via a technique of [3]. The problem of showing $\overline{D(B)}=L^1(R)$ is left open in [10] and is claimed in [9] to be solved by Benilan in an unpublished paper. One of the implications of this result is that for any initial data $u_0 \in L^\infty \cap L^1$ the semigroup generated by B provides a solution of the conservation law in the sense of Kruzkov. But we know that $B_0 u = \{\alpha(u)_x\}$ for $u \in D(B_0) \cap L^\infty(R)$, so the Kruzkov solution agrees with our generalized solution for all $u_0 \in L^1 \cap L^\infty$. If α is not continuous the definition of B_0 is no longer meaningful. The preceding development can be regarded as assigning a meaning to the Cauchy problem

$$\begin{aligned} u_t + \alpha(u)_x &\ni 0, \\ u(x, 0) &= u_0 \end{aligned} \tag{3.5}$$

and solving the resulting problem at the same time. Finally, the solution of (3.5) in the sense of Kruzkov is always unique provided α is continuous and $\alpha(0) = 0$.

4. $(d/dx)_\varepsilon \circ \alpha$ IN L^1

We begin with some suggestive remarks on the system (1.1). For this to be resolved as an abstract Cauchy problem of the form (1.4) we would need to show that the stationary problem

$$\begin{aligned} u + w_x = f \qquad &\text{in } L^1(b), \qquad w(-\infty) = 0, \\ v = w + \varepsilon w_x \in \alpha(u) \qquad &\text{a.e.} \quad x < b \end{aligned} \tag{4.1}$$

has a solution. Implicit in (4.1) is the condition $w_x \in L^1(b)$. Assume for the moment that $v \in L^1(b)$, hence $w \in W^{1,1}(b)$, and denote by ∂ the L^1 realization of d/dx with domain $\{w \in W^{1,1}(b) \colon w(-\infty) = 0\}$. Then (4.1) is given by

$$u + \partial(I + \varepsilon\,\partial)^{-1} v = f \qquad \text{in } L^1(b), \qquad v \in \alpha(u) \text{ a.e. } x < b. \tag{4.2}$$

The resolvent of ∂ is given by

$$(I + \varepsilon\,\partial)^{-1} v(x) = \frac{1}{\varepsilon} \int_{-\infty}^{x} \exp\left(\frac{s - x}{\varepsilon}\right) v(s)\, ds, \qquad x < b,$$

and for such operators we have the following fundamental results from [7].

THEOREM C. *Let L be a densely-defined linear m-accretive operator on $L^1(b)$. Then L satisfies*

$$\|[(I + \varepsilon L)^{-1} v]^+\|_{L^\infty(b)} \leq \|v^+\|_{L^\infty(b)}, \qquad v \in L^1(b), \quad \varepsilon > 0$$

if and only if for every maximal monotone graph β with $[0, 0] \in \beta \subset \mathbb{R} \times \mathbb{R}$, if $Lu \in L^p(b)$, $v \in L^{p'}(b)$, $v(x) \in \beta(u(x))$ a.e., then

$$\int_{-\infty}^{b} Lu(x) \cdot v(x)\, dx \geqslant 0.$$

Furthermore, if L is such an operator as above, then so also is $\lambda I + L_\varepsilon$ for every $\lambda \geqslant 0$, $\varepsilon \geqslant 0$, where $L_\varepsilon = (1/\varepsilon)(I - (I + \varepsilon L)^{-1})$ for $\varepsilon > 0$. Finally, if L is such an operator and if α is a maximal monotone graph with $0 \in \alpha(0)$, then $L \circ \alpha$ is accretive, and $(\lambda I + L) \circ \alpha$ is m-accretive in $L^1(b)$ for each $\lambda > 0$.

This abstract result fails to resolve (4.1) in two ways. First, the operator $\partial_\varepsilon \circ \alpha$ of (4.2) is accretive in L^1, but this applies to (4.1) only if $v \in L^1(b)$, a regularity assumption not always true. Second, $\partial_\varepsilon \circ \alpha$ is not necessarily m-accretive; $\lambda > 0$ is required in Theorem C, whereas we need $\lambda = 0$ for (4.2). Thus, we shall first resolve the stationary problem with $\lambda > 0$, show these solutions converge as $\lambda \to 0$ to a solution of (4.1), and finally show *directly* that the corresponding operator, A^ε, is accretive in $L^1(b)$.

Let $f \in L^1(b)$, $b < +\infty$, and $\lambda > 0$. From Theorem C it follows that there is a unique triple $u_\lambda \in L^1(b)$, $v_\lambda \in L^1(b)$ and $w_\lambda = (I + \varepsilon\, \partial)^{-1}\, v_\lambda$ for which

$$(\lambda I + \partial(I + \varepsilon\, \partial)^{-1}) v_\lambda + u_\lambda = f, \qquad v_\lambda \in \alpha(u_\lambda). \tag{4.3}$$

Multiply (4.3) by $\operatorname{sgn}_0(u_\lambda + v_\lambda)$, and note that this function belongs at a.e. point to $\operatorname{sgn}(u_\lambda)$ and to $\operatorname{sgn}(v_\lambda)$, since α is monotone. Integrate the product and use Theorem C to obtain

$$\lambda \|v_\lambda\|_{L^1(R)} + \|u_\lambda\|_{L^1(R)} \leqslant \|f\|_{L^1(R)}, \qquad -\infty < R \leqslant b.$$

Apply the translation τ_h to (4.3), subtract the result from (4.3), multiply the result by $\operatorname{sgn}_0(\tau_h - I)(\lambda v_\lambda + u_\lambda) \in \operatorname{sgn}((\tau_h - I)\lambda v_\lambda) \cap \operatorname{sgn}((\tau_h - I) u_\lambda)$. By an integration and application of Theorem C we obtain

$$\lambda \|\tau_h v_\lambda - v_\lambda\|_{L^1(b)} + \|\tau_h u_\lambda - u_\lambda\|_{L^1(b)} \leqslant \|\tau_h f - f\|_{L^1(b)}, \qquad h > 0.$$

From these two estimates it follows that $\{u_\lambda\}$ and $\{\lambda v_\lambda\}$ are relatively compact in $L^1(b)$. Also, ∂^{-1} is continuous from $L^1(b)$ to $C_0(b)$, so $\{w_\lambda\}$ is relatively compact in $C_0(b)$. By passing to a subsequence, which we denote by the same notation, we have $(1/\varepsilon) v_\lambda = f - u_\lambda + (1/\varepsilon) w_\lambda - \lambda v_\lambda$ converges in $L^1_{\mathrm{loc}}(b)$, and so we have the limits

$$u_\lambda \to u, \qquad \lambda v_\lambda \to 0, \qquad \partial w_\lambda \to \partial w \qquad \text{in } L^1(b),$$

and $w_\lambda \to w$ in $C_0(b)$. Thus there exists $u \in L^1(b)$, $w \in C_0(b)$ with $Dw \in L^1(b)$ such that $v \equiv w + \varepsilon\, Dw \in \alpha(u)$ a.e. $x < b$, and $u + Dw = f$ in $L^1(b)$. That is, $u + A^\varepsilon(u) \ni f$ where A^ε is given as follows.

DEFINITION. $f \in A^{\varepsilon}(u)$ if $u, f \in L^1(b)$ and there exists $w \in C_0(b)$ such that $Dw = f$ and $w + \varepsilon\, Dw \in \alpha(u)$ at a.e. $x < b$.

Next we establish directly that A^{ε} is accretive, hence, m-accretive in $L^1(b)$. For $j = 1, 2$, let $u_j + A^{\varepsilon}(u_j) \ni f_j$, and let $w_j \in C_0(b)$ with $Dw_j \in L^1(b)$ as above with $v_j = w_j + \varepsilon\, Dw_j$. Denote the respective *differences* by $u = u_1 - u_2$, $v = v_1 - v_2$, $w = w_1 - w_2$, and $f = f_1 - f_2$. Thus we have

$$u + Dw = f, \qquad v = w + \varepsilon\, Dw, \qquad v_j \in \alpha(u_j) \qquad \text{for} \quad j = 1, 2.$$

Choose $\sigma = \operatorname{sgn}_0^+(u + v) \in \operatorname{sgn}^+(u) \cap \operatorname{sgn}^+(v)$, since α is monotone, so that we have

$$u \cdot \sigma + \frac{1}{\varepsilon}(v - w)\sigma = f\sigma \leqslant f^+, \qquad \text{a.e.} \quad x < b. \tag{4.4}$$

In order to estimate the integral of the middle term, we note $w(x) = -(1/\varepsilon) \int_{-\infty}^{x} \exp((s - x)/\varepsilon)\, v(s)\, ds$, and that for each $a < b$ we have

$$\begin{aligned} &\int_a^b \frac{1}{\varepsilon} \int_a^x \exp\left(\frac{s - x}{\varepsilon}\right) v(s)\, ds \cdot \sigma(x)\, dx \\ &\leqslant \int_a^b \frac{1}{\varepsilon} \int_a^x \exp\left(\frac{s - x}{\varepsilon}\right) v(s)^+\, ds\, dx \\ &= \int_a^b \frac{1}{\varepsilon} \int_s^b \exp\left(\frac{s - x}{\varepsilon}\right) dx v(s)^+\, dx \leqslant \int_a^b v(s)^+\, ds. \end{aligned}$$

Since $v^+ = v \cdot \sigma$, a.e., this shows that

$$\begin{aligned} 0 &\leqslant \int_a^b \frac{1}{\varepsilon} \left(v(x) - \frac{1}{\varepsilon} \int_a^x \exp\left(\frac{s - x}{\varepsilon}\right) v(s)\, ds \right) \sigma(x)\, dx \\ &= \int_a^b Dw(x) \cdot \sigma(x)\, dx + V(a), \end{aligned} \tag{4.5}$$

where

$$V(a) \equiv \int_a^b \frac{1}{\varepsilon^2} \int_{-\infty}^{a} \exp\left(\frac{s - x}{\varepsilon}\right) v(s)\, ds \cdot \sigma(x)\, dx.$$

This is estimated by

$$
\begin{aligned}
|V(a)| &\leqslant \frac{1}{\varepsilon^2} \int_a^b \int_{-\infty}^a \exp\left(\frac{s-x}{\varepsilon}\right) |v(s)| \, ds \, dx \leqslant \frac{1}{\varepsilon} \int_{-\infty}^a \exp\left(\frac{s-x}{\varepsilon}\right) |v(s)| \, ds \\
&\leqslant \frac{1}{\varepsilon} \int_{-\infty}^a \exp\left(\frac{s-x}{\varepsilon}\right) (|w(s)| + \varepsilon |Dw(s)|) \, ds \\
&\leqslant \|w\|_{C_0(a)} + \|Dw\|_{L^1(a)},
\end{aligned}
$$

and it converges to zero as $a \to -\infty$. Thus, by taking the limit in (4.5) we obtain

$$0 \leqslant \int_{-\infty}^b Dw \cdot \sigma \, dx.$$

From this and (4.4) we have

$$\int_{-\infty}^b u^+(x) \, dx \leqslant \int_{-\infty}^b f^+(x) \, dx,$$

and this shows A^ε is accretive and order-preserving.

THEOREM 3. *Let α be a maximal monotone graph in $\mathbb{R}$ with $0 \in \alpha(0)$, $\varepsilon > 0$, and $b \leqslant +\infty$. The operator A^ε is m-accretive in $L^1(b)$ and the resolvents, $J_\mu^\varepsilon \equiv (I + \mu A^\varepsilon)^{-1}$, $\mu > 0$, satisfy the following: J_μ^ε commutes with each translation τ_h, $h > 0$, and*

$$\|(J_\mu^\varepsilon(f_1) - J_\mu^\varepsilon(f_2))^+\|_{L^1(b)} \leqslant \|(f_1 - f_2)^+\|_{L^1(b)}, \qquad f_1, f_2 \in L^1(b); \quad (4.6a)$$

$$\|J_\mu^\varepsilon(f)\|_{L^p(b)} \leqslant \|f\|_{L^p(b)}, \qquad f \in L^p(b), \qquad 1 \leqslant p \leqslant +\infty; \quad (4.6b)$$

$$-\|f^-\|_{L^\infty(b)} \leqslant J_\mu^\varepsilon(f)(x) \leqslant \|f^+\|_{L^\infty(b)}, \qquad \text{a.e.} \quad x < b. \quad (4.6c)$$

Proof. The first part is finished for $b < \infty$. For the case $b = +\infty$, the direct proof of (4.6a) stands as it is, while the existence follows easily by noting the preceding convergence in $L^1(b)$, $C_0(b)$ for every $b > 0$, the domain of dependence of u_λ on f, and the dominated convergence theorem. By [7, Lemma 3], (4.6b) will follow from (4.6c). Thus it suffices to prove (4.6c) in the case $\mu = 1$.

For this, let $f \in L^1(b)$ with $k = \|f^+\|_{L^\infty} < \infty$ and $u + A^\varepsilon(u) \ni f$. That is, we assume

$$u \in L^1(b), \qquad w \in C_0(b), \qquad v \equiv w + \varepsilon \, Dw,$$

$$u + Dw = f, \qquad \text{and} \qquad v \in \alpha(u), \qquad \text{a.e.} \quad x < b.$$

If $u(x) \leqslant k$, a.e. $x < b$, we are done. Otherwise $k \in \operatorname{dom}(\alpha)$ and so $\alpha_0(k) \in \mathbb{R}$. Choose

$$\sigma = \operatorname{sgn}_0^+(u - k + v - \alpha_0(k)) \in \operatorname{sgn}^+(u-k) \cap \operatorname{sgn}^+(v - \alpha_0(k))$$

and note then that

$$(u-k)\sigma + \frac{1}{\varepsilon}(v-w)\sigma \leqslant (f-k)^+ \sigma = 0, \qquad \text{a.e.} \quad x < b.$$

Proceeding as above, we obtain for $a < b$

$$\begin{aligned}
&\int_a^b \left(\frac{1}{\varepsilon} \int_a^x \exp\left(\frac{s-x}{\varepsilon}\right) v(s)\, ds - \alpha_0(k) \right) \sigma(x)\, dx \\
&\quad = \int_a^b \left(\frac{1}{\varepsilon} \int_a^x \exp\left(\frac{s-x}{\varepsilon}\right) (v(s) - \alpha_0(k))\, ds - \exp\left(\frac{a-x}{\varepsilon}\right) \alpha_0(k) \right) \sigma(x)\, dx \\
&\quad \leqslant \int_a^b \frac{1}{\varepsilon} \int_a^x \exp\left(\frac{s-x}{\varepsilon}\right) (v(s) - \alpha_0(k))^+ \, ds\, dx \\
&\quad \leqslant \int_a^b (v(s) - \alpha_0(k))^+ \, ds.
\end{aligned}$$

Thus after deleting $\pm \alpha_0(k)\, \sigma(x)$ from the integrand we have

$$0 \leqslant \int_a^b \frac{1}{\varepsilon} \left(v(s) - \frac{1}{\varepsilon} \int_a^x \exp\left(\frac{s-x}{\varepsilon}\right) v(s)\, ds \right) \sigma(x)\, dx$$

in which we let $a \to -\infty$ and obtain

$$\int_{-\infty}^b (u-k)^+ \leqslant 0.$$

This implies $u(x) \leqslant k$ a.e., and finishes the proof of Theorem 3.

THEOREM 4. *Let α be a maximal monotone graph in $\mathbb{R}$ with $0 \in \alpha(0)$ and $b \leqslant +\infty$. Assume $b < +\infty$ or $\alpha(0) = \{0\}$. Let A^ε be given as above for $\varepsilon > 0$ and let A be given as in Section 2. Then $A^\varepsilon \to A$; that is,*

$$\lim_{\varepsilon \to 0} (I + \mu A^\varepsilon)^{-1} f = (I + \mu A)^{-1} f, \qquad \mu > 0, \quad f \in L^1(b). \tag{4.7}$$

Proof. Consider the case $b < +\infty$. For $\varepsilon > 0$, let $u^\varepsilon \in L^1(b)$, $w^\varepsilon \in C_0(b)$ with $v^\varepsilon = w^\varepsilon + \varepsilon\, Dw^\varepsilon \in \alpha(u^\varepsilon)$ and $u^\varepsilon + Dw^\varepsilon = f$ in $L^1(b)$. From the proof of

Theorem 3 we obtain

$$\int_{-\infty}^{R} |u^\varepsilon| \leqslant \int_{-\infty}^{R} |f|, \qquad R \leqslant b,$$

$$\int_{-\infty}^{b} |\tau_h u^\varepsilon - u^\varepsilon| \leqslant \int_{-\infty}^{b} |\tau_h f - f|, \qquad h > 0,$$

so it follows that u^ε has a subsequence $u^{\varepsilon'} \to u$ in $L^1(b)$. It follows from above that $Dw^{\varepsilon'} \to Dw$ in $L^1(b)$ and $w^{\varepsilon'} \to w$ in $C_0(b)$, hence, $v^\varepsilon \to w$ in L^1_{loc} and we have

$$u + Dw = f \qquad \text{in } L^1(b), \qquad w \in \alpha(u) \text{ a.e. } x < b$$

by Lemma 0. That is, $(I + A)(u) \ni f$. By uniqueness, the original sequence u^ε converges to u as desired. This proves (4.7) with $\mu = 1$; the case $\mu > 0$ is immediate from this.

In the case of $b = +\infty$, $\alpha(0) = \{0\}$ it suffices to show, as in the proof of Theorem 2, that

$$\lim_{R \to +\infty} \int_{R}^{+\infty} |u^\varepsilon| = 0, \qquad \text{uniformly in} \quad \varepsilon > 0.$$

From the pointwise estimates

$$J_1^\varepsilon(f^-) \leqslant J_1^\varepsilon(f)^- \leqslant 0 \leqslant J_1^\varepsilon(f)^+ \leqslant J_1^\varepsilon(f^+),$$

it follows we need only consider the case of $f \geqslant 0$. But then we have, successively,

$$u^\varepsilon \geqslant 0, \qquad v^\varepsilon \geqslant 0, \qquad w^\varepsilon \geqslant 0.$$

Moreover, this shows

$$\int_{-\infty}^{\infty} u^\varepsilon \leqslant \int_{-\infty}^{\infty} u^\varepsilon + w^\varepsilon(+\infty) = \int_{-\infty}^{\infty} f.$$

By standard arguments based on uniqueness of u, we may assume $u^\varepsilon \to u$ a.e. on $\mathbb{R}$ and use Fatou's Theorem to show

$$\begin{aligned} 2 \int_{-\infty}^{\infty} u &\leqslant \liminf_{\varepsilon \to 0} \int_{-\infty}^{\infty} (u^\varepsilon + u - |u^\varepsilon - u|) \\ &\leqslant \int_{-\infty}^{\infty} f + \int_{-\infty}^{\infty} u - \limsup_{\varepsilon \to 0} \int_{-\infty}^{\infty} |u^\varepsilon - u|, \end{aligned}$$

and now (2.7c) implies $\lim_{\varepsilon \to 0} \|u^\varepsilon - u\|_{L^1(\mathbb{R})} = 0$.

5. Remarks

(i) The continuous dependence on α of the operator A^ε follows immediately from our preceding proofs.

THEOREM 5. *For each $n = 1, 2, ..., +\infty$, assume α_n is a maximal monotone graph in $\mathbb{R}$ with $0 \in \alpha_n(0)$, and assume $b < +\infty$. Let $\varepsilon > 0$ and A_n^ε be the m-accretive operator in $L^1(b)$ constructed from α_n as in Section 4. If $\alpha_n \to \alpha_\infty$ in $\mathbb{R}$, then $A_n^\varepsilon \to A_\infty^\varepsilon$ in $L^1(b)$.*

The preceding proofs do not suffice for the case $b = +\infty$ since we do *not* have $\int (I + A^\varepsilon)^{-1} f = \int f$ for $\varepsilon > 0$.

(ii) Suppose the maximal monotone graph α is bounded, i.e., it maps bounded sets into bounded sets, and that it grows at most linearly near zero. That is, suppose there is a pair $r > 0$, $K > 0$ such that

$$|v| \leqslant K|u| \qquad \text{for all} \quad [u, v] \in \alpha, \quad |u| \leqslant r. \tag{5.1}$$

Then for $u \in L^1(b) \cap L^\infty(b)$ and measureable $v \in \alpha(u)$, it follows $v \in L^1(b) \cap L^\infty(b)$. The L^∞ claim is clear; for the L^1 claim, set

$$E_r = \{x < b \colon |u(x)| \leqslant r\}, \qquad \tilde{E}_r = \{x < b \colon |u(x)| > r\}$$

and note that

$$\|v\|_{L^1(b)} = \int_{E_r} |v| + \int_{\tilde{E}_r} |v| \leqslant K \int_{E_r} |u| + \frac{\|v\|_{L^\infty}}{r} \int_{\tilde{E}_r} |u|. \tag{5.2}$$

It follows that

$$L^1(b) \cap L^\infty(b) \subset \operatorname{dom}(\partial(I + \varepsilon\,\partial)^{-1} \circ \alpha) \subset \operatorname{dom}(A^\varepsilon),$$

so the domain of A^ε is dense in $L^1(b)$.

(iii) Now let the maximal monotone α be a continuous function on $\mathbb{R}$ which satisfies (5.1). Let $u_0 \in L^1(b) \cap L^\infty(b)$ and denote by u the generalized solution of

$$u'(t) + A^\varepsilon(u(t)) = 0, \qquad 0 < t \leqslant T, \qquad u(0) = u_0. \tag{5.3}$$

Thus, $u = \lim_{\delta \to 0} u_\delta$ in $C(0, T \colon L^1(b))$ where u_δ is the solution to

$$\frac{1}{\delta}(u_\delta(t) - u_\delta(t - \delta)) + A^\varepsilon(u^\delta(t)) = 0, \qquad 0 < t \leqslant T, \qquad u_\delta(t) = u_0, \qquad t \leqslant 0,$$

according to Theorem A, and we have

$$\|u_\delta(t)\|_{L^p(b)} \leqslant \|u_0\|_{L^p(b)}, \qquad 0 \leqslant t \leqslant T, \qquad 1 \leqslant p \leqslant +\infty.$$

by Theorem 3. By splitting the domain of integration as in (5.2) we obtain for each $R > 0$

$$\int_0^T \int_{|x| \geqslant R}^b |\alpha(u_\delta)|\, dx\, dt \leqslant \left(K + \frac{M}{r}\right) \int_0^T \int_{|x| \geqslant R}^b |u_\delta|\, dx\, dt, \qquad M = \|v\|_{L^\infty}.$$

Thus $\alpha(u_\delta) \to \alpha(u)$ in L^1 and it follows that

$$u'(t) + \partial(I + \varepsilon\, \partial)^{-1}\, \alpha(u) = 0 \qquad \text{in } L^1(0, T; L^1(b)) \tag{5.4}$$

by our preceding remark (ii) and the continuity of ∂_ε on $L^1(b)$. Thus, when α is continuous and satisfies (5.1), the generalized solution of (5.3) with $u_0 \in L^1 \cap L^\infty$ is a *strong* solution in $W^{1,1}(0, T; L^1(b))$ of (5.4).

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