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Single-phase flow in composite poroelastic media

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SUMMARY

The mathematical formulation and analysis of the Barenblatt–Biot model of elastic deformation and laminar flow in a heterogeneous porous medium is discussed. This describes consolidation processes in a fluid-saturated double-diffusion model of fractured rock. The model includes various degenerate cases, such as incompressible constituents or totally fissured components, and it is extended to include boundary conditions arising from partially exposed pores. The quasi-static initial–boundary problem is shown to have a unique weak solution, and this solution is strong when the data are smoother.

1. INTRODUCTION

Any model of fluid flow through a deformable solid matrix must account for the coupling between the mechanical behaviour of the matrix and the fluid dynamics. For example, compression of the medium leads to increased pore pressure, if the compression is fast relative to the fluid flow rate. Conversely, an increase in pore pressure induces a dilation of the matrix in response to the added stress. The concept of *total stress* is the essence of coupled deformation-flow behaviour within porous media and sets it apart from the theory of flow through a rigid structure. This coupled pressure–deformation interaction is the basis of the development of poroelasticity starting with the work of Terzaghi [1, 2]. The first detailed studies of the coupling between the pore-fluid pressure and solid stress fields were described by Biot [3]. The basic constitutive equations relate the total stress to both the *effective stress* given by the strain of the structure and to the *pressure* arising from the pore fluid. Time-dependent fluid flow is incorporated by combining the fluid mass conservation with Darcy's law, and the displacement of the structure is described by combining Hooke's law for elastic deformation with the momentum balance equations. The transient flow and deformation behaviour in a deformable porous medium may result from changes in either the fluid pressure, flux, displacements, or traction conditions applied to the boundary of the medium. The model for consolidation requires the *quasi-static* assumption that the dynamic momentum equations be replaced by the corresponding equilibrium equations.

The representation of porosity and permeability in heterogeneous media often requires several distinct spatial scales. Thus, the need arises for more general models incorporating

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qualitatively different characteristics. For example, in problems of fluid flow in subsurface reservoirs and aquifers, the simplest and most frequently used model is the double-porosity/double-permeability medium which consists of the combined effects of two distinct components in parallel. These components occur locally in any representative volume element, and they behave as two independent diffusion processes, which are coupled by a distributed exchange term. This construction and its application to the description of composite diffusion processes are generally attributed to Barenblatt *et al.* [4]. In the special case which is used to model fractured media, the first component of the model is the fracture system and the second is the porous matrix structure.

As originally formulated, Biot's theory applies to a homogeneous porous medium, but the basic ideas of this fundamental poroelasticity model continue to play an important role in the more complex double-diffusion models. Since the pressure fields contribute to the stress field of the structure, it is necessary to incorporate Biot's concepts of poroelasticity into the Barenblatt double-diffusion deformation model. First the equilibrium momentum equations must be formulated with the contributions to total stress from the two pressure fields. Then the equations of fluid transport can be obtained from the continuity of fluid mass and consideration of the effects of *dilation* of the structure on the flow in each of the components. The fluid transport within this composite deformable medium is described by a pair of pressure equations for diffusion in the respective components of the medium together with an exchange term that, in its simplest form, is proportional to the difference in pressure between fluids in the two components. This simplistic combination of the Barenblatt double-diffusion model with the Biot diffusion–deformation model has been developed and used extensively in the engineering literature.

Our objective is to develop the mathematical analysis of the initial–boundary-value problem for the Barenblatt–Biot system representing double diffusion in elastic porous media. This system takes the form

$$-(\lambda+\mu)\nabla(\nabla\cdot\mathbf{u})-\mu\Delta\mathbf{u}+\alpha_1\nabla p_1+\alpha_2\nabla p_2=\mathbf{f}(x,t) \tag{1a}$$

$$c_1\dot{p}_1-\nabla\cdot(k_1\nabla p_1)+\alpha_1\nabla\cdot\dot{\mathbf{u}}+\kappa(p_1-p_2)=h_1(x,t) \tag{1b}$$

$$c_2\dot{p}_2-\nabla\cdot(k_2\nabla p_2)+\alpha_2\nabla\cdot\dot{\mathbf{u}}+\kappa(p_2-p_1)=h_2(x,t) \tag{1c}$$

in which $\mathbf{u}$ is the displacement of the solid skeleton and p_1 and p_2 are the fluid pressures in the respective components. The constant λ is the drained Lamé modulus, μ is the shear modulus, and the constants α_1 and α_2 measure changes of porosities due to an applied volumetric strain. We will develop this theory for the Barenblatt–Biot system (1) as an example of the application of the theory of implicit evolution equations in Hilbert spaces. Thereby we will not only obtain optimal results on the appropriate spaces and definitions for the solution, but we will also obtain corresponding estimates directly from the abstract theory. We are especially interested in determining when the evolution is *parabolic*. This case will lead to sharp estimates of order $O(1/t)$, additional *regularity* of the solution, and a larger class of data for which the initial-value problem is well posed. We are also interested in the behaviour of $\lim_{t\to 0}\mathbf{u}(t)$ and $\lim_{t\to 0} p_i(t)$.

We begin in Section 2 with a description of the extension of the Biot diffusion–deformation model to include the Barenblatt double-diffusion system. This is followed in Section 3 by some remarks on the literature of these and related systems. After introducing appropriate mathematical preliminaries in Section 4, we describe in Section 5 the quasi-static evolution system and the corresponding initial–boundary-value problem for this system. In particular, we include new boundary conditions which model the case of partially exposed pores, and we include the case of degenerate coefficients corresponding to an incompressible fluid and solid material.

2. THE DOUBLE-DIFFUSION-DEFORMATION MODEL

2.1. The rigid case

We first describe a two-component model for the flow of a single phase, slightly compressible fluid in a *rigid composite medium*. This is defined to be a porous medium composed of two interwoven (and possibly connected) components. For the case of a fractured medium, the first component is a system of highly permeable fractures, and the second component is a matrix of porous and somewhat permeable material, so both double-porosity and double-permeability characteristics are exhibited. In the special case of disjoint porous blocks which are separated by the system of fractures, it is called a *totally fractured medium*. The common characteristics of fractured media are that the solid matrix occupies a much larger volume than the fractures and that it is relatively much more resistant to fluid flow than is the fracture system. As a consequence, most of the flow passes through the system of fractures, while the bulk storage of fluid takes place primarily inside the porous matrix. The flow in the composite is enhanced by the exchange of fluid which takes place on the matrix–fracture interface. Limiting cases of the geometry arise when one of the two components of the medium becomes disconnected. In the case of a totally fractured medium, the global flow in the blocks is induced only indirectly by the exchange of fluid which takes place on the block–fracture interfaces, and any interaction between the blocks is possible only via the neighbouring system of fractures, which separate the blocks. The more general case of a connected matrix is called a *partially fractured medium*. In this model, some part of the flow passes directly through the matrix interconnections, but the primary flow still continues to be that from the matrix into fractures followed by flow within the fractures.

More generally, the *parallel flow model* is a classical description of diffusion in a heterogeneous medium. The idea is to introduce at each point in space a density, pressure or concentration for each component, each being obtained by averaging in the respective medium over a generic neighbourhood sufficiently large to contain a representative sample of each component. The rate of exchange between the components must be expressed in terms of these quantities, and the resulting expressions become distributed source and sink terms for the diffusion equations in the individual components. Thus, one obtains a *system* of diffusion equations, one for each component. The classical linear *double-diffusion* model for the flow of slightly compressible fluid in a general heterogeneous medium consisting of two components is the system

$$\frac{\partial}{\partial t}c_1 p_1 - \nabla\cdot(k_1\nabla p_1) + \kappa(p_1 - p_2) = h_1 \tag{2a}$$

$$\frac{\partial}{\partial t}c_2 p_2 - \nabla\cdot(k_2\nabla p_2) + \kappa(p_2 - p_1) = h_2 \tag{2b}$$

For the model of a fractured medium, the first equation describes the flow in the fracture system, which consists of regions of small relative volume but large permeability. The second equation describes the flow in the matrix, which consists of regions of large porosity or volume. System (2) was developed by Barenblatt *et al.* [4], and it has come to be known as the *Barenblatt system*. It is *parabolic* when all constants are positive, but the applications frequently require certain degenerate cases such as those described below.

Although the components of this system are structured symmetrically, fractured media characteristics are necessarily modelled by the use of very small coefficients. The fracture and matrix phases are distinctly different in both porosity and permeability. For the case of a totally fractured medium, the coefficient c_1 is almost zero, because the relative volume of the fractures is small, and $k_2 = 0$ because there is so little direct flow within the matrix, i.e. it may consist of individual cells which are isolated from each other by the fracture system. The last term on the left of each equation represents the exchange of fluid between the cells and the fractures. The parameter $1/\kappa$ represents the resistance of the interface to this exchange. When $\kappa = 0$, no exchange flow is possible. An alternative interpretation is that κ represents the degree of fissuring of the medium. When the degree of fissuring is infinite, the exchange flow encounters no resistance and $p_1 = p_2$. The external sources of fluid represented by $h_1(\cdot)$ and $h_2(\cdot)$ are located in the fractures and in the cells, respectively. By eliminating $p_2(\cdot)$ from system (2) with $c_1 = 0$ and $k_2 = 0$, we obtain the *fractured medium equation*

$$c_2 \frac{\partial}{\partial t}\left(p_1 - \frac{1}{\kappa}\nabla\cdot(k_1\nabla p_1)\right) - \nabla\cdot(k_1\nabla p_1) = \left(I + \frac{c_2}{\kappa}\frac{\partial}{\partial t}\right)h_1 + h_2$$

This equation is of *pseudo-parabolic type*.

2.2. *The deformable case*

We shall develop the model of deformable porous media with double diffusion. In order to formulate such a model correctly, we first describe a representative element of volume, V, of the two-component composite medium. Denote by V_j the volume of fluid in component j of V, and $\phi_j \equiv V_j/V$, the *porosity* of component j, for $j = 1, 2$. Let V_s be the remaining volume of the solid part of V, and $\phi_s \equiv V_s/V$ the corresponding volume fraction of solid. Of course, we have $V = V_1 + V_2 + V_s$. The *total porosity* of the medium, i.e. the volume fraction available to the fluid, is given by $\phi \equiv \phi_1 + \phi_2$, and the remaining volume fraction of solid will be denoted by $\phi_s \equiv 1 - \phi$.

2.2.1. Conservation equations. Let Ω be a smoothly bounded region which represents the porous and permeable elastic matrix with density ρ, and assume it is saturated by a slightly compressible and viscous fluid which diffuses through it. The *displacement* of the solid matrix is denoted by $\mathbf{u}(x, t)$ for each point $x \in \Omega$ and time $t > 0$. Let ρ_s be the density of the solid. The volume fraction of solid is given by $1 - \phi$, so the quantity of solid in each subdomain B of Ω is given by $\int_B (1 - \phi)\rho_s \, dx$. The rate at which solid mass moves across the boundary ∂B is given by

$$\int_{\partial B} (1 - \phi)\rho_s \dot{\mathbf{u}} \cdot \mathbf{n} \, dS$$

so the *conservation of solid mass* takes the integral form

$$\frac{\partial}{\partial t}\int_B (1-\phi)\rho_s \,dx + \int_{\partial B}(1-\phi)\rho_s \dot{\mathbf{u}}\cdot\mathbf{n}\,dS = 0, \quad B\subset\Omega$$

When these quantities are differentiable, we obtain the equations of *solid mass balance* in the differential form

$$\frac{\partial}{\partial t}(1-\phi)\rho_s + \nabla\cdot((1-\phi)\rho_s\dot{\mathbf{u}}) = 0$$

If we expand these derivatives and express the result in terms of the material derivative,

$$\frac{\mathrm{D}}{\mathrm{D}t} \equiv \frac{\partial}{\partial t} + \dot{\mathbf{u}}\cdot\nabla$$

then this is written as

$$\rho_s\frac{\mathrm{D}\phi}{\mathrm{D}t} = (1-\phi)\frac{\mathrm{D}\rho_s}{\mathrm{D}t} + \rho_s(1-\phi)\nabla\cdot\dot{\mathbf{u}} \tag{3}$$

Let ρ_1 and ρ_2 be the densities of fluid and $\mathbf{w}_1$ and $\mathbf{w}_2$ the displacement of the fluid in the respective components. Then the *continuity of fluid mass* in the first component is given by

$$\frac{\partial}{\partial t}\phi_1\rho_1 + \nabla\cdot(\phi_1\rho_1\dot{\mathbf{w}}_1) + \Gamma = \rho_1 h_1$$

where h_1 is the fluid volume source and Γ is the fluid mass exchange from the first component to the second component. Using the *Darcy relative velocity* of the first component fluid,

$$\mathbf{v}_1 \equiv \phi_1(\dot{\mathbf{w}}_1 - \dot{\mathbf{u}})$$

we write this as

$$\frac{\partial}{\partial t}\phi_1\rho_1 + \nabla\cdot\rho_1(\phi_1\dot{\mathbf{u}} + \mathbf{v}_1) + \Gamma = \rho_1 h_1$$

If we expand these derivatives, we obtain as before

$$\phi_1\frac{\mathrm{D}\rho_1}{\mathrm{D}t} + \rho_1\frac{\mathrm{D}\phi_1}{\mathrm{D}t} + \rho_1\phi_1\nabla\cdot\dot{\mathbf{u}} + \nabla\cdot(\rho_1\mathbf{v}_1) + \Gamma = \rho_1 h_1$$

By substituting from the solid conservation equation (3), we obtain

$$\phi_1\frac{\mathrm{D}\rho_1}{\mathrm{D}t} + \rho_1\left(\frac{1-\phi}{\rho_s}\frac{\mathrm{D}\rho_s}{\mathrm{D}t} + (1-\phi)\nabla\cdot\dot{\mathbf{u}}\right) - \rho_1\frac{\mathrm{D}\phi_2}{\mathrm{D}t} + \rho_1\phi_1\nabla\cdot\dot{\mathbf{u}} + \nabla\cdot(\rho_1\mathbf{v}_1) + \Gamma = \rho_1 h_1$$

and this simplifies to

$$\phi_1\frac{\mathrm{D}\rho_1}{\mathrm{D}t} + \rho_1\left(\frac{1-\phi}{\rho_s}\frac{\mathrm{D}\rho_s}{\mathrm{D}t} + (1-\phi_2)\nabla\cdot\dot{\mathbf{u}}\right) - \rho_1\frac{\mathrm{D}\phi_2}{\mathrm{D}t} + \nabla\cdot(\rho_1\mathbf{v}) + \Gamma = \rho_1 h_1 \tag{4}$$

We need to express $\mathrm{D}\phi_2/\mathrm{D}t$ above. Since $\phi_2 \equiv V_2/V$, we have

$$\mathrm{D}\phi_2 = \frac{1}{V}(\mathrm{D}V_2 - \phi_2\,\mathrm{D}V) = \frac{1}{V}((1-\phi_2)\,\mathrm{D}V_2 - \phi_2\mathrm{D}V_1 - \phi_2\mathrm{D}V_s)$$

Let p_1 and p_2 denote the *fluid pressure* in the respective components, and let p_s be the *effective pressure* on the solid. This will be prescribed below as a linear combination of p_1 and p_2.

2.2.2. Structural constitutive equations. The fundamental variables in the system will be the fluid pressures p_1 and p_2, so we consider the dependence on variations in these local pressures as the mean effective stress on the volume element. Due in part to the exchange of fluid between V_1 and V_2 and to an increment in the dilation of the solid, there results a change in volume of the respective components of the form

$$\frac{\mathrm{D}V_j}{V} = \gamma_j \mathrm{D}p_j + \phi_j \mathrm{D}\nabla\cdot\mathbf{u}, \quad j = 1,2$$
$$\frac{\mathrm{D}V_s}{V} = -\gamma_s \mathrm{D}p_s + \phi_s \mathrm{D}\nabla\cdot\mathbf{u}$$

These account for the coupling of adjacent components in the medium, e.g. an increase of p_1 corresponds to an increase in V_1 and a decrease of V_s corresponding to flow of fluid into V_1, but the direct relation with V_2 is not postulated. This gives from above

$$\mathrm{D}\phi_2 = (1-\phi_2)(\gamma_2\mathrm{D}p_2 + \phi_2\mathrm{D}\nabla\cdot\mathbf{u}) - \phi_2(\gamma_1\mathrm{D}p_1 + \phi_1\mathrm{D}\nabla\cdot\mathbf{u}) + \phi_2(\gamma_s\mathrm{D}p_s - \phi_s\mathrm{D}\nabla\cdot\mathbf{u})$$

Since the coefficients of $\mathrm{D}\nabla\cdot\mathbf{u}$ are $\phi_2(1-\phi_2-\phi_1-\phi_s) = 0$, this simplifies to

$$\mathrm{D}\phi_2 = (1-\phi_2)\gamma_2\mathrm{D}p_2 - \phi_2\gamma_1\mathrm{D}p_1 + \phi_2\gamma_s\mathrm{D}p_s \tag{6}$$

Thus, the dilation plays no role in the variation of porosity in the two-component medium.

2.2.3. Material constitutive equations. Changes in density are given by the *compressibilities*

$$\frac{1}{\rho_f}\frac{\mathrm{D}\rho_f}{\mathrm{D}p} = c_f, \quad \frac{1}{\rho_s}\frac{\mathrm{D}\rho_s}{\mathrm{D}p} = c_s$$

of the fluid and solid, respectively. Substitution of these and (6) into (4) leads to

$$\rho_1\phi_1 c_f \frac{\mathrm{D}p_1}{\mathrm{D}t} + \rho_1(1-\phi)c_s\frac{\mathrm{D}p_s}{\mathrm{D}t} + \rho_1(1-\phi_2)\nabla\cdot\dot{\mathbf{u}}$$
$$-\rho_1\left((1-\phi_2)\gamma_2\frac{\mathrm{D}p_2}{\mathrm{D}t} - \phi_2\gamma_1\frac{\mathrm{D}p_1}{\mathrm{D}t} + \phi_2\gamma_s\frac{\mathrm{D}p_s}{\mathrm{D}t}\right) + \nabla\cdot(\rho_1\mathbf{v}_1) + \Gamma = \rho_1 h_1$$

Finally, we write the *effective solid pressure* as

$$p_s = \alpha_1 p_1 + \alpha_2 p_2$$

and then replace the densities and porosities by their nearly constant values to obtain the *linearized storage equation*

$$\rho_{\mathrm{f}}\{\phi_1 c_{\mathrm{f}} + \phi_{\mathrm{s}} c_{\mathrm{s}}\alpha_1 + \phi_2\gamma_1 - \phi_2\gamma_{\mathrm{s}}\alpha_1\}\frac{\mathrm{D}p_1}{\mathrm{D}t} + \rho_{\mathrm{f}}\{\alpha_2\phi_{\mathrm{s}}c_{\mathrm{s}} - \phi_2\gamma_{\mathrm{s}}\alpha_2 - (1-\phi_2)\gamma_2\}\frac{\mathrm{D}p_2}{\mathrm{D}t}$$
$$+\rho_{\mathrm{f}}(1-\phi_2)\nabla\cdot\dot{\mathbf{u}} + \nabla\cdot(\rho_{\mathrm{f}}\mathbf{v}_1) + \Gamma = \rho_{\mathrm{f}}h_1$$

Similarly, we obtain for the second component

$$\rho_{\mathrm{f}}\{\alpha_1\phi_{\mathrm{s}}c_{\mathrm{s}} - \phi_1\gamma_{\mathrm{s}}\alpha_1 - (1-\phi_1)\gamma_1\}\frac{\mathrm{D}p_1}{\mathrm{D}t} + \rho_{\mathrm{f}}\{\phi_2 c_{\mathrm{f}} + \phi_{\mathrm{s}}c_{\mathrm{s}}\alpha_2 + \phi_1\gamma_2 - \phi_1\gamma_{\mathrm{s}}\alpha_2\}\frac{\mathrm{D}p_2}{\mathrm{D}t}$$
$$+\rho_{\mathrm{f}}(1-\phi_1)\nabla\cdot\dot{\mathbf{u}} + \nabla\cdot(\rho_{\mathrm{f}}\mathbf{v}_2) - \Gamma = \rho_{\mathrm{f}}h_2$$

Since $\rho_{\mathrm{s}}V_{\mathrm{s}}$ is constant,

$$\frac{\mathrm{D}\rho_{\mathrm{s}}}{\rho_{\mathrm{s}}} + \frac{\mathrm{D}V_{\mathrm{s}}}{V_{\mathrm{s}}} = 0$$

and by comparing the structural and material equations we obtain the *consistency condition* $\gamma_{\mathrm{s}} = \phi_{\mathrm{s}}c_{\mathrm{s}}$. The exchange is given by $\Gamma = \rho_{\mathrm{f}}\kappa(p_1 - p_2)$, and *Darcy's law* gives

$$\mathbf{v}_1 = -\frac{k_1}{\mu}(\nabla p_1 + \rho_{\mathrm{f}}\mathbf{g})$$

$$\mathbf{v}_2 = -\frac{k_2}{\mu}(\nabla p_2 + \rho_{\mathrm{f}}\mathbf{g})$$

where $\mathbf{g}$ is the gravitational acceleration. Then the system of storage equations simplifies to the form

$$\rho_{\mathrm{f}}(\phi_1 c_{\mathrm{f}} + \alpha_1\gamma_{\mathrm{s}}(1-\phi_2) + \gamma_1\phi_2)\frac{\mathrm{D}p_1}{\mathrm{D}t} + \rho_{\mathrm{f}}(\alpha_2\gamma_{\mathrm{s}} - \gamma_2)(1-\phi_2)\frac{\mathrm{D}p_2}{\mathrm{D}t}$$
$$+\rho_{\mathrm{f}}(1-\phi_2)\nabla\cdot\dot{\mathbf{u}} - \nabla\cdot\rho_{\mathrm{f}}\left(\frac{k_1}{\mu}(\nabla p_1 + \rho_{\mathrm{f}}\mathbf{g})\right) + \rho_{\mathrm{f}}\kappa(p_1 - p_2) = \rho_{\mathrm{f}}h_1$$
$$\rho_{\mathrm{f}}(\alpha_1\gamma_{\mathrm{s}} - \gamma_1)(1-\phi_1)\frac{\mathrm{D}p_1}{\mathrm{D}t} + \rho_{\mathrm{f}}(\phi_2 c_{\mathrm{f}} + \alpha_2\gamma_{\mathrm{s}}(1-\phi_1) + \gamma_2\phi_1)\frac{\mathrm{D}p_2}{\mathrm{D}t}$$
$$+\rho_{\mathrm{f}}(1-\phi_1)\nabla\cdot\dot{\mathbf{u}} - \nabla\cdot\rho_{\mathrm{f}}\left(\frac{k_2}{\mu}(\nabla p_2 + \rho_{\mathrm{f}}\mathbf{g})\right) - \rho_{\mathrm{f}}\kappa(p_1 - p_2) = \rho_{\mathrm{f}}h_2$$

By combining these with the momentum equation, we obtain the final form of our model for fluid flow and deformation in a saturated composite elastic porous medium. This is the *Barenblatt–Biot system*

$$\rho\ddot{\mathbf{u}} - (\lambda+\mu)\nabla(\nabla\cdot\mathbf{u}) - \mu\Delta\mathbf{u} + \alpha_1\nabla p_1 + \alpha_2\nabla p_2 = \mathbf{f}(x,t) \tag{7a}$$

$$\rho_f(\phi_1 c_f + \alpha_1\gamma_s(1-\phi_2) + \gamma_1\phi_2)\frac{Dp_1}{Dt} + \rho_f(\alpha_2\gamma_s - \gamma_2)(1-\phi_2)\frac{Dp_2}{Dt}$$
$$+ \rho_f(1-\phi_2)\nabla\cdot\dot{\mathbf{u}} - \nabla\cdot\rho_f\left(\frac{k_1}{\mu}(\nabla p_1 + \rho_f\mathbf{g})\right) + \rho_f\kappa(p_1 - p_2) = \rho_f h_1 \tag{7b}$$

$$\rho_f(\alpha_1\gamma_s - \gamma_1)(1-\phi_1)\frac{Dp_1}{Dt} + \rho_f(\phi_2 c_f + \alpha_2\gamma_s(1-\phi_1) + \gamma_2\phi_1)\frac{Dp_2}{Dt}$$
$$+\rho_f(1-\phi_1)\nabla\cdot\dot{\mathbf{u}} - \nabla\cdot\rho_f\left(\frac{k_2}{\mu}(\nabla p_2 + \rho_f\mathbf{g})\right) - \rho_f\kappa(p_1 - p_2) = \rho_f h_2 \tag{7c}$$

The choice of parameters $\alpha_1 = (1-\phi_2)/(1+\phi_s)$, $\alpha_2 = (1-\phi_1)/(1+\phi_s)$ is a convex combination which gives the *reversibility* of the stress–strain relation for the material. That is, all elastic energy of the system is recovered upon release of the load. Otherwise it would be possible to extract energy from the system during a cycle of loading and unloading.

In most developments of the Barenblatt system for double diffusion, the matrix of coefficients for the pressures is assumed to be *diagonal*. This is usually justified by a statement that the cross effects of storage are negligible. However, our discussion above shows that this is tantamount to assuming that $\alpha_2\gamma_s \approx \gamma_2$ and $\alpha_1\gamma_s \approx \gamma_1$, i.e. that the compressibility of the components is approximated by a scaled compressibility of the solid. Such an assumption simplifies the system to the form

$$\rho\ddot{\mathbf{u}} - (\lambda+\mu)\nabla(\nabla\cdot\mathbf{u}) - \mu\Delta\mathbf{u} + \alpha_1\nabla p_1 + \alpha_2\nabla p_2 = \mathbf{f}(x,t) \tag{8a}$$

$$\rho_f(\phi_1 c_f + \alpha_1\gamma_s)\frac{Dp_1}{Dt} + \rho_f(1-\phi_2)\nabla\cdot\dot{\mathbf{u}} - \nabla\cdot\rho_f\left(\frac{k_1}{\mu}(\nabla p_1 + \rho_f\mathbf{g})\right) + \rho_f\kappa(p_1 - p_2) = \rho_f h_1 \tag{8b}$$

$$\rho_f(\phi_2 c_f + \alpha_2\gamma_s)\frac{Dp_2}{Dt} + \rho_f(1-\phi_1)\nabla\cdot\dot{\mathbf{u}} - \nabla\cdot\rho_f\left(\frac{k_2}{\mu}(\nabla p_2 + \rho_f\mathbf{g})\right) - \rho_f\kappa(p_1 - p_2) = \rho_f h_2 \tag{8c}$$

For additional discussion of this point, we refer to Berryman and Wang [5]. We shall assume this form of the system in the following, and furthermore we shall neglect the gravity term, since the structure is not at all changed for the theory developed. Note, finally, that (8) contains the Barenblatt system (2) for the rigid case, $\mathbf{u} = \mathbf{0}$. On the other hand, by letting $\kappa \to \infty$ and deleting the gravity terms in (8), we recover the classical *Biot system*

$$\rho\ddot{\mathbf{u}} - (\lambda+\mu)\nabla(\nabla\cdot\mathbf{u}) - \mu\Delta\mathbf{u} + \nabla p = \mathbf{f}(x,t) \tag{9a}$$

$$\rho_f(\phi_f c_f + \phi_s c_s)\frac{Dp}{Dt} + \rho_f\nabla\cdot\dot{\mathbf{u}} - \nabla\cdot\rho_f\left(\frac{k_1}{\mu}(\nabla p)\right) = \rho_f h \tag{9b}$$

for a single-component material.

3. REMARKS ON THE LITERATURE

For a small sample of fundamental work on the storage equation and its application in reservoir simulation, see Bear [6], Collins [7], Peaceman [8], and Huyakorn-Pinder [9]. Flow in porous media and connections with the theory of homogenization are developed in the monograph of Hornung [10].

The *fully dynamic* system (9) with $\rho > 0$ was introduced by Biot [11–14], to describe higher-frequency deformation in porous media. For the theory of this system in the context of thermo-elasticity, see the fundamental work of Dafermos [15], the exhaustive and complementary accounts of Carlson [16] and Kupradze [17], and the development in the context of strongly elliptic systems by Fichera [18]. By contrast, very few references are to be found in the thermo-elasticity literature for the mathematical well posedness of even the simplest linear problem for the coupled *quasi-static* case, $\rho = 0$, in which the system degenerates to a mixed elliptic–parabolic type.

The *poroelastic consolidation model* of Biot *requires* the quasi-static case; see Biot [3] and [19], Rice and Cleary [20], Zienkiewicz *et al.* [21]. An additional degeneracy occurs in the *incompressible* case in which we have also $c_{\mathrm{f}} = c_{\mathrm{s}} = 0$, and then the system is formally of elliptic type. A deeper study using homogenization methods reveals that the macroscopic equations of small-amplitude motion of a linearly elastic porous medium saturated with an elastic Newtonian fluid will have, in general, an integro-differential character. The coefficients can depend on the frequency, and the classical Biot system appears as a special limiting case. In the case of a periodic structure, the macroscopic coefficients are deduced by homogenization from the assumed local structure. In general, anisotropy appears in the macromodel, provided the local structure is anisotropic. See Auriault and Sanchez-Palencia [22], Levy [23], Sanchez-Hubert [24], and the review by Auriault [10]. However, isotropic cases will be developed below for simplicity, since the results do not depend on the specific forms.

The mathematical issues of well posedness for the quasi-static case of (9) were first studied in the fundamental work of Auriault and Sanchez-Palencia [22]. They derived a non-isotropic form of the Biot system by homogenization and then proved existence and uniqueness of a *strong* solution for which the equations hold in $L^2(\Omega)$. In the later paper of Zenisek [25], a *weak* solution is obtained in the first-order Sobolev space $H^1(\Omega)$, so the equations hold in the dual space, $H^{-1}(\Omega)$ (see below). The existence, uniqueness, and regularity theory for the Biot system together with extensions to include the possibility of viscous terms arising from secondary consolidation and the introduction of appropriate boundary conditions at both closed and drained interfaces were recently given by Showalter [26]. The case of partially saturated flow is developed by Showalter and Su [27]. All of the preceding are restricted to *homogeneous* media.

The introduction of *double-diffusion composite models* in order to model flow in *rigid heterogeneous media* is generally attributed to Barenblatt *et al.* [4]; this construction and its application to the description of *fissured media* were further developed by Warren and Root [28]. Such multiporosity or multipermeability systems have been used extensively to model various types of composite media; see Bear [29]. The mathematical theory of such systems is obtained as an application of the theory of degenerate evolution equations by Showalter [30, 31]. Such processes in *deforming* composite media are discussed in [5, 9, 32–37].

4. MATHEMATICAL PRELIMINARIES

In order to obtain the mathematical formulation of system (1), we first recall some appropriate function spaces. Then we construct operators in these spaces to represent the variational formulation of the initial–boundary-value problem for this system.

4.1. Sobolev spaces

Let Ω be a smoothly bounded domain in $\Re^3$, and denote by Γ_0 and Γ_{tr} the two complementary parts of a partition of the boundary, $\partial\Omega$. Denote by $C_0^\infty(\Omega)$ the space of infinitely differentiable functions with support contained in Ω and by $L^2(\Omega)$ the Lebesgue space of (equivalence classes of) complex valued functions whose modulus squared is integrable on Ω, with the usual inner product and induced norm

$$(f,g) = \int_\Omega f\bar{g}\,\mathrm{d}\mathbf{x}, \qquad \|f\| = (f,f)^{1/2}$$

For any $p(\cdot)\in L^2(\Omega)$ and j, $1\leqslant j\leqslant 3$, we denote by $\partial_j p$ its distributional derivative,

$$\langle \partial_j p, \varphi\rangle = -\int_\Omega p(x)\overline{\partial_j\varphi(x)}\,\mathrm{d}s, \qquad \varphi\in C_0^\infty(\Omega)$$

For integer $m\geqslant 1$, let $H^m(\Omega)$ be the *Sobolev space* consisting of those functions in $L^2(\Omega)$ having all derivatives up to order m also in $L^2(\Omega)$. Each $H^m(\Omega)$ is a Hilbert space, and we define $H^s(\Omega)$ for real numbers $s\geqslant 0$ by interpolation. The *trace* map $\gamma: H^1(\Omega)\rightarrow Rg(\gamma) = H^{1/2}(\partial\Omega)\subset L^2(\partial\Omega)$ is the operator defined by $\gamma(w) = w|_{\partial\Omega}$ as restriction to the boundary, $\partial\Omega$. The space $H_0^1(\Omega)$ is the closure in $H^1(\Omega)$ of $C_0^\infty(\Omega)$ and is characterized as the subspace of $H^1(\Omega)$ consisting of those functions whose *trace* is zero. The dual of $H_0^1(\Omega)$ is the space $H^{-1}(\Omega)$ of distributions on Ω which are first-order derivatives of functions in $L^2(\Omega)$. We shall also use the quotient space $L^2(\Omega)/\Re$ with the norm $\inf_{c\in\Re}\|p+c\|_{L^2}$. Corresponding spaces of vector-valued functions will be denoted by bold face symbols. For example, we denote the product space $(L^2(\Omega))^3$ by $\mathbf{L}^2(\Omega)$ and the corresponding triple of Sobolev spaces by $\mathbf{H}^1(\Omega)\equiv(H^1(\Omega))^3$. Finally, with I an interval in time, we denote by

$$L^2(I;H^m) = \left\{ f: I\rightarrow H^m, \int_I \|f(\tau)\|_m^2\,\mathrm{d}\tau\leqslant\infty\right\}$$

the indicated space of Bochner square-integrable vector-valued functions.

4.2. Elasticity operator

The Navier system of partial differential equations describes the small displacements of a purely elastic structure. The (small) *displacement* $\mathbf{u}(\mathbf{x}) = (u_1(\mathbf{x}), u_2(\mathbf{x}), u_3(\mathbf{x}))$ from the position $\mathbf{x}\in\Omega$ gives the (linearized) *strain* tensor $\varepsilon_{kl}(\mathbf{u})\equiv\frac{1}{2}(\partial_k u_l + \partial_l u_k)$ which provides a measure of the local deformation of the body. The *stress* σ_{ij} is a symmetric tensor that represents the internal forces on surface elements. We assume that these are related by *Hooke's law* for an *isotropic medium*,

$$\sigma_{ij} = \lambda\delta_{ij}\varepsilon_{kk} + 2\mu\varepsilon_{ij}$$

with positive Lamé constant λ and shear modulus μ. Let Γ_0 and Γ_{tr} be the complementary subsets of the boundary as given above. The stationary elasticity system is given by the equations of equilibrium

$$-\partial_j \sigma_{ij} = -\partial_j(\lambda\delta_{ij}(\partial_k u_k) + \mu(\partial_i u_j + \partial_j u_i)) = f_i \text{ in } \Omega \tag{10a}$$

$$u_i = 0 \text{ on } \Gamma_0, \quad \sigma_{ij} n_j = g_i \text{ on } \Gamma_{tr} \tag{10b}$$

for each $1 \leqslant i \leqslant 3$.† Thus, the boundary condition on Γ_0 is a constraint on displacement, and on Γ_{tr} it involves the surface density of forces or *traction* $\sigma(\mathbf{n})$ with ith component given by $\sigma_{ij} n_j$ and value determined by the unit outward normal vector $\mathbf{n} = (n_1, n_2, n_3)$ on Γ_{tr}.

In order to obtain the weak formulation of this boundary-value problem, we define the Sobolev space

$$\mathbf{V} = \{\mathbf{v} \in \mathbf{H}^1(\Omega): \ \mathbf{v} = \mathbf{0} \text{ on } \Gamma_0\}$$

of *admissable displacements*. We shall assume that *measure* $(\Gamma_0) > 0$. The variational form of the elasticity system (10) is given by

$$\mathbf{u} \in \mathbf{V}: \ e(\mathbf{u}, \mathbf{v}) = \mathbf{h}(\mathbf{v}) \quad \forall \mathbf{v} \in \mathbf{V} \tag{11}$$

where the sesquilinear form $e(\cdot,\cdot): \mathbf{V} \times \mathbf{V} \longrightarrow \Re$ and the conjugate linear functional $\mathbf{h}(\cdot)$ on the Hilbert space $\mathbf{V}$ are defined by

$$e(\mathbf{u}, \mathbf{v}) = \int_\Omega (\lambda(\partial_k u_k)\overline{(\partial_i v_i)} + 2\mu\varepsilon_{ij}(\mathbf{u})\overline{\varepsilon_{ij}(\mathbf{v}))}\, \mathrm{d}\mathbf{x}, \quad \mathbf{h}(\mathbf{v}) = \int_\Omega f_i \bar{v}_i \, \mathrm{d}\mathbf{x} + \int_{\Gamma_{tr}} g_i \bar{v}_i \, \mathrm{d}s$$

Hereafter we denote the corresponding *elasticity operator* by $\mathscr{E}: \mathbf{V} \longrightarrow \mathbf{V}'$; this is the linear operator determined by the sesquilinear form $e(\cdot,\cdot)$ on $\mathbf{V}$. The variational formulation (11) is equivalent to $\mathscr{E}(\mathbf{u}) = \mathbf{h}$. It follows from the Korn's inequality and Poincare's theorem that $e(\cdot,\cdot)$ is a $\mathbf{V}$ coercive form, and hence that $\mathscr{E}$ is an isomorphism [38, 39].

For $\mathbf{u} \in \mathbf{V}$ we define the restriction of $\mathscr{E}(\mathbf{u}) \in \mathbf{V}'$ to $\mathbf{C}_0^\infty(\Omega)$ by $\mathscr{E}_0(\mathbf{u})$. This is given by the distributions $\mathscr{E}_0(\mathbf{u}) \equiv -(\lambda + \mu)\nabla(\nabla \cdot \mathbf{u}) - \mu\Delta\mathbf{u}$. Then we can recover the boundary-value problem (10) from $\mathscr{E}$ as follows. If the closures of Γ_0 and Γ_{tr} do not intersect, and if the boundary is sufficiently smooth, then the regularity theory for strongly elliptic systems shows that whenever $\mathscr{E}_0(\mathbf{u}) \in \mathbf{L}^2(\Omega)$ we have $\mathbf{u} \in \mathbf{H}^2(\Omega) \cap \mathbf{V}$ (see References [39, 40], and then from Stokes' theorem there follows

$$\mathscr{E}(\mathbf{u})(\mathbf{v}) = (\mathscr{E}_0(\mathbf{u}), \mathbf{v})_{\mathbf{L}^2(\Omega)} + (\sigma_{ij} n_j, v_i)_{L^2(\Gamma_{tr})}, \quad \mathbf{v} \in \mathbf{V} \tag{12}$$

This shows how $\mathscr{E}$ decouples into the sum of its *formal* part $\mathscr{E}_0$ on Ω and its *boundary* part $\sigma(\mathbf{n})$ on Γ_{tr}.

†Throughout the following, we adopt the convention that *repeated subscripts* are to be summed.

4.3. Double-diffusion operator

We define the appropriate spaces and operators to describe the Barenblatt double-diffusion system (2). Suppose we are given the pair of functions $k_1(\cdot), k_2(\cdot) \in L^\infty(\Omega)$ satisfying

$$k_j(x) \geqslant c_0 > 0, \quad x \in \Omega, \ j = 1, \ 2$$

Let $V = H^1(\Omega)$ and define the sesquilinear form $a(\cdot, \cdot) : V^2 \times V^2 \longrightarrow \Re$ by

$$a\left(\begin{bmatrix} p_1 \\ p_2 \end{bmatrix}, \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}\right) = \int_\Omega (k_1 \nabla p_1 \cdot \overline{\nabla q_1} + k_2 \nabla p_2 \cdot \overline{\nabla q_2} + \kappa (p_1 - p_2)\overline{(q_1 - q_2)})\, \mathrm{d}\mathbf{x}$$

for $[p_1, p_2]$ and $[q_1, q_2] \in V^2$. The corresponding symmetric and monotone operator $A : V^2 \longrightarrow (V^2)'$ is of the form

$$A = \begin{pmatrix} A^1 & 0 \\ 0 & A^2 \end{pmatrix} + \kappa \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

where the component operators $A^i : V \longrightarrow V'$, for $i = 1, 2$, are defined by

$$A^i p(q) = \int_\Omega k_i \nabla p \cdot \overline{\nabla q}\, \mathrm{d}\mathbf{x}, \qquad p, q \in V$$

Specifically, each A^i has a formal part in $H^{-1}(\Omega)$ given by the elliptic operator

$$A_0^i\, p = -\nabla \cdot k_i \nabla p \quad \text{for} \quad i = 1, 2$$

If $p \in V$, $A_0^i p \in L^2(\Omega)$, and $k_i(\cdot)$ is smooth, then the elliptic regularity theory implies that $p \in V \cap H^2(\Omega)$, and we obtain the decoupling of A^i into

$$A^i p(q) = (A_0^i p, q)_{L^2(\Omega)} + \left(k_i \frac{\partial p}{\partial n}, \gamma q\right)_{L^2(\partial\Omega)}, \quad q \in V$$

Combining the above component operators, we find that the decoupling of the symmetric double-diffusion operator $A : V^2 \longrightarrow (V^2)'$ into a formal part in Ω and a boundary part on $\partial\Omega$ can be represented by

$$A \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = \begin{bmatrix} A_0^1(p_1) + \kappa(p_1 - p_2), & k_1 \dfrac{\partial p_1}{\partial n} \\ A_0^2(p_2) + \kappa(p_2 - p_1), & k_2 \dfrac{\partial p_2}{\partial n} \end{bmatrix} \tag{13}$$

in $(L^2(\Omega) \oplus L^2(\partial\Omega))^2$.

In order to compute the kernel of A, we note that for a pair $[p_1, p_2] \in V^2$, the equation

$$A \begin{bmatrix} p_1 \\ p_2 \end{bmatrix}\left(\begin{bmatrix} p_1 \\ p_2 \end{bmatrix}\right) = \int_\Omega k_1 \nabla p_1 \cdot \overline{\nabla p_1} + k_2 \nabla p_2 \cdot \overline{\nabla p_2} + \kappa\, |\, p_1 - p_2|^2\, \mathrm{d}\mathbf{x} = 0$$

implies ∇p_1, ∇p_2 and $\|p_1 - p_2\|$ are equal to zero, since k_1, k_2 and κ are all positive. Then we have $p_1 = p_2 = c$ for some $c \in \Re$.

4.4. Pressure-dilation operators

Let the function $\beta(\cdot) \in L^\infty(\Gamma_{tr})$ be given; we shall assume that $0 \leqslant \beta(s) \leqslant 1,\ s \in \Gamma_{tr}$. Then define the corresponding *gradient* operator, $\vec{\nabla} : V \to \mathbf{L}^2(\Omega) \oplus \mathbf{L}^2(\Gamma_{tr})$, by

$$\langle \vec{\nabla} p, [\mathbf{f}, \mathbf{g}] \rangle \equiv \int_\Omega \partial_j p \bar{f}_j \, \mathrm{d}x - \int_{\Gamma_{tr}} \beta \, p n_j \, \overline{g_j} \, \mathrm{d}s, \quad p \in V, \ [\mathbf{f}, \mathbf{g}] \in \mathbf{L}^2(\Omega) \oplus \mathbf{L}^2(\Gamma_{tr})$$

This consists of the *formal* part ∇p in Ω and the *boundary* part $-\beta \, p \, \mathbf{n}$ on Γ_{tr}, and we denote this representation by

$$\vec{\nabla} p = [\nabla p, -\beta \, p \, \mathbf{n}] \tag{14}$$

Define $\vec{\nabla} \cdot : \mathbf{L}^2(\Omega) \oplus \mathbf{L}^2(\Gamma_{tr}) \to V'$ to be the negative of the corresponding dual operator. This is the *divergence* operator $\vec{\nabla} \cdot = -\vec{\nabla}'$ given by

$$\langle \vec{\nabla} \cdot [\mathbf{f}, \mathbf{g}], \ p \rangle \equiv -\overline{\langle \vec{\nabla} p, [\mathbf{f}, \mathbf{g}] \rangle} \quad [\mathbf{f}, \mathbf{g}] \in \mathbf{L}^2(\Omega) \oplus \mathbf{L}^2(\Gamma_{tr}), \ \ p \in V$$

The trace map gives a natural identification $\mathbf{v} \mapsto [\mathbf{v}, \gamma(\mathbf{v})|_{\Gamma_{tr}}]$ of

$$\mathbf{V} \subset \mathbf{L}^2(\Omega) \oplus \mathbf{L}^2(\Gamma_{tr})$$

and this identification will be employed throughout the following. It also gives the identification $p \mapsto [p, \gamma(p)|_{\Gamma_{tr}}]$ of

$$V \subset L^2(\Omega) \oplus L^2(\Gamma_{tr})$$

We note that both of these identifications have dense range, and so the corresponding duals can be identified. That is, we have

$$\mathbf{L}^2(\Omega) \oplus \mathbf{L}^2(\Gamma_{tr}) \subset \mathbf{V}', \qquad L^2(\Omega) \oplus L^2(\Gamma_{tr}) \subset V'$$

For smoother functions $\mathbf{v} \in \mathbf{V} \subset \mathbf{L}^2(\Omega) \oplus \mathbf{L}^2(\Gamma_{tr})$ we have the *Stokes' formula*

$$\begin{aligned} \langle \vec{\nabla} \cdot \mathbf{v}, \ p \rangle &= -\int_\Omega \overline{\partial_j p} \, v_j \, \mathrm{d}x + \int_{\Gamma_{tr}} \beta \, \bar{p} v_j n_j \, \mathrm{d}s \\ &= \int_\Omega \partial_j v_j \, \bar{p} \, \mathrm{d}x - \int_{\Gamma_{tr}} (1 - \beta) \mathbf{v} \cdot \mathbf{n} \, \bar{p} \, \mathrm{d}s, \quad p \in V \end{aligned}$$

This shows the restriction satisfies

$$\vec{\nabla} \cdot : \mathbf{V} \to L^2(\Omega) \oplus L^2(\Gamma_{tr})$$

and that the divergence operator has a *formal* part in Ω as well as a *boundary* part on Γ_{tr}. We denote the part in $L^2(\Omega)$ by $\nabla \cdot$, that is, $\nabla \cdot \mathbf{v} = \partial_j v_j$, and the identity above is indicated by

$$\vec{\nabla} \cdot \mathbf{v} = [\nabla \cdot \mathbf{v}, -(1 - \beta) \mathbf{v} \cdot \mathbf{n}] \in L^2(\Omega) \oplus L^2(\Gamma_{tr}), \quad \mathbf{v} \in \mathbf{V} \tag{15}$$

Now we can extend the definition of $\vec{\nabla}$ from V up to $L^2(\Omega) \oplus L^2(\Gamma_{tr})$. This extension is obtained as $-(\vec{\nabla} \cdot)'$, the negative of the dual of the restriction to $\mathbf{V}$ of the divergence. This dual operator

$$(\vec{\nabla} \cdot)' : L^2(\Omega) \oplus L^2(\Gamma_{tr}) \to \mathbf{V}'$$

is defined for each $[f, g] \in L^2(\Omega) \oplus L^2(\Gamma_{\mathrm{tr}})$ by

$$\begin{aligned}
\langle (\vec{\nabla}\cdot)'[f, g], \mathbf{v}\rangle &= \overline{(\vec{\nabla}\cdot \mathbf{v},\, [f, g])}_{L^2(\Omega)\oplus L^2(\Gamma_{\mathrm{tr}})} \\
&= \overline{(\partial_j v_j,\, f)}_{L^2(\Omega)} - \overline{((1-\beta)\mathbf{v}\cdot\mathbf{n},\ g)}_{L^2(\Gamma_{\mathrm{tr}})} \\
&= (f,\, \nabla\cdot\mathbf{v})_{L^2(\Omega)} - (g,\, (1-\beta)\mathbf{v}\cdot\mathbf{n})_{L^2(\Gamma_{\mathrm{tr}})}, \quad \mathbf{v}\in\mathbf{V}
\end{aligned}$$

For the smoother case of $[f, g] = [w, w|_{\Gamma_{\mathrm{tr}}}]$, with the indicated $w \in V$ identified as a function on Ω and its trace on Γ_{tr}, the Stokes' formula shows that

$$\begin{aligned}
-\langle (\vec{\nabla}\cdot)'[w, w|_{\Gamma_{\mathrm{tr}}}], \mathbf{v}\rangle &= -(w,\, \nabla\cdot\mathbf{v})_{L^2(\Omega)} + (w,\, (1-\beta)\mathbf{v}\cdot\mathbf{n})_{L^2(\Gamma_{\mathrm{tr}})} \\
&= (\partial_j w,\, v_j)_{L^2(\Omega)} - (\beta w,\, \mathbf{v}\cdot\mathbf{n})_{L^2(\Gamma_{\mathrm{tr}})} \\
&= (\vec{\nabla} w, \mathbf{v})_{L^2(\Omega)\oplus L^2(\Gamma_{\mathrm{tr}})}, \qquad w\in V,\ \mathbf{v}\in\mathbf{V}
\end{aligned}$$

and this shows that $-\vec{\nabla}\cdot'$ provides the desired extension of $\vec{\nabla}$ from V to $L^2(\Omega)\oplus L^2(\Gamma_{\mathrm{tr}})$. The preceding constructions are summarized in the following diagram.

$$\begin{array}{ccccc}
\mathbf{L}^2(\Omega)\oplus\mathbf{L}^2(\Gamma_{\mathrm{tr}}) & \xrightarrow{\vec{\nabla}\cdot\, =\, -\vec{\nabla}'} & V' & & \\
\bigcup & & \bigcup & & \\
\mathbf{V} & \xrightarrow{\vec{\nabla}\cdot} & L^2(\Omega)\oplus L^2(\Gamma_{\mathrm{tr}}) & \xrightarrow{\vec{\nabla}\, =\, -(\vec{\nabla}\cdot)'} & \mathbf{V}' \\
 & & \bigcup & & \bigcup \\
 & & V & \xrightarrow{\vec{\nabla}} & \mathbf{L}^2(\Omega)\oplus\mathbf{L}^2(\Gamma_{\mathrm{tr}})
\end{array}$$

Remark. Here the space $L^2(\Omega)\oplus L^2(\Gamma_{\mathrm{tr}})$ is identified with its dual through the Riesz representation map, and since the space V is dense in $L^2(\Omega)\oplus L^2(\Gamma_{\mathrm{tr}})$ the inclusion map to the dual space V' is injective.

It will be necessary to characterize the kernels of both the gradient operator $\vec{\nabla}$ and the formal gradient $\nabla : V \to \mathbf{L}^2(\Omega)$. Recall that if $\nabla p = 0$ in $\mathbf{H}^{-1}(\Omega)$, then $p(x) = c$, a constant, for $x\in\Omega$.

Suppose that $[f, g] \in \mathrm{Ker}(\vec{\nabla})$. We have $f(x) = c$, and then from Stokes' theorem we find that

$$\int_{\Gamma_{\mathrm{tr}}} (c - (1-\beta)g)\,\overline{\mathbf{v}\cdot\mathbf{n}}\,\mathrm{d}s = 0 \tag{16}$$

for all $\mathbf{v}\in\mathbf{V}$. Therefore, we have

$$f(x) = c,\ x\in\Omega, \quad c = (1-\beta(s))g(s),\ s\in\Gamma_{\mathrm{tr}} \tag{17}$$

But from the identification of the space V as a subspace of $L^2(\Omega)\oplus L^2(\Gamma_{\mathrm{tr}})$, it follows that $[f, g]=[w, w|_{\Gamma_{\mathrm{tr}}}]\in V\cap \mathrm{Ker}(\vec{\nabla})$, where

$$w(x)=c,\ x\in\Omega,\quad c\,\beta(s)=0,\ s\in\Gamma_{\mathrm{tr}} \tag{18}$$

We summarize this calculation as the following.

Lemma 4.1. $\mathrm{Ker}(\vec{\nabla})=\Re$, the constant functions, if $\beta\equiv 0$. Otherwise, $\mathrm{Ker}(\vec{\nabla})=\{0\}$.

Hereafter we shall denote the gradient and divergence operators above by $\vec{\nabla}^{\beta}$ and $\vec{\nabla}\cdot^{\beta}$, respectively, in order to display their dependence on the function $\beta(\cdot)$.

Now let β_1 and β_2 be a pair of functions in $L^\infty(\Gamma_{\mathrm{tr}})$ as above. These determine the *coupling operators* which will be used to write down the quasi-static system for double diffusion in an elastic medium. Let the pair of numbers $\alpha_1,\ \alpha_2\geqslant 0$ be given. Using the notation introduced in the previous section, we define the linear operator $\Lambda:(L^2(\Omega)\oplus L^2(\Gamma_{\mathrm{tr}}))^2\longrightarrow \mathbf{V}'$ by

$$\begin{gathered}\Lambda[(f_1,g_1),(f_2,g_2)]=\alpha_1\vec{\nabla}^{\beta_1}(f_1,g_1)+\alpha_2\vec{\nabla}^{\beta_2}(f_2,g_2)\\ [(f_1,g_1),(f_2,g_2)]\in(L^2(\Omega)\oplus L^2(\Gamma_{\mathrm{tr}}))^2\end{gathered} \tag{19}$$

Then its dual operator $\Lambda':\mathbf{V}\longrightarrow(L^2(\Omega)\oplus L^2(\Gamma_{\mathrm{tr}}))^2$ is given by

$$\Lambda'(\mathbf{v})=-[\alpha_1\vec{\nabla}\cdot^{\beta_1}\mathbf{v},\alpha_2\vec{\nabla}\cdot^{\beta_2}\mathbf{v}],\quad \mathbf{v}\in\mathbf{V} \tag{20}$$

These operators will determine the coupling of fluid pressure to stress and of displacement to dilation, respectively.

5. THE QUASI-STATIC SYSTEM

We formulate our problem as an evolution system in the appropriate Hilbert spaces. Then we characterize the corresponding Cauchy problem as an initial–boundary-value problem for a system of partial differential equations of mixed types and discuss its relation to the Barenblatt–Biot consolidation problem. Finally, we prove that the Cauchy problem for this Barenblatt–Biot evolution system has a unique solution in two situations. With L^2-type data prescribed, it has a *strong solution*, and when H^{-1}-type data is prescribed, it has a *weak solution*.

5.1. Initial–boundary-value problem

Let $P:(L^2(\Omega)\oplus L^2(\Gamma_{\mathrm{tr}}))^2\longrightarrow(L^2(\Omega)\oplus\{0\})^2$ be the indicated *projection* operator onto the first components. In terms of the operators constructed in Section 4, the quasi-static system (1) suggests the form

$$\mathscr{E}(\mathbf{u}(t))+\Lambda[p_1(t),p_2(t)]=\mathbf{f}(t) \tag{21a}$$

$$P\begin{bmatrix}c_1\dot{p}_1(t)\\ c_2\dot{p}_2(t)\end{bmatrix}+A\begin{bmatrix}p_1(t)\\ p_2(t)\end{bmatrix}-\Lambda'\dot{\mathbf{u}}(t)=\begin{bmatrix}h_1(t)\\ h_2(t)\end{bmatrix} \tag{21b}$$

The first equation (21a) corresponds to the *equilibrium system* for momentum and the second system (21b) is the *mass balance* for double diffusion. The first equation in the space $\mathbf{V}'$ is elliptic, and the second system in $V' \times V'$ is of mixed elliptic–parabolic type with $c_1 \geqslant 0, c_2 \geqslant 0$. The forcing terms $\mathbf{f}(\cdot)$, $h_1(t)$ and $h_2(t)$ represent any externally applied forces and sources, respectively. Note that we can eliminate the non-homogeneous term $\mathbf{f}(t)$ from this system by a simple translation. That is, for each $t \geqslant 0$, let $\mathbf{u}_0(t)$ be the solution of the stationary elasticity problem $\mathscr{E}(\mathbf{u}_0(t)) = \mathbf{f}(t)$, and replace $\mathbf{u}(t)$ in (21) by $\mathbf{u}(t) + \mathbf{u}_0(t)$ to obtain the equivalent system

$$\mathscr{E}(\mathbf{u}(t)) + \Lambda[p_1(t), p_2(t)] = \mathbf{0} \text{ in } \mathbf{V}' \tag{22a}$$

$$P \begin{bmatrix} c_1 \dot{p}_1(t) \\ c_2 \dot{p}_2(t) \end{bmatrix} + A \begin{bmatrix} p_1(t) \\ p_2(t) \end{bmatrix} - \Lambda' \dot{\mathbf{u}}(t) = \begin{bmatrix} h_1(t) \\ h_2(t) \end{bmatrix} + \Lambda' \dot{\mathbf{u}}_0(t) \quad \text{in } V' \times V' \tag{22b}$$

Then rename $[h_1(t), h_2(t)]$ to be $[h_1(t), h_2(t)] + \Lambda' \dot{\mathbf{u}}_0(t)$. Thus, any non-homogeneous internal or boundary distributed stresses can be replaced by corresponding null data.

We consider the quasi-static system (22) and show below that it is essentially a *parabolic* system which has a *strong* solution under minimal smoothness requirements on the initial data and sources $h_1(\cdot)$ and $h_2(\cdot)$. Note that (22b) requires that each $p_i(t) \in V$, so both terms of (22a) are necessarily in $(\mathbf{L}^2(\Omega) \oplus \mathbf{L}^2(\Gamma_{\mathrm{tr}}))$, and this forces additional regularity on the displacement $\mathbf{u}(t)$. By a *strong solution*, we mean that Equation (22b) holds in the smaller space $(L^2(\Omega) \oplus L^2(\Gamma_{\mathrm{tr}}))^2 \subset (V')^2$, so this solution has the additional regularity necessary to decouple the partial differential equations and the boundary conditions implicit in (22b).

We shall display system (22) explicitly in its parts as an initial–boundary-value problem for the system of partial differential equations and boundary conditions. Denote by χ_{tr} the characteristic function of the traction boundary, Γ_{tr}. Using the decompositions of the operators constructed in Section 4, we find that system (22) takes the form

$$\mathscr{E}_0(\mathbf{u}(t)) + \alpha_1 \nabla p_1(t) + \alpha_2 \nabla p_2(t) = \mathbf{0} \quad \text{in } \Omega \tag{23a}$$

$$\frac{\partial}{\partial t}(c_1 p_1(t) + \alpha_1 \nabla \cdot \mathbf{u}(t)) + A_0^1(p_1(t)) + \kappa(p_1(t) - p_2(t)) = h_1(t) \quad \text{in } \Omega \tag{23b}$$

$$\frac{\partial}{\partial t}(c_2 p_2(t) + \alpha_2 \nabla \cdot \mathbf{u}(t)) + A_0^2(p_2(t)) + \kappa(p_2(t) - p_1(t)) = h_2(t) \quad \text{in } \Omega \tag{23c}$$

$$\mathbf{u}(t) = \mathbf{0} \quad \text{on } \Gamma_0, \quad \sigma(\mathbf{n}) - \alpha_1 \beta_1 p_1 \mathbf{n} - \alpha_2 \beta_2 p_2 \mathbf{n} = 0 \quad \text{on } \Gamma_{\mathrm{tr}} \tag{23d}$$

$$\left.\begin{aligned} -\alpha_1(1-\beta_1)\chi_{\mathrm{tr}} \dot{\mathbf{u}}(t) \cdot \mathbf{n} + k_1 \frac{\partial p_1(t)}{\partial n} &= \eta_1(t) \\ -\alpha_2(1-\beta_2)\chi_{\mathrm{tr}} \dot{\mathbf{u}}(t) \cdot \mathbf{n} + k_2 \frac{\partial p_2(t)}{\partial n} &= \eta_2(t) \end{aligned}\right\} \text{ on } \partial\Omega \tag{23e}$$

for each $t > 0$, and

$$\begin{aligned}
\lim_{t\to 0^+} [c_1 p_1(t) + \alpha_1 \nabla\cdot \mathbf{u}(t)] &= \upsilon_1 \\
\lim_{t\to 0^+} [c_2 p_2(t) + \alpha_2 \nabla\cdot \mathbf{u}(t)] &= \upsilon_2 \text{ in } L^2(\Omega) \\
\lim_{t\to 0^+} \alpha_1(1-\beta_1)\mathbf{u}(t)\cdot\mathbf{n} &= v_1 \\
\lim_{t\to 0^+} \alpha_2(1-\beta_2)\mathbf{u}(t)\cdot\mathbf{n} &= v_2 \text{ in } L^2(\Gamma_{\rm tr})
\end{aligned} \tag{23f}$$

where each $[h_i(t), \eta_i(t)] \in L^2(\Omega)\oplus L^2(\Gamma_{\rm tr})$ and the initial functions $[\upsilon_i, v_i]$ are given similarly. Note that Equation (22a) is equivalent to pair (23a) and (23d), because $p_1(t)$ and $p_2(t)$ both belong to V. Furthermore, for the strong solution, we have sufficient additional regularity to guarantee that $A_0^i(p_i(t)) \in L^2(\Omega)$ for $i = 1, 2$, and then (22b) is equivalent to (23b), (23c), and (23e).

Let us consider the meaning of the boundary conditions in the context of this poroelasticity model. Equations (23d) consist of the complementary pair requiring null displacement on the *clamped boundary*, Γ_0, and a balance of forces on the *traction boundary*, $\Gamma_{\rm tr}$. The boundary conditions (23e) require a balance of fluid mass. For each $j = 1,\ 2$, the function $\beta_j(\cdot)$ is defined on that portion $\Gamma_{\rm tr}$ of the boundary which is neither drained nor clamped, and it specifies the surface fraction of the pores from component j which are *sealed* along $\Gamma_{\rm tr}$. For these the hydraulic pressure contributes to the total stress within the structure. The remaining portion $1-\beta_j(\cdot)$ of the pores are *exposed* along $\Gamma_{\rm tr}$, and these contribute to the flux. On any portion of $\Gamma_{\rm tr}$ which is completely exposed, that is, where $\beta_j = 0$ for $j = 1,\ 2$, only the *effective* or elastic component of stress is specified, since there the fluid pressures do not contribute to the support of the matrix. On the entire boundary there is a transverse flow into component j that is given by the input $\eta_j(\cdot)$ and the relative normal displacement of the structure. This input could be specified in the form $\eta_j(t) = -(1-\beta_j)\mathbf{v}(t)\cdot\mathbf{n}$, where $\mathbf{v}(t)$ is the given velocity of fluid or boundary flux on $\Gamma_{\rm tr}$. The first term and right side of these flux balances are null where $\beta_j = 1$, so the same holds for the second terms in (23e), that is, we have the *impermeable* conditions $k\partial p_j(t)/\partial n = 0$ on a completely sealed portion of $\Gamma_{\rm tr}$. We also note that in (23e) the first term on the left side and the right side of each equation are null on Γ_0, so the same necessarily holds for the second term on the left side of each. That is, we always have the *null flux* conditions $k_1\partial p_1/\partial n = k_2 \partial p_2/\partial n = 0$ on Γ_0.

5.2. Strong solution

In order to write the system as a single equation, we solve (22a) for $\mathbf{u}(t)$ and substitute it into (22b) to obtain the equivalent form

$$\frac{\mathrm{d}}{\mathrm{d}t}\left((\mathbf{c}P + \Lambda'\mathscr{E}^{-1}\Lambda)\begin{bmatrix} p_1(t) \\ p_2(t)\end{bmatrix}\right) + A\begin{bmatrix} p_1(t) \\ p_2(t)\end{bmatrix} = \begin{bmatrix} h_1(t) \\ h_2(t)\end{bmatrix} \tag{24}$$

which holds in $(L^2(\Omega)\oplus L^2(\Gamma_{\rm tr}))^2$. Here we have let

$$\mathbf{c} = \begin{pmatrix} c_1 & 0 \\ 0 & c_2 \end{pmatrix}$$

be the indicated matrix. Equation (24) suggests the construction of the operator B defined on $(L^2(\Omega)\oplus L^2(\Gamma_{\mathrm{tr}}))^2$ by

$$B[p_1(t), p_2(t)] = -\Lambda'(\mathbf{u})$$

where

$$\mathscr{E}(\mathbf{u}) = -\Lambda([p_1, p_2]), \quad \text{and} \quad p_i \in L^2(\Omega)\oplus L^2(\Gamma_{\mathrm{tr}}),\ i = 1, 2$$

In terms of the operator B, the system has the form

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{c}P + B)\begin{bmatrix} p_1(t) \\ p_2(t) \end{bmatrix} + A\begin{bmatrix} p_1(t) \\ p_2(t) \end{bmatrix} = \begin{bmatrix} h_1(t) \\ h_2(t) \end{bmatrix}$$

The time derivative of the solution occurs implicitly, so this is an evolution equation of generalized *Sobolev type*, an *implicit evolution equation.*

Lemma 5.1. The operator

$$B = \Lambda'\mathscr{E}^{-1}\Lambda : (L^2(\Omega)\oplus L^2(\Gamma_{\mathrm{tr}}))^2 \longrightarrow (L^2(\Omega)\oplus L^2(\Gamma_{\mathrm{tr}}))^2$$

is continuous, self-adjoint and accretive.‡

Proof. The linear operator B is a composition of continous maps. Since $\mathscr{E}^{-1}$ is symmetric and monotone, it follows that B is self-adjoint and accretive. □

First, we consider the *non-degenerate case* in which both of the functions c_1 and c_2 are uniformly positive and bounded on Ω. The operator obtained by restricting B to $V\times V$,

$$M \equiv \mathbf{c}P + B : V\times V \longrightarrow V'\times V'$$

is symmetric and strictly monotone, so it determines a norm on the space $V\times V$. Let W_m be the space $V\times V$ with this norm and the corresponding scalar product, $(\cdot,\cdot)_m \equiv M(\cdot,\cdot)$. Then from the estimates

$$C'\|[p_1, p_2]\|_{(L^2(\Omega)\oplus, L^2(\Gamma_{\mathrm{tr}}))^2} \geqslant \|[p_1, p_2]\|_{W_m} \geqslant C''\|[p_1, p_2]\|_{(L^2(\Omega))^2}$$

for some constants C' and $C''>0$, it follows that W_m' is a Hilbert space for which we have the continuous imbeddings

$$(L^2(\Omega)\oplus\{0\})^2 \to W_m' \to (L^2(\Omega)\oplus L^2(\Gamma_{\mathrm{tr}}))^2$$

and the identity

$$f([p_1, p_2]) = (f, M[p_1, p_2])_{W_m'}, \quad f\in W_m',\ [p_1, p_2]\in V\times V$$

Let $D(A)$ be the subspace $D(A) = \{[p_1, p_2]\in V^2 : A[p_1, p_2]\in W_m'\}$ of W_m. To obtain an (explicit) evolution equation in W_m' which is equivalent to

$$\frac{\mathrm{d}}{\mathrm{d}t}M[p_1(t), p_2(t)] + A[p_1(t), p_2(t)] = [h_1(t), h_2(t)] \tag{25}$$

‡See [30] or [31] for definitions of *accretive*, *m-accretive*, *sectorial* and *m-sectorial* operators.

we define an unbounded operator $\mathscr{C}$ on W_m' by

$$\mathscr{C}[q_1, q_2] = A[p_1, p_2] \quad \text{iff} \quad [q_1, q_2] = M[p_1, p_2] \quad \text{for some } [p_1, p_2] \in D(A)$$

This $\mathscr{C}(\cdot)$ is defined on $\mathrm{Dom}(\mathscr{C}) = M[D(A)]$, and the equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\begin{bmatrix} q_1(t) \\ q_2(t) \end{bmatrix}\right) + \mathscr{C}\left(\begin{bmatrix} q_1(t) \\ q_2(t) \end{bmatrix}\right) = \begin{bmatrix} h_1(t) \\ h_2(t) \end{bmatrix} \tag{26}$$

is equivalent to (25). We have

$$(\mathscr{C}[q_1, q_2], [q_1, q_2])_{W_m'} = (A[p_1, p_2], M[p_1, p_2])_{W_m'} = A[p_1, p_2]([p_1, p_2])$$

and $\mathrm{Rg}(I + \mathscr{C}) = \mathrm{Rg}(M + A)$ in W_m'. We shall prove the following.

Lemma 5.2. The unbounded operator $\mathscr{C}$ is self-adjoint and accretive on the Hilbert space W_m'.

Proof. The operator A is symmetric and monotone from W_m to W_m'; hence, the operator $\mathscr{C}$ is symmetric and accretive. To show $\mathscr{C}$ is self-adjoint, we will check that $M + A$ is V^2 coercive and therefore $\mathrm{Rg}(M + A) = W_m'$. This will imply that $I + \mathscr{C}$ is *onto* W_m', so $\mathscr{C}$ is maximal symmetric, hence, self-adjoint. Let $[p_1, p_2] \in D(A)$. Then

$$\begin{aligned}(M + A)[p_1, p_2]([p_1, p_2]) &= (\mathbf{c}P + B + A)[p_1, p_2]([p_1, p_2]) \\ &= \int_\Omega (c_1|p_1|^2 + c_2|p_2^2|)\,\mathrm{d}\mathbf{x} + \langle \mathbf{u}, \mathscr{E}(\mathbf{u})\rangle \\ &\quad + \int_\Omega (k_1|\nabla p_1|^2 + k_2|\nabla p_2|^2 + \kappa|p_1 - p_2|^2)\,\mathrm{d}\mathbf{x} \\ &\geqslant c_0\|[p_1, p_2]\|_{V^2}^2\end{aligned}$$

for some $c_0 > 0$, since $c_1 > 0$ and $c_2 > 0$. □

Lemma 5.2 shows that the operator $-\mathscr{C}$ is the generator of an holomorphic semigroup on the space W_m', [30, 31, 41]. This implies that the initial-value problem for the evolution equation (26) is well posed. That is, for each $\mathbf{q} \in W_m'$ (26) has a unique solution $[q_1(\cdot), q_2(\cdot)] \in C^0([0, T], W_m') \cap C^1((0, T], W_m')$ with $[q_1(0), q_2(0)] = \mathbf{q}$. Moreover, we obtain the following.

Theorem 1. Assume the functions $c_1(\cdot)$ and $c_2(\cdot)$ are uniformly strictly positive and bounded on Ω. Let $T > 0$, υ_1, $\upsilon_2 \in (L^2(\Omega))^2$, and the Hölder continuous functions $h_1(\cdot)$, $h_2(\cdot) \in C^\alpha([0, T], L^2(\Omega))$ be given. Then there exists a unique triple of functions $p_1(\cdot)$, $p_2(\cdot) : (0, T] \to V$ and $\mathbf{u}(\cdot) : (0, T] \to \mathbf{V}$ for which each $p_i \in C^0([0, T], L^2(\Omega)) \cap C^1((0, T], L^2(\Omega))$ and

$$\alpha_i[\nabla_0 \cdot \mathbf{u}(\cdot), (1 - \beta_i)\mathbf{u}(\cdot) \cdot \mathbf{n}] \in C^0([0, T], L^2(\Omega) \oplus L^2(\Gamma_{\mathrm{tr}})) \cap C^1((0, T], L^2(\Omega) \oplus L^2(\Gamma_{\mathrm{tr}}))$$

for $i = 1, 2$, and they satisfy the initial–boundary-value problem (23) with $\eta_i = v_i = 0$. Moreover, the solution satisfies

$$t \longrightarrow tA([p_1(t), p_2(t)]) \in L^\infty([0, T], (L^2(\Omega) \oplus L^2(\Gamma_{\mathrm{tr}}))^2)$$

From Equation (22a) and the continuity of Λ and $\mathscr{E}^{-1}$, it follows that

$$\mathbf{u}(\cdot) \in C^0([0,T],\mathbf{V}) \cap C^1((0,T],\mathbf{V})$$

and from *regularity* of A and $\mathscr{E}$ we obtain

$$\|p(t)\|_{H^2(\Omega)},\ \ \|\mathbf{u}(t)\|_{(H^2(\Omega))^3} \leqslant C/t, \quad 0<t\leqslant T$$

We also note that the initial condition (23f) is equivalent to specifying at the initial time, $t=0$, the combinations

$$[c_i\, p_i(t) + \alpha_i \nabla_0 \cdot \mathbf{u}(t),\ \ \alpha_i(1-\beta_i)\mathbf{u}(t)\cdot\mathbf{n}],\ \ i=1,2$$

which are just the *fluid content* in the respective components.

Theorem 2. When the given functions $h_1(\cdot), h_2(\cdot)$ are smooth, the solution to the system is C^∞ for $t>0$.

Proof. Since the linear operators A and $\mathscr{E}$ are regularizing, we can show that

$$(I+\mathscr{C})^{-1} : (H^k(\Omega))^2 \longrightarrow (H^{k+2}(\Omega))^2$$

Consider $[q_1,q_2] \in (H^k(\Omega)\oplus L^2(\Gamma_{\mathrm{tr}}))^2$ for which $(I+\mathscr{C})([q_1,q_2]) \in (H^k(\Omega)\oplus L^2(\Gamma_{\mathrm{tr}}))^2$, that is, $\mathscr{C}([q_1,q_2]) = A([p_1,p_2]) \in (H^k(\Omega)\oplus L^2(\Gamma_{\mathrm{tr}}))^2$. Then the regularity theory for the elliptic operator A implies that $p_i \in H^{k+2}(\Omega)\oplus L^2(\Gamma_{\mathrm{tr}})$. Also by Section 4.2 the regularity effect of $\mathscr{E}$ on the solution implies that the spaces $H^{k+2}(\Omega)\oplus L^2(\Gamma_{\mathrm{tr}})$ are invariant under B, and hence $q_i = c_i P p_i + B(p_i) \in H^{k+2}(\Omega)\oplus L^2(\Gamma_{\mathrm{tr}})$ for $i=1,2$. Repeated application of the above will give the smoothness of the solution. □

5.3. Degenerate case

Here we consider the special case, in which one of the storage coefficients can vanish, e.g., $c_1=0$. In this case system (21b) is of mixed elliptic–parabolic type. In the context of the poroelasticity model, this means the fracture component of the porous medium consists of regions of negligible relative volume, and as a result the storage coefficient for this component vanishes. In this degenerate case, Equation (24) will be

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\left(\begin{pmatrix} 0 & 0 \\ 0 & c_2 \end{pmatrix} P + \Lambda' \mathscr{E}^{-1}\Lambda\right)\begin{bmatrix} p_1(t) \\ p_2(t) \end{bmatrix}\right) + A\begin{bmatrix} p_1(t) \\ p_2(t) \end{bmatrix} = \begin{bmatrix} h_1(t) \\ h_2(t) \end{bmatrix}$$

and the operator $M \equiv \mathbf{c} + B : V\times V \to V'\times V'$ with

$$\mathbf{c} = \begin{pmatrix} 0 & 0 \\ 0 & c_2 \end{pmatrix}$$

is symmetric and monotone. In order to construct the evolution operator $\mathscr{C}$ as before, it is sufficient that M be one-to-one. In order to check that condition, let $[p_1,p_2] \in V^2$. Then

$$M[p_1,p_2]([p_1,p_2]) = \int_\Omega c_2|p_2|^2\,\mathrm{d}\mathbf{x} + \langle \mathbf{u}, \mathscr{E}(\mathbf{u})\rangle = 0$$

implies $p_2 = 0$, and $\Lambda[p_1, p_2] = \alpha_1 \vec{\nabla}^{\beta_1}(p_1) + \alpha_2 \vec{\nabla}^{\beta_2}(p_2) = \mathbf{0}$. By Lemma 4.1, if $\beta_1 \neq 0$, then $p_1 = 0$. This shows that $\ker(M) = \{\mathbf{0}\}$ and, hence, M is strictly monotone. The above system is well posed by the same argument presented for the non-degenerate case. More generally, we have the following *kernel condition*.

Lemma 5.3. The condition $\ker(M) = \{\mathbf{0}\}$ holds if either $c_1(\cdot) > 0$ and $\alpha_2\, \beta_2(\cdot) \not\equiv 0$, or if $c_2(\cdot) > 0$ and $\alpha_1\, \beta_1(\cdot) \not\equiv 0$.

As described above, the function β_1 specifies the fraction of the fissures that are sealed along the boundary $\Gamma_{\rm tr}$ Two extremes are the situationss in which a uniformly parallel fracture system is perpendicular to $\Gamma_{\rm tr}$ so all of the fractures are cut by $\Gamma_{\rm tr}$, or in which all the fracture system is parallel to $\Gamma_{\rm tr}$ and all the fissures are sealed on the boundary surface. In the first scenario $\beta_1 = 0$, and in the latter $\beta_1 = 1$. Most previous works have considered only the latter case. The more typical scenario is the situation in which β_1 takes on a value between 0 and 1.

5.4. Weak solution

Let us differentiate the first equation of the quasi-static system (22) to obtain

$$\mathscr{E}(\dot{\mathbf{u}}(t)) + \Lambda([\dot{p}_1(t),\ \dot{p}_2(t)]) = \mathbf{0} \quad \text{in } \mathbf{V}' \tag{27a}$$

$$\mathbf{c}P[\dot{p}_1(t),\ \dot{p}_2(t)] + A[p_1(t),\ p_2(t)] - \Lambda'\dot{\mathbf{u}}(t) = [h_1(t), h_2(t)] \quad \text{in } V' \tag{27b}$$

This puts the system in the form of an *implicit evolution equation*

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} \mathscr{E} & \Lambda \\ -\Lambda' & \mathbf{c}P \end{pmatrix}\begin{bmatrix} \mathbf{u}(t) \\ p_1(t),\ p_2(t) \end{bmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & A \end{pmatrix}\begin{bmatrix} \mathbf{u}(t) \\ p_1(t),\ p_2(t) \end{bmatrix} = \begin{bmatrix} 0 \\ h_1(t), h_2(t) \end{bmatrix}$$

with the indicated matrix operators. This clearly displays the symmetric as well as anti-symmetric terms in the system and suggests the following structure of our operators.

On the product space $\mathscr{V} \equiv \mathbf{V} \times V^2$, we define the bilinear form $b(\cdot\,,\cdot) : \mathscr{V} \times \mathscr{V} \longrightarrow \Re$ by

$$b([\mathbf{u}, p_1, p_2], [\mathbf{v}, q_1, q_2]) = e(\mathbf{u}, \mathbf{v}) + \langle \Lambda[p_1, p_2], \mathbf{v}\rangle - \langle \Lambda'\mathbf{u}, [q_1, q_2]\rangle + \int_\Omega c_1 p_1 q_1 + c_2 p_2 q_2 \,\mathrm{d}\mathbf{x}$$

Lemma 5.4. The operator $\mathscr{B} : \mathscr{V} \to \mathscr{V}'$ is continuous, monotone, and sectorial with $\mathrm{Ker}(\mathscr{B}) = \{\mathbf{0}, 0, 0\}$.

Proof. On the diagonal we have

$$\mathrm{Re}\langle \mathscr{B}[\mathbf{u}, p_1, p_2], [\mathbf{u}, p_1, p_2]\rangle = e(\mathbf{u}, \mathbf{u}) + \int_\Omega (c_1|p_1|^2 + c_2|p_2|^2)\,\mathrm{d}x$$

Since the sesquilinear $e(\mathbf{u}, \mathbf{u})$ is $\mathbf{V}$-coercive and c_1, c_2 are uniformly positive, it follows that $\mathrm{Ker}(\mathscr{B})$ is null and there is a constant $c > 0$ such that

$$\mathrm{Re}\langle \mathscr{B}[\mathbf{u}, p_1, p_2], [\mathbf{u}, p_1, p_2]\rangle \geqslant c|\mathrm{Im}\langle \mathscr{B}[\mathbf{u}, p_1, p_2], [\mathbf{u}, p_1, p_2]\rangle| \qquad \square$$

Similarly, we define on $\mathscr{V}$ the sesquilinear form $a(\cdot,\cdot):\mathscr{V}\times\mathscr{V}\longrightarrow\Re$ by

$$a([\mathbf{u},p_1,p_2],[\mathbf{v},q_1,q_2]) = A[p_1,p_2]([q_1,q_2]) = \int_\Omega k_1\nabla p_1\cdot\overline{\nabla q_1} + k_2\nabla q_2\cdot\overline{\nabla q_2} + \kappa(p_1-p_2)\overline{(q_1-q_2)}\,\mathrm{d}\mathbf{x}$$

and this gives a symmetric and monotone operator $\mathscr{A}:\mathscr{V}\longrightarrow\mathscr{V}'$ in terms of the component operator $A: V^2\longrightarrow(V')^2$. From the matrix representations of these operators,

$$\mathcal{B} = \begin{pmatrix} \mathscr{E} & \Lambda \\ -\Lambda' & \mathbf{c}P \end{pmatrix}, \quad \text{and} \quad \mathscr{A} = \begin{pmatrix} 0 & 0 \\ 0 & A \end{pmatrix}$$

we see that they are precisely the ones which represent the quasi-static system (27) in the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{B}(\vec{u}(t)) + \mathscr{A}(\vec{u}(t)) = \vec{f}(t) \tag{28}$$

with $\vec{u}=[\mathbf{u},p_1,p_2]\in\mathscr{V}$.

Being a continuous, symmetric and monotone linear operator, $\mathscr{A}$ defines a seminorm on the space $\mathscr{V}$. We denote the space $\mathscr{V}$ with this seminorm $\mathscr{A}(\cdot,\cdot)^{1/2}$ by $\mathscr{V}_a$. Then the injection $\mathscr{V}\to\mathscr{V}_a$ is continous, and we have $\mathscr{V}_a'\subset\mathscr{V}'$. The space $\mathscr{V}_a'$ is a Hilbert space, for which we have the identity $\vec{f}(\vec{u})=(\vec{f},\mathscr{A}\vec{u})_{\mathscr{V}_a'}$, $\vec{f}\in\mathscr{V}_a'$, $\vec{u}\in\mathscr{V}$. Define the domain

$$D(\mathscr{C}) = \{\mathscr{B}(\vec{u})\in\mathscr{V}_a' :\ \text{for some } \vec{u}\in\mathscr{V}\}$$

and the linear operator $\mathscr{C}: D(\mathscr{C})\longrightarrow\mathscr{V}_a'$ by

$$\mathscr{C}(\vec{v}) = \mathscr{A}\vec{u} \quad \text{iff } \vec{v}=\mathscr{B}(\vec{u})\in\mathscr{V}_a' \text{ for some } \vec{u}\in\mathscr{V} \tag{29}$$

Then for any $\vec{v}\in D(\mathscr{C})$ we obtain $(\mathscr{C}\vec{v},\vec{v})_{\mathscr{V}_a'}=(\mathscr{A}\vec{u},\mathscr{B}\vec{u})_{\mathscr{V}_a'}=\overline{(\mathscr{B}\vec{u})(\vec{u})}$, and since $\mathscr{B}$ is sectorial on $\mathscr{V}$, it follows that $\mathscr{C}$ is sectorial on $\mathscr{V}_a'$. Moreover, we see that $\mathrm{Rg}(I+\mathscr{C})=\mathrm{Rg}(\mathscr{A}+\mathscr{B})\cap\mathscr{V}_a'$. This leads to the following lemma.

Lemma 5.5. The operator $\mathscr{C}$ is m-sectorial on $\mathscr{V}_a'$.

Proof. It suffices to show that $\mathscr{V}_a'\subset\mathrm{Rg}(\mathscr{A}+\mathscr{B})$. But we have $\mathrm{Rg}(\mathscr{A}+\mathscr{B})=\mathscr{V}'$, since the operator $\mathscr{B}+\mathscr{A}$ is $\mathscr{V}$-coercive. □

The equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{v} + \mathscr{C}(\vec{v}) = \vec{f} \quad \text{in } \mathscr{V}_a' \tag{30}$$

is equivalent to (28). By Lemma 5.5, the operator $-\mathscr{C}$ is the generator of an holomorphic semigroup on the Hilbert space $\mathscr{V}_a'$, which implies that the *Cauchy Problem* for (30) is well posed. Specifically, we obtain from this the following results for the corresponding weak solution of (28) and (27).

Theorem 3 (*Holomorphic case*). Let $T>0$. Let the pair $[\upsilon_1,\upsilon_2]\in(V_a')^2$ and a pair of Hölder continuous functions $h_1(\cdot)$, $h_2(\cdot)\in C^{\alpha}([0,T],V_a')$ be given. Then there exist a unique triple of functions $p_1(\cdot)$, $p_2(\cdot)$: $(0,T]\longrightarrow V$ and $\mathbf{u}(\cdot)$: $(0,T]\longrightarrow \mathbf{V}$ for which $c_i\,p_i(\cdot)+\alpha_i\vec{\nabla}\cdot\mathbf{u}(\cdot)\in C^0([0,T],V_a')\cap C^1((0,T],V_a')$, $i=1,2$, and they satisfy the initial-value problem

$$\mathscr{E}(\mathbf{u}(t))+\alpha_1\vec{\nabla}p_1(t)+\alpha_2\vec{\nabla}p_2(t)=\mathbf{0} \tag{31a}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}(c_1Pp_1(t)+\alpha_1\vec{\nabla}\cdot\mathbf{u}(t))+A(p_1(t))=h_1(t) \tag{31b}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}(c_2Pp_2(t)+\alpha_2\vec{\nabla}\cdot\mathbf{u}(t))+A(p_2(t))=h_2(t)\ \text{ for }\ t\in(0,T] \tag{31c}$$

$$\lim_{t\to0^+}(c_1Pp_1(t)+\alpha_1\vec{\nabla}\cdot\mathbf{u}(t))=\upsilon_1\ \text{ in }\ V_a' \tag{31d}$$

$$\lim_{t\to0^+}(c_2Pp_2(t)+\alpha_2\vec{\nabla}\cdot\mathbf{u}(t))=\upsilon_2\ \text{ in }\ V_a' \tag{31e}$$

In addition the solution satisfies $\|\mathscr{A}(\vec{u}(t))\|_{\mathscr{V}_a'}\leqslant C/t$, $0<t\leqslant T$.

The above condition gives the following bounds on the solution:

$$\|p_i\|_{V_a}=\frac{\langle A^i p_i,p_i\rangle}{\|p_i\|_{V_a}}\leqslant\|A^i p_i\|_{V_a'}\leqslant C/t,\quad 0<t\leqslant T\ \text{ for }\ i=1,2$$

and also since $\mathscr{E}$ is $\mathbf{V}$-coercive we have

$$\|\mathbf{u}(t)\|_{\mathbf{V}}\leqslant\|\mathscr{E}(\mathbf{u}(t))\|_{\mathbf{V}'}\leqslant\frac{C}{t},\quad 0<t\leqslant T$$

We also have the following regularity results for the solution:

$$\mathscr{A}([\mathbf{u},p_1,p_2])\in C^0((0,T],\mathscr{V}_a'),\quad \mathbf{u}\in C^0([0,T],\mathbf{V}\cap H^2(\Omega))\cap C^1((0,T],\mathbf{V}\cap H^2(\Omega))$$

$$p_1(\cdot)\in C^0([0,T],V/\mathrm{Ker}(A^1))\cap C^1((0,T],V/\mathrm{Ker}(A^1))$$

$$p_2(\cdot)\in C^0([0,T],V/\mathrm{Ker}(A^2))\cap C^1((0,T],V/\mathrm{Ker}(A^2))$$

5.4.1. The boundary-value problem. We note finally that in the case of the weak solution, Equation (31a) in $(\mathbf{L}^2(\Omega)\oplus\mathbf{L}^2(\Gamma_{\mathrm{tr}}))^2$ is equivalent to the pair of equations

$$\mathscr{E}_0(\mathbf{u}(t))+\alpha_1\nabla p(t)+\alpha_2\nabla p(t)=\mathbf{0}\ \text{ in }\ \Omega \tag{32a}$$

$$\mathbf{u}(t)=\mathbf{0}\ \text{ on }\ \Gamma_0,\ \ \sigma(\mathbf{n})-\alpha_1\beta_1p_1\mathbf{n}-\alpha_2\beta_2p_2\mathbf{n}=0\ \text{ on }\ \Gamma_{\mathrm{tr}} \tag{32b}$$

However, we cannot similarly decompose (31b) and (31c) into partial differential equations and boundary conditions, as they hold in the space $(V_a')^2$. In this situation the weak solution is not smooth enough to apply Stokes' theorem.

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