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SEMILINEAR DEGENERATE BIOT-SIGNORINI SYSTEM

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Abstract. Nonlinear extensions of the quasi-static Biot model of consolidation are studied with emphasis on boundary conditions, attainment of initial values, and parabolic regularizing effects. The local fluid content is monotone and possibly nonlinear or degenerate with respect to pressure, and the stress of the solid in the fully-saturated porous medium is strictly monotone in strain. In addition to boundary conditions of classical Dirichlet, Neumann, or Robin type, the medium may have a singular or degenerate semipermeable interface with the exterior fluid at a known pressure, and the monotone dependence of traction on boundary displacement includes unilateral constraints of Signorini type given by a variational inequality. The initial-boundary-value problem for this general system is formulated as a Cauchy problem in Hilbert space for a semilinear implicit evolution equation that is nonlinear in the time derivative, and it is shown to be well-posed with regularity of the solution dependent on the data. When the stress is the derivative of a convex strain energy function, the evolution equation is a gradient flow with corresponding *parabolic* regularizing effects on the solution.

1. Introduction

A section of the fluid-filled vertical cylinder $G \times \mathbb{R}$ with bounded open $G \subset \mathbb{R}^2$ is occupied by the fully-saturated and deformable porous medium $\Omega \equiv \{\mathbf{x} = (x_1, x_2, x_3) : (x_1, x_2) \in G,\ \varphi_0(x_1, x_2) < x_3 < \varphi_1(x_1, x_2)\}$ with $\varphi_0 < \varphi_1$. At time $t > 0$ and position $\mathbf{x} \in \Omega$, we denote *pressure* and *flux* of the fluid by $p(\mathbf{x}, t)$ and $\mathbf{q}(\mathbf{x}, t)$, *displacement* and *stress* of the medium by $\mathbf{u}(\mathbf{x}, t)$ and $\sigma(\mathbf{x}, t)$, respectively. The *quasi-static* Biot system for this poroelastic medium is

$$\frac{\partial}{\partial t}\big(c(p) + \alpha \boldsymbol{\nabla} \cdot \mathbf{u}\big) + \boldsymbol{\nabla} \cdot \mathbf{q} = F\,, \tag{1.1a}$$

$$-\boldsymbol{\nabla} \cdot \sigma + \alpha \boldsymbol{\nabla} p = \mathbf{f}\,, \tag{1.1b}$$

$$\mathbf{q} + \kappa \boldsymbol{\nabla} p = \kappa \mathbf{g}\,, \text{ and} \tag{1.1c}$$

$$\sigma = E(\varepsilon(\mathbf{u})) \text{ in } \Omega,\ 0 < t \le T. \tag{1.1d}$$

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Partial derivatives are denoted by $\partial_i = \frac{\partial}{\partial x_i}, 1 \leq i \leq 3$, $\boldsymbol{\nabla} s = (\partial_1 s, \partial_2 s, \partial_3 s)$ is the *gradient* of the real-valued function $s(\mathbf{x})$, $\boldsymbol{\nabla} \cdot \mathbf{r} = \partial_i r_i$ is the *divergence* of the vector-valued function $\mathbf{r}(\mathbf{x})$, and $\varepsilon_{ij}(\mathbf{r}) = \frac{1}{2}(\partial_i r_j + \partial_j r_i)$ is the *symmetric gradient* tensor. We employ the use of bold-face letters for vectors, lower-case Greek letters for symmetric second-order tensors, and summation on repeated indices is understood. Accordingly, we use $\mathbf{v} \cdot \mathbf{w} = v_i w_i$ for the scalar product of vectors and $\sigma : \tau = \sigma_{ij}\tau_{ij}$ for the product of second-order tensors. Denote by Σ the linear space of symmetric second-order tensors, and let $\delta = (\delta_{ij})$ be the identity tensor.

Equation (1.1a) is the *conservation of fluid mass* and (1.1b) is the *conservation of solid momentum*. The rate of change of the momentum has been deleted from the latter since its magnitude is assumed to be substantially smaller than the remaining terms in the equation. In other words, the inertial effects are negligible, and the hyperbolic-parabolic fully-dynamic Biot system reduces to an elliptic-parabolic system. This is the *quasi-static* assumption. The constitutive equations (1.1c) and (1.1d) are *Darcy's law* and a nonlinear form of *Hooke's law*, respectively. By taking these as definitions of $\mathbf{q}$ and σ, we can rewrite (1.1) as the equivalent classical *two field Biot system*

$$\frac{\partial}{\partial t}(c(p) + \alpha \boldsymbol{\nabla} \cdot \mathbf{u}) - \boldsymbol{\nabla} \cdot \kappa(\boldsymbol{\nabla} p - \mathbf{g}) = F\,, \text{ and} \tag{1.2a}$$

$$-\boldsymbol{\nabla} \cdot E(\varepsilon(\mathbf{u})) + \alpha \boldsymbol{\nabla} p = \mathbf{f} \text{ in } \Omega,\ 0 < t \leq T. \tag{1.2b}$$

The boundary $\partial\Omega = \Gamma$ of the deformable porous medium consists of the side $\Gamma_S \equiv \{\mathbf{x} = (x_1, x_2, x_3) : (x_1, x_2) \in \partial G,\ \varphi_0(x_1, x_2) < x_3 < \varphi_1(x_1, x_2)\}$, bottom $\Gamma_0 \equiv \{\mathbf{x} : (x_1, x_2) \in G,\ x_3 = \varphi_0(x_1, x_2)\}$, and top $\Gamma_1 \equiv \{\mathbf{x} : (x_1, x_2) \in G,\ x_3 = \varphi_1(x_1, x_2)\}$. Denote the unit outward normal on $\partial\Omega$ by $\mathbf{n} = (n_1, n_2, n_3)$. The *normal coordinate* of the displacement field on the boundary is $u_n = \mathbf{u} \cdot \mathbf{n}$, and the *tangential component* is $\mathbf{u}_T = \mathbf{u} - u_n \mathbf{n}$. Likewise, $\sigma_n = \sigma(\mathbf{n}) \cdot \mathbf{n}$ and $\sigma(\mathbf{n})_T = \sigma(\mathbf{n}) - \sigma_n \mathbf{n}$ are the *normal coordinate* and *tangential component* of the *normal stress* or *traction* $\sigma(\mathbf{n}) = (\sigma_{ij} n_j)$ on the boundary.

We assume the medium Ω is sealed and fixed on the side. At the bottom, it is in partial contact with the fluid below at pressure $P_0 \in \mathbb{R}$ and the displacement is free. At the top, it is in partial contact with the fluid above at pressure $P_1 \in \mathbb{R}$, and displacement is constrained to lie *below* a rigid and totally permeable constraint located at $(x_1, x_2, \varphi_1(x_1, x_2)) + h(x_1, x_2)\mathbf{n}(x_1, x_2)$, $(x_1, x_2) \in G$. Here $h \geq 0$ is the distance measured along the normal direction from Γ_1 to the constraint. We assume there is no tangential friction on the constraint. Therefore the boundary conditions on the side are

$$q_n = 0, \text{ and } \mathbf{u} = \mathbf{0} \text{ on } \Gamma_S, \tag{1.3a}$$

boundary conditions at the bottom are

$$q_n = \kappa_0(p - P_0), \ \sigma(\mathbf{n})_T = \mathbf{0}, \text{ and} \tag{1.3b}$$

$$\sigma_n - \alpha p + P_0 = 0 \text{ on } \Gamma_0, \tag{1.3c}$$

and at the top, they are

$$q_n = \kappa_1(p - P_1), \ \sigma(\mathbf{n})_T = \mathbf{0}, \text{ and} \tag{1.3d}$$

$$\begin{cases} u_n \le h, \ \sigma_n - \alpha p + P_1 \le 0, \\ (u_n - h)(\sigma_n - \alpha p + P_1) = 0 \text{ on } \Gamma_1, \end{cases} \tag{1.3e}$$

where the function $\kappa_j(\mathbf{x}) \ge 0$ is the *interface permeability* on Γ_j for $j = 0, 1$. These are reduced by the *flux resistance* due to clogging or damage of the boundary pores [28]. (Wherever $\kappa_j(\mathbf{x}) = +\infty$, we could replace the flux balance by $p = P_j$; see [31].) The remaining conditions are the balance of normal stress $\sigma(\mathbf{n})$. The tangential component $\sigma(\mathbf{n})_T$ vanishes on Γ_0 where the medium is in contact with the stationary fluid and on Γ_1 due to the lack of friction with the constraint. Equation (1.3c) is the balance of the normal component of normal stress on Γ_0, and the *unilateral constraints* (1.3e) are the classical *Signorini contact conditions*. That is, where there is contact we have

$$u_n = h \text{ and } \sigma_n \le \alpha p - P_1,$$

and then $\sigma_n - \alpha p + P_1$ is the additional traction imposed on the medium by the rigid constraint. Outside of the contact zone

$$u_n < h \text{ and } \sigma_n = \alpha p - P_1.$$

In either case, the product in (1.3e) is equal to 0. The unilateral boundary conditions on $\mathbf{u}$ in (1.3e) will be characterized by a subdifferential. We shall recall this in detail in Section 2. Figure 1 depicts a cross-section of the cylinder in successive stages when excess pressure from below pushes the medium into the upper constraint. Our objectives are to determine an appropriate variational formulation of the partial differential equations

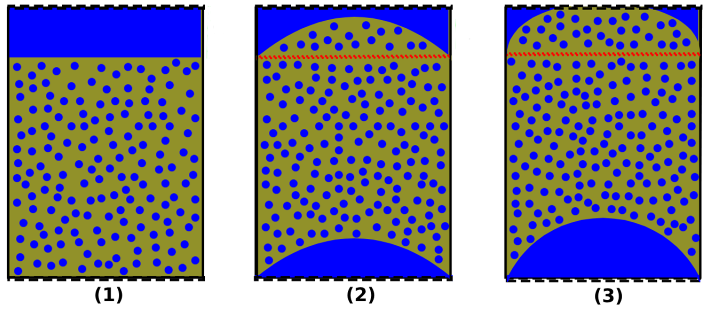

FIGURE 1. In (1), $u_n = 0$ on Γ_1, in (2), $0 < u_n < h$, and in (3), $u_n = h$ on part of Γ_1.

(1.2) and boundary conditions (1.3), and to prove the well-posedness of the corresponding *initial-boundary-value problem* with an initial condition

$$(c(p) + \alpha \boldsymbol{\nabla} \cdot \mathbf{u})\,|_{t=0^+} = \zeta \text{ on } \Omega \tag{1.4}$$

for a given initial fluid content $\zeta(\mathbf{x})$ as indicated, or from a limit of such functions.

The system (1.2) consists of a (possibly degenerate parabolic-elliptic) porous medium equation and a nonlinear elliptic elasticity system coupled by the grad-div dual pair of differential operators. As the coefficients or other functions in these equations are introduced below, we continue to indicate whether they depend variously on $\mathbf{x}$ and t, only $\mathbf{x}$, or neither. The equations hold in function spaces on Ω or its boundary $\partial\Omega$ during the time interval $(0, T)$, so we shall frequently suppress the spatial variables. The function spaces for the domain and its boundary are introduced in Appendix A. In (1.2a), the function $c(\mathbf{x}, p) \geq 0$ denotes variations of the fluid mass concentration that result from the combined *compressibility* of fluid or solid particles. This is permitted to be a non-decreasing function $c(\mathbf{x}, s)$ of pressure $s = p$ at each $\mathbf{x} \in \Omega$. The *dilation* $\boldsymbol{\nabla} \cdot \mathbf{u}$ is a measure of the *porosity* of the structure, *i.e.*, the volume fraction available to the fluid, and $\alpha(\mathbf{x})$ is the Biot-Willis function. In the situation considered here, for a given function p on Ω there is a unique solution $\mathbf{u}$ of (1.2b) with appropriate boundary conditions from (1.3), and with this we define the *Biot function* on Ω by $\mathbf{B}(p) = c(p) + \alpha \boldsymbol{\nabla} \cdot \mathbf{u}$. This combination gives the total variation in local *fluid content* or *storage*, and it plays a central role in the development to follow. When both the fluid and the material of the medium are incompressible, we have the degenerate case $c(p) = 0$, so any change in local fluid content results from distortion of the medium. The function $F(\mathbf{x},t)$ denotes volume-distributed fluid sources, and $\mathbf{f}(\mathbf{x})$ and $\mathbf{g}(\mathbf{x})$ are autonomous volume-distributed forces on solid and fluid within Ω, respectively. The viscosity of the fluid is normalized to unity. The *permeability* for the Darcy flow of fluid in the medium is a bounded function $\kappa(\mathbf{x})$ with a positive lower bound. We do not consider the variation of α or κ with small strains of the medium. The problem is formulated in the context of small strain theory, but the *stress* is permitted to be a strictly monotone nonlinear function $\sigma = E(\mathbf{x},\tau)$ of *linearized strain* $\tau = \varepsilon(\mathbf{u})$ for each $\mathbf{x} \in \Omega$.

In the special case of homogeneous *linear elasticity*, Hooke's law takes the form $\sigma_{ij}(\mathbf{u}) = E_{ijk\ell}\varepsilon_{k\ell}(\mathbf{u})$ corresponding to the linearized strain $\varepsilon_{ij}(\mathbf{u}) = \frac{1}{2}(\partial_i u_j + \partial_j u_i)$ of the solid matrix of the porous medium. The *elasticity tensor* E is symmetric and positive-definite: $E_{ijk\ell} = E_{jik\ell} = E_{k\ell ij}$, $E_{ijk\ell}\tau_{ij}\tau_{k\ell} \geq k_0\tau_{ij}\tau_{ij}$ for $\tau \in \Sigma$, where $k_0 > 0$. If the medium is homogeneous and isotropic, the elasticity tensor has components $E_{ijk\ell} = \lambda\delta_{ij}\delta_{k\ell} + \mu(\delta_{ik}\delta_{j\ell} + \delta_{i\ell}\delta_{jk})$ where the constants $\lambda > 0$ and $\mu > 0$ are the *Lamé coefficients*, the *dilation* and *shear* moduli of elasticity, respectively. Then the stress is given by $E_{ijk\ell}\,\varepsilon_{k\ell}(\mathbf{u}) \equiv \lambda\delta_{ij}\varepsilon_{kk}(\mathbf{u}) + 2\mu\varepsilon_{ij}(\mathbf{u})$, and its divergence is $\boldsymbol{\nabla}\cdot\sigma = (\lambda+\mu)\boldsymbol{\nabla}\boldsymbol{\nabla}\cdot\mathbf{u} + \mu\Delta\mathbf{u}$.

We shall obtain well-posedness results for initial-boundary-value problems for the system (1.2) that include nonlinear partial differential equations as indicated above and rather general boundary conditions (1.3). These can be singular or degenerate coefficients, such as linear interface permeability, and nonlinear monotone relations between stress and displacement on the boundary, in particular, the unilateral Signorini constraint on boundary displacement from contact mechanics. We shall apply standard results on maximal monotone operators and the Cauchy problem for nonlinear evolution equations in Hilbert space. These results are summarized in Appendix B. They contain two notions of solution that are part of the theory of nonlinear evolution equations, a differentiable-in-time *strong solution* and a continuous-in-time *generalized solution*. It will be shown that the initial-boundary-value problem uniquely determines the Biot function of pressure, $\mathbf{B}(p(t))$ for $0 < t \leq T$, and this, in turn, determines the displacement $\mathbf{u}(t)$ for a generalized solution, and it determines both $\mathbf{u}(t)$ and the pressure $p(t)$ for a strong solution. The generalized solution evolves from more general initial data than does the strong solution. Moreover, in the *hyperelastic* case in which the elasticity operator is the derivative of a convex potential function, the system has the parabolic regularizing effect that every generalized solution is a strong solution that satisfies additional $\sqrt{t}$-weighted parabolic estimates.

The variational formulation of the system of partial differential equations (1.2) and boundary conditions(1.3) will be reduced to a single abstract nonlinear evolution equation by means of an intermediate construction, an *implicit evolution equation* for the time derivative of the (nonlinear) Biot function of the pressure and a (linear) elliptic function of the pressure. The strong and generalized solutions of the initial-value problems for the variational formulation and for the implicit evolution equation will be defined to correspond to the respective solutions for the classical abstract evolution equation. Precise assumptions on all the data in the problem will be summarized in Assumptions 2.1 in Section 2.

The results obtained here go beyond those of [31, 49] in various ways.

- We address basic problems of contact mechanics in poro-hyperelastic structures [6, 26, 29, 32, 34, 40, 60]. They extend those of [31] in which the partial differential equations (1.2) are linear, and they show that the major results of [49] do not depend on linearity. In particular, (1.2a) contains the general porous medium equation, and (1.2b) describes monotone nonlinear elasticity.
- A (differentiable) strong solution exists for initial fluid content ζ from the range of the Biot function. Such a function will depend on initial data (p_0, u_0) satisfying (1.4). A generalized solution exists for initial value ζ from a *closure* of the range of the Biot function, a much larger set.
- For a generalized solution, the total fluid content $c(p) + \alpha\boldsymbol{\nabla} \cdot \mathbf{u}$ and displacement $\mathbf{u}$ are uniquely determined by the initial data. Pressure is unique for a strong solution, and it is also unique for a generalized solution if $s \mapsto c(\mathbf{x}, s)$ is strictly monotone.
- In the hyperelastic case for which the elasticity operator is the derivative of a potential energy function, the dynamics is governed by a *gradient flow*. This yields parabolic regularizing effects: a generalized solution is a (differentiable) strong solution, and then pressure p is unique, even with the more general initial data. For the linear case, this was shown in [49] to correspond to an analytic semigroup.

- Even in the non-hyperelastic case, we obtain existence and uniqueness of the generalized solution, but with less regularity. Nonetheless, we show *backward difference approximations* have sufficient spatial regularity of their solutions to characterize the boundary conditions, and their piecewise-linear-in-time interpolants are generalized solutions of the approximating equation which satisfy the initial condition.
- The results for the problem above will be proven first in the abstract form for the implicit evolution equation, which is easily modified to apply to a wide variety of configurations and boundary conditions from contact mechanics for the the Biot system.

It is instructive to compare the results here with those obtained in [37, 49, 62] for the linear case and in [13, 14, 20, 21, 56, 63] for the nonlinear case of dilation-dependent permeability.

Remarks on Literature. The fluid flow through elastic porous media is described by *poroelasticity* [8, 22]. The theory began with the work of K. Terzaghi [58] in soil science and was extensively developed in the engineering community by M. Biot for subsurface acoustic problems and for filtration-consolidation problems. The former problems lead to a system of diffusion and wave equations [10, 11], while the latter consist of the coupled diffusion and elliptic equations given by (1.2) [9]. Due to their many applications in geomechanics and more recently in biomechanics [6, 32, 40], these equations have been confirmed and extensively refined by homogenization and mixture theories [3, 19, 27, 36, 56, 59]. These techniques also characterize the scalings that lead to quasi-static case considered here. Moreover, these and additional applications have driven their use in fundamental fluid-structure problems [4, 5, 57].

Mathematical theory for the fully-dynamic poroelastic acoustic problem was developed in [27, 51] and (in the context of coupled thermoelasticity) in [24, 30, 35], and for the quasi-static consolidation problem in [3] and later in [37, 38, 49, 51, 62]. Additional estimates and alternative formulations have been obtained for the needs of numerical analysis and computation [1, 18, 37, 39, 41, 61]. These works have revealed a tremendous variety of structures for the Biot system. See recent works of the authors and references therein of [15] for a study of the well-posedness of thermo-poro-visco-elastic structures and of [54] for extensions to nonlinear models of Cahn-Hilliard type with two solid phases by exploiting their gradient flow structure. Existence of a solution has been established for nonlinear problems that arise from partially saturated media [16, 52] with a unilateral boundary constraint on pressure and for poro-plastic media [51]. The approach of [51] is limited to constraints on the velocity $\frac{\partial \mathbf{u}}{\partial t}$. See [44] for additional remarks on history and an extensive development of various multi-component multiphase models that satisfy the second law of thermodynamics and a corresponding dissipation inequality.

Nonlinear models from biomechanics use linear dilation for small porosity variations that affect permeability $\kappa(\boldsymbol{\nabla}\cdot\mathbf{u})$ [13, 14, 20, 21, 63]. In these works, it is assumed that the solution satisfies $0 < \kappa_0 \leq \kappa(\boldsymbol{\nabla}\cdot\mathbf{u}) \leq \kappa_1$, so the diffusion is nondegenerate. But see [56], which describes physical laws for which the permeability is not necessarily bounded below, and numerical simulations that show the problem can degenerate in a very challenging manner. Also, some models use κ(fluid-content).

The case of pressure-dependent permeability $\kappa(p)$ arises in geomechanics, but this is not significant until very large values of pressure are reached [43]; we do not include this case. In the context of quasi-static thermoelasticity, a Signorini-type constraint on the displacement with interface conductivity dependent on the distance to the constraint $\kappa_1(h - u_n)$ was obtained in [60]. Also, see the earlier works [2, 45] for a discussion of similar thermal boundary conditions. None of these nonlinear problems are directly amenable to methods of monotonicity, but they are structurally similar and can be handled by additional compactness and fixed-point methods or by methods of pseudo-monotone operators.

Problems with nonlinearity in gradients are frequently monotone [50, 51], and here we permit the stress to be a strictly monotone nonlinear function of the linearized elastic strain in (1.1d). Such a constitutive law is an unacceptable assumption in any nonlinear theory of elasticity since the strain-energy function can not be everywhere convex. However, this assumption has been used for phenomenological models intended for limited extension beyond the linear range of Hooke's law, even to replicate plastic or locking behavior [4, 29, 51]. See [25, 34, 53] for more perspectives on the constitutive relations in elasticity. Also, see [1] for a coupled Biot-Stokes system with a quasi-Newtonian fluid. Here, as in [49], we show the quasi-static hyperelastic Biot system is parabolic, that is, it is a *gradient flow*.

Extensive discussions of the effects of compressibility on uniqueness and its use as a regularizing parameter $c(p) = c_0 p,\ c_0 \to 0^+$, for existence have been given in [13, 14]. Other works have regularized the momentum equation with the addition of viscosity [12, 16] to establish existence. Note that such regularizations suppress the parabolic effects that are present in the purely elastic case considered here. The reference [13] contains a new uniqueness proof for the non-autonomous linearized problem, and this leads to a more efficient iteration step to resolve the nonlinear problem with dilation-dependent permeability. It also develops the application of classical results on implicit degenerate linear evolution equations [46] to initial-boundary-value problems for the linear degenerate *non-autonomous* Biot system.

The Plan. Background material is summarized in Appendix A for Sobolev spaces and boundary trace results and in Appendix B for monotone operators and nonlinear evolution equations in Hilbert space. In Section 2, appropriate operators on Sobolev spaces are introduced to prescribe a *variational formulation* of the boundary-value problem. These operators make precise the sense in which solutions of the partial differential equations (1.2) and boundary conditions (1.3) satisfy the variational form of the problem at each time $t \in (0, T]$.

Section 3 begins with the construction of the pressure-to-fluid-content *Biot function* and the single semilinear *implicit evolution equation* that represents the variational formulation. Then the Biot function of pressure is characterized as the solution of a standard *evolution equation* in Hilbert space. Two notions of solution are known for this abstract evolution equation, and these determine the corresponding notions for the semilinear implicit evolution equation and, thus, for the variational formulation of the initial-boundary-value problem. The *strong solution* is differentiable-in-time, and first estimates characterize a *generalized solution* that is a continuous-in-time limit of strong solutions. This latter notion arises naturally in the theory of evolution equations. The differentiability is lost, but the generalized solution still maintains a meaningful notion of dynamics. In fact, it corresponds to the classical continuous semigroup representation of solutions. See Section III.2 in [7] and Section III.2 in [17]. In the absence of additional estimates available in special cases or regularizations, it remains to show the sense in which even the strong solution is related to the partial differential equations (1.2) and boundary conditions (1.3). This will be done for the generalized solution at the end of Section 3.

2. The Variational Formulation

Now we construct the spaces that will be used to formulate the initial-boundary-value problem. For $j = 0, 1$, let γ_j denote the pointwise a.e. restriction of the trace γ to Γ_j, and assume the interface permeability $\kappa_j \in L^1(\Gamma_j)$ is non-negative. Define the spaces (Appendix A)

$$W \equiv \{s \in H^1(\Omega) : \ \kappa_j^{1/2} \gamma_j s \in L^2(\Gamma_j), \ j = 0, 1\},$$
$$\mathbf{V} \equiv \{\mathbf{v} \in \mathbf{H}^1(\Omega) : \gamma\mathbf{v} = \mathbf{0} \text{ in } \mathbf{H}^{1/2}(\Gamma_S)\}$$

for pressure $p \in W$ and displacement $\mathbf{u} \in \mathbf{V}$. The scalar product on W is characterized in Proposition 2.2. Note that we identify $W \subset L^2(\Omega) \simeq L^2(\Omega)' \subset W'$. The set of *admissible displacements* $\mathbf{K} = \{\mathbf{v} \in \mathbf{V} : \gamma_n \mathbf{v} \leq h \text{ in } H^{1/2}(\Gamma_1)\}$ is closed, convex, and contains $\mathbf{0}$. Symmetric tensors Σ and Σ-valued functions $\sigma_{ij} = \sigma_{ji} \in L^2(\Omega, \Sigma)$ are regarded as columns of vectors or vector-valued functions, respectively, so $\boldsymbol{\nabla} \cdot \sigma = (\partial_i \sigma_{ij})$ and $\sigma_n = \sigma(\mathbf{n}) \cdot \mathbf{n}$. When the columns of σ belong to $\mathbf{H}(\text{div}, \Omega)$, we have $\sigma_n \mathbf{n} \cdot \gamma_1 \in \mathbf{V}'$. Then we can define the linear functional

$$\ell(\mathbf{v}) = -(\sigma_n - \alpha\gamma_1 p + P_1)\mathbf{n} \cdot \gamma_1(\mathbf{v}), \ \mathbf{v} \in \mathbf{V},$$

which will be used to characterize the boundary conditions on $\mathbf{u}$ in (1.3e) as a subdifferential (Definition B.5). The subdifferential of the indicator function is characterized by $\ell \in \partial I_{\mathbf{K}}(\mathbf{u})$ if and only if $\mathbf{u} \in \mathbf{K}$, $\ell(\mathbf{v} - \mathbf{u}) \leq 0$ for $\mathbf{v} \in \mathbf{K}$, and for the particular set $\mathbf{K}$ above this is equivalent to (1.3e), namely,

$$u_n \leq h, \ \ell \geq 0, \ \ell(u_n - h) = 0 \text{ on } \Gamma_1. \tag{2.5}$$

See Section 8.1 in [29], Chapter 2 in [34], or Section II.6 in [48].

Let's develop an appropriate variational formulation of the partial differential equations (1.2) and boundary conditions (1.3) at a time $t \in (0, T)$. If $(p(t), \mathbf{u}(t)) \in W \times \mathbf{V}$ is a solution of (1.2), multiply (1.2a) by $s \in W$ and (1.2b) by $\mathbf{v} - \mathbf{u}(t) \in \mathbf{V}$ with $\mathbf{v} \in \mathbf{K}$, integrate with the Divergence Theorem (A.19) and (1.3a), and subtract $P(v_n - u_n) \equiv \int_{\Gamma_0} P_0(v_n - u_n)\, dS +$

$\int_{\Gamma_1} P_1(v_n - u_n)\, dS$ from both sides of the second equation to obtain

$$\int_\Omega \frac{\partial}{\partial t}\left(c(p) + \alpha\delta : \varepsilon(\mathbf{u})\right) s\, dx + \int_\Omega \kappa(\boldsymbol{\nabla} p - \mathbf{g}) \cdot \boldsymbol{\nabla} s\, dx$$
$$+ q_n(\gamma_0 s + \gamma_1 s) = \int_\Omega F\, s\, dx,\ \ s \in W,$$

$$\int_\Omega \big(E(\varepsilon(\mathbf{u})) : \varepsilon(\mathbf{v} - \mathbf{u}) - \alpha\, p\, \delta : \varepsilon(\mathbf{v} - \mathbf{u})\big)\, dx - (\sigma_n - \alpha\gamma_1 p + P_1)(v_n - u_n)$$
$$= \int_\Omega \mathbf{f} \cdot (\mathbf{v} - \mathbf{u})\, dx - \int_{\Gamma_0} P_0\, (v_n - u_n)\, dS - \int_{\Gamma_1} P_1\, (v_n - u_n)\, dS,\ \ \mathbf{v} \in \mathbf{V},$$
$$\text{and } \ -(\sigma_n - \alpha\gamma_1 p + P_1) \in \partial I_{\mathbf{K}}(\mathbf{u}).$$

Note that if $\frac{\partial}{\partial t}\left(c(p) + \alpha\delta : \varepsilon(\mathbf{u})\right) \in L^2(\Omega)$, then $\mathbf{q} \in \mathbf{H}(\mathrm{div}, \Omega)$, so the divergence theorem (A.19) holds for $q_n \in H^{-1/2}(\Gamma)$ with $q_n|_{\Gamma_S} = 0$. Similar calculations hold for σ_n without additional assumptions. Also, $(v_n - u_n)|_{\Gamma_S} = 0$. The t-dependence has been suppressed as noted above.

Assume the interface permeability satisfies $\kappa_j \in L^1(\Gamma_j)$ for $j = 0, 1$, and that $\mathbf{f} \in \mathbf{L}^2(\Omega)$. We use (1.3b), (3.18c), and (2.5) to rewrite this system in the form

(2.6a) $\ p(t) \in W$:

$$\int_\Omega \frac{\partial}{\partial t}\big(c(p) + \alpha\delta : \varepsilon(\mathbf{u})\big) s\, dx + \int_\Omega \kappa \boldsymbol{\nabla} p \cdot \boldsymbol{\nabla} s\, dx + \int_{\Gamma_0} \kappa_0 p\, s\, dS + \int_{\Gamma_1} \kappa_1 p\, s\, dS$$
$$= \int_\Omega F\, s\, dx + \int_\Omega \kappa \mathbf{g} \cdot \boldsymbol{\nabla} s\, dx + \int_{\Gamma_0} \kappa_0 P_0 s\, dS + \int_{\Gamma_1} \kappa_1 P_1 s\, dS,\ \ s \in W,$$

(2.6b) $\ \mathbf{u}(t) \in \mathbf{K}$:

$$\int_\Omega \left(E(\varepsilon(\mathbf{u})) : \varepsilon(\mathbf{v} - \mathbf{u}) - \alpha\, p\, \delta : \varepsilon(\mathbf{v} - \mathbf{u})\right)\, dx$$
$$\geq \int_\Omega \mathbf{f} \cdot (\mathbf{v} - \mathbf{u})\, dx - \int_{\Gamma_0} P_0\, (v_n - u_n)\, dS - \int_{\Gamma_1} P_1\, (v_n - u_n)\, dS,\ \ \mathbf{v} \in \mathbf{K}.$$

This consists of a nonlinear porous medium equation coupled by first-order spatial derivatives to a variational inequality for a nonlinear elasticity system.

The Operators. Equation (2.6a) suggests we define linear operators $\mathcal{A}_1 : W \to W'$ and $\mathcal{B} : L^2(\Omega) \to \mathbf{H}(\mathrm{div}, \Omega)'$, the nonlinear Nemytskii operator $\mathcal{C} : L^2(\Omega) \to L^2(\Omega)'$, and the linear functional $\mathcal{F}_1(t) \in W'$ by

$$\begin{aligned}
\mathcal{A}_1 p(s) &= \int_\Omega (\kappa(\mathbf{x})\boldsymbol{\nabla} p(\mathbf{x})) \cdot \boldsymbol{\nabla} s(\mathbf{x})\, dx \\
&\qquad + \int_{\Gamma_0} \kappa_0(\mathbf{x}) p(\mathbf{x}) s(\mathbf{x})\, dS + \int_{\Gamma_1} \kappa_1(\mathbf{x}) p(\mathbf{x}) s(\mathbf{x})\, dS \\
\mathcal{B} p(\mathbf{v}) &= \int_\Omega \alpha\, p(\mathbf{x}) \boldsymbol{\nabla} \cdot \mathbf{v}(\mathbf{x})\, dx, \\
\mathcal{C}(p)(s) &= \int_\Omega c(\mathbf{x}, p(\mathbf{x})) s(\mathbf{x})\, dx, \\
\mathcal{F}_1(t)(s) &= \int_\Omega F(\mathbf{x}, t) s(\mathbf{x})\, dx + \int_\Omega \kappa(\mathbf{x}) \mathbf{g}(\mathbf{x}) \cdot \boldsymbol{\nabla} s(\mathbf{x})\, dx \\
&\qquad + \int_{\Gamma_0} \kappa_0(\mathbf{x}) P_0 s(\mathbf{x})\, dS + \int_{\Gamma_1} \kappa_1(\mathbf{x}) P_1 s(\mathbf{x})\, dS.
\end{aligned}$$

Likewise, (2.6b) leads us to define the nonlinear Nemytskii operator $\mathcal{E} : \mathbf{V} \to \mathbf{V}'$, the sum of $\mathcal{E}$ with the constraint, $\mathcal{A}_2 = \mathcal{E} + \partial I_{\mathbf{K}} : \mathbf{V} \to \mathbf{V}'$, and the linear functional $\mathcal{F}_2 \in \mathbf{V}'$ by

$$\begin{aligned}
\mathcal{E}(\mathbf{u})(\mathbf{v}) &= \int_\Omega E(\mathbf{x}, \varepsilon(\mathbf{u}(\mathbf{x}))) : \varepsilon(\mathbf{v}(\mathbf{x}))\, dx, \\
\mathcal{A}_2(\mathbf{u})(\mathbf{v}) &= \mathcal{E}(\mathbf{u})(\mathbf{v}) + \partial I_{\mathbf{K}}(\mathbf{u})(\mathbf{v}) = \mathcal{E}(\mathbf{u})(\mathbf{v}) + \{\ell(\mathbf{v}) :\ \ell \in \partial I_K(\mathbf{u})\}, \\
\mathcal{F}_2(\mathbf{v}) &= \int_\Omega \mathbf{f}(\mathbf{x}) \cdot \mathbf{v}(\mathbf{x})\, dx - \int_{\Gamma_0} P_0\, \mathbf{v}(\mathbf{x}) \cdot \mathbf{n}(\mathbf{x})\, dS - \int_{\Gamma_1} P_1\, \mathbf{v}(\mathbf{x}) \cdot \mathbf{n}(\mathbf{x})\, dS.
\end{aligned}$$

Note that the operator $\mathcal{A}_2$ is multivalued, and its values are characterized in part by (2.5). Also, in the system (2.6) the linear operator $\mathcal{B} : L^2(\Omega) \to \mathbf{H}(\mathrm{div}, \Omega)'$ and its dual $\mathcal{B}' : \mathbf{H}(\mathrm{div}, \Omega) \to L^2(\Omega)'$ occur only in the respective compositions $W \hookrightarrow L^2(\Omega) \xrightarrow{\mathcal{B}} \mathbf{H}(\mathrm{div}, \Omega)' \subset \mathbf{V}'$ and $\mathbf{V} \hookrightarrow \mathbf{H}(\mathrm{div}, \Omega) \xrightarrow{\mathcal{B}'} L^2(\Omega)' \subset W'$ with the indicated inclusion and restriction operators.

We list sufficient conditions on the data for these operators and functionals to be well-defined.

Assumptions 2.1. *Let constants M, $k_0 > 0$ and a function $K(\cdot) \in L^2(\Omega)$ be given. Assume that the data satisfies the following conditions.*

(1) *The region G is bounded and connected with a Lipschitz continuous boundary ∂G, and the functions φ_0, φ_1, and h are Lipschitz continuous with $\varphi_0 + k_0 \le \varphi_1$ on G.*
(2) *The constants $P_0, P_1 \in \mathbb{R}$ and the function $\alpha \in L^\infty(\Omega)$ are given.*
(3) *The source and force functions are integrable:*

$$F \in L^1(0,T; L^2(\Omega)),\ \mathbf{f}, \mathbf{g} \in \mathbf{L}^2(\Omega).$$

(4) *The interface permeability is positive, integrable, and not all zero:*

$$\kappa_j \ge 0 \ \text{in}\ L^1(\Gamma_j),\ j = 0,1,\ \text{and}\ \int_{\Gamma_0} \kappa_0\, dS + \int_{\Gamma_1} \kappa_1\, dS > 0.$$

(5) *The permeability of the medium satisfies $\kappa \in L^\infty(\Omega)$ and $\kappa(\mathbf{x}) \ge k_0$.*
(6) *The Carathéodory function $c : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfies*
  (a) $c(\mathbf{x}, r)$ is measurable in $\mathbf{x}$, continuous in r, and $c(\mathbf{x}, 0) = 0$,
  (b) $(c(\mathbf{x}, r_1) - c(\mathbf{x}, r_2))\,(r_1 - r_2) \ge 0,\ r_1, r_2 \in \mathbb{R}$, and
  (c) $|c(\mathbf{x}, r)| \le M|r| + K(\mathbf{x}),\ r \in \mathbb{R},\ a.e.\ \mathbf{x} \in \Omega$.
(7) *The Carathéodory function $E : \Omega \times \Sigma \to \Sigma$ satisfies*
  (a) $E(\mathbf{x}, \tau)$ is measurable in $\mathbf{x}$, continuous in τ, and $E(\mathbf{x}, 0) = 0$,
  (b) $(E(\mathbf{x}, \tau_1) - E(\mathbf{x}, \tau_2)) : (\tau_1 - \tau_2) > 0,\ \tau_1, \tau_2 \in \Sigma,\ \tau_1 \ne \tau_2$,
  (c) $\|E(\mathbf{x}, \tau)\|_\Sigma \le M\|\tau\|_\Sigma + K(\mathbf{x}),\ \tau \in \Sigma,\ a.e.\ \mathbf{x} \in \Omega$, and
  (d) $E(\mathbf{x}, \tau) : \tau \ge k_0 \|\tau\|_\Sigma^2$.

These assumptions give the essential properties of the operators and functionals.

Proposition 2.2. *From the Assumptions 2.1, we obtain the following.*

(a) *The linear operator $\mathcal{A}_1$ is continuous, linear, symmetric, and W-coercive, so it determines an equivalent scalar product on W for which $\mathcal{A}_1$ is the Riesz isomorphism of W onto W'.*
(b) *The linear functionals satisfy $\mathcal{F}_1(t) \in L^1(0,T; W')$ and $\mathcal{F}_2 \in \mathbf{V}'$.*
(c) *The operators $\mathcal{C}$ and $\mathcal{E}$ are monotone, continuous, and affine bounded. Moreover, $\mathcal{E}$ and $\mathcal{A}_2$ are strictly monotone and $\mathbf{V}$-coercive.*
(d) *The multi-valued sum $\mathcal{A}_2$ is invertible, and the inverse $\mathcal{A}_2^{-1}$ is a bounded monotone* demi-continuous *function. That is, it is continuous from $\mathbf{V}'$ to the space $\mathbf{V}$ with weak convergence.*

Proof. (a) The coercivity of $\mathcal{A}_1$ follows from the Poincaré inequality, Assumption 2.1(4), and Sobolev embedding theorems. (See also Proposition II.5.2 in [48]).

(b) The boundary integrals in $\mathcal{F}_1(t)$ are finite since $\kappa_j^{1/2}$ and $\kappa_j^{1/2}s$ are both in $L^2(\Gamma_j)$. Likewise, the continuity of those in $\mathcal{F}_2$ follows from that of the boundary trace operator on $\mathbf{V}$. The remaining terms are bounded by Assumptions 2.1(3) and (5). Assumption 2.1(3) gives the time dependence.

(c) Both $\mathcal{C}$ and $\mathcal{E}$ are Nemytskii operators (Theorem II.3.2 in [48]) for which the estimates in Assumptions 2.1(6) and (7) apply. Also Korn's inequality holds since $\gamma(\mathbf{v}) = \mathbf{0}$ on Γ_S for $\mathbf{v} \in \mathbf{V}$, and Γ_S has positive measure.

(d) This follows from Corollary II.2.5 in [17], since the sum $\mathcal{A}_2$ is strictly monotone and coercive, hence, $\mathcal{A}_2^{-1}$ is a maximal monotone function with domain $\mathbf{V}'$.

□

Additional regularity of data yields stronger conditions on the functionals and operators.

Corollary 2.3. *If $F \in W^{1,1}(0,T;L^2(\Omega))$, then $\mathcal{F}_1 \in W^{1,1}(0,T;W')$.*

If E is strongly monotone*, i.e.,*

$$(E(\mathbf{x},\tau_1) - E(\mathbf{x},\tau_2)) : (\tau_1 - \tau_2) \geq k_0 \|\tau_1 - \tau_2\|^2,\ \ \tau_1, \tau_2 \in \Sigma,$$

then $\mathcal{E}$ and $\mathcal{A}_2$ are strongly monotone and have Lipschitz continuous inverses.

Corollary 2.4. *Assume in addition to* (7) *that there exists a* potential function *$G : \Omega \times \Sigma \to \mathbb{R}$ such that $G(\mathbf{x},\tau)$ is measurable in $\mathbf{x}$, strictly convex and continuously differentiable in τ, and $E_{ij}(\mathbf{x},\tau) = \partial_{\tau_{ij}} G(\mathbf{x},\tau)$, $1 \leq i,j \leq 3$. Then the nonlinear operator $\mathcal{E}$ is the Fréchet derivative of the convex function $\mathcal{G}(\mathbf{v}) = \int_\Omega G(\mathbf{x}, \varepsilon(\mathbf{v}(\mathbf{x})))\,dx, \mathbf{v} \in \mathbf{V}$.*

(See the chain rule Proposition II.7.8 and Examples II.8 in [48] and Chapter II of [55].) Note that in the linear case, $G(\mathbf{x},\tau)$ is quadratic in τ for $\mathbf{x} \in \Omega$, and the elasticity tensor is given by $E_{ijk\ell}(\mathbf{x}) = \partial_{\tau_{ij}}\partial_{\tau_{k\ell}} G(\mathbf{x},\tau)$.

In summary, the *variational formulation* of the system of partial differential equations (1.2) and boundary conditions (1.3) is the system (2.6). With the indicated Sobolev spaces and operators, this system takes the equivalent mixed form

$$p(t) \in W :\ \frac{d}{dt}(\mathcal{C}(p(t)) + \mathcal{B}'\mathbf{u}(t)) + \mathcal{A}_1 p(t) = \mathcal{F}_1(t) \text{ in } W', \tag{2.7a}$$

$$\mathbf{u}(t) \in \mathbf{V} :\ \mathcal{E}(\mathbf{u}(t)) + \partial I_K(\mathbf{u}(t)) - \mathcal{B}p(t) \ni \mathcal{F}_2 \text{ in } \mathbf{V}',\ 0 < t < T. \tag{2.7b}$$

Definition 2.5. *A* strong solution *of* (2.7) *is a pair of functions* $p : (0,T) \to W,\ \mathbf{u} : (0,T) \to \mathbf{V}$ *for which the sum* $\mathcal{C}(p) + \mathcal{B}'\mathbf{u} \in C([0,T], W')$ *with* $p \in L^1(\delta, T; W)$ *for each* $0 < \delta < T$*, and the equations* (2.7) *are satisfied at a.e.* $t \in [0,T]$.

It is understood that (2.7a) implies $\frac{d}{dt}(\mathcal{C}(p(t)) + \mathcal{B}'\mathbf{u}(t)) \in L^1(\delta, T; W')$ for each $0 < \delta < T$, since $\mathcal{A}_1$ is an isomorphism. This is not enough to obtain $\mathcal{A}_1 p(t) \in L^2(\Omega)$, hence, $\mathbf{q}(t) \in \mathbf{H}(div, \Omega)$, in order to specify the boundary conditions on normal flux q_n. Without further restrictions, even a strong solution is not 'regular enough' to reverse the calculations that led from (1.2) and (1.3) to (2.7). We return to this issue in Proposition 3.9.

Note that for a strong solution, the time-derivative in (2.7a) belongs to W'. This is the criterion that distinguishes the two types of solutions.

Definition 2.6. *A* generalized solution *of* (2.7) *is a pair of functions* $p : (0,T) \to W,\ \mathbf{u} : (0,T) \to \mathbf{V}$ *for which the sum* $\mathcal{C}(p) + \mathcal{B}'\mathbf{u} \in C([0,T], W')$ *and there is a sequence* $\{p_n,\ \mathbf{u}_n\}$ *of strong solutions of* (2.7) *with corresponding right-sides* $\mathcal{F}_n \in L^1(0,T; W')$ *such that we have convergence* $\mathcal{C}(p_n) + \mathcal{B}'\mathbf{u}_n \to \mathcal{C}(p) + \mathcal{B}'\mathbf{u}$ *in* $C([0,T], W')$ *and* $\mathcal{F}_n \to \mathcal{F}_1$ *in* $L^1(0,T; W')$.

3. The Implicit Evolution Equation

Hereafter the Assumptions 2.1 hold. From the consequences of Proposition 2.2 we shall show that the system (2.7) is equivalent to a single semilinear evolution equation of implicit type in Hilbert space for which the appropriate initial-value problem is well-posed. Moreover, in the hyperelastic situation of Corollary 2.4, i.e., when not only $\mathcal{A}_1$ and $\mathcal{C}$ but also $\mathcal{A}_2$ is a *subdifferential operator*, the evolution equation is driven by a subgradient, so (2.7) is a *gradient flow*. It is in this sense that the evolution equation is *parabolic*, and then regularizing effects follow for the solutions.

The Biot Function. We begin by considering the evolution part of the system (2.7), namely, the time-differentiated combination in (2.7a) of $(p, \mathbf{u}) \in L^2(\Omega) \times \mathbf{V}$ and the equation (2.7b) satisfying

$$\mathcal{C}(p) + \mathcal{B}'\mathbf{u} = \mathcal{F} \text{ in } L^2(\Omega)', \tag{3.8a}$$

$$\mathcal{A}_2(\mathbf{u}) - \mathcal{B}p \ni \mathcal{F}_2 \text{ in } \mathbf{V}'. \tag{3.8b}$$

Notice that the nonlinear operators in the system (3.8) are the Nemytskii operator $\mathcal{C}$ and the operator $\mathcal{A}_2$. For each $p \in L^2(\Omega)$ solve (3.8b) for $\mathbf{u}$ to define the value of the nonlinear *Biot function* $\mathbf{B} : L^2(\Omega) \to L^2(\Omega)'$ as

$$\mathbf{B}(p) \equiv \mathcal{C}(p) + \mathcal{B}'\mathbf{u}, \quad \mathbf{u} = \mathcal{A}_2^{-1}(\mathcal{B}p + \mathcal{F}_2). \tag{3.9}$$

Then (3.8) is equivalent to $p \in L^2(\Omega),\ \mathbf{B}(p) = \mathcal{F}$ in $L^2(\Omega)'$. However, as noted above, the function $\mathbf{B}$ occurs in the evolution system (2.7) only as the composition $W \hookrightarrow L^2(\Omega) \xrightarrow{\mathbf{B}} L^2(\Omega)' \subset W'$ with inclusion and restriction. Note that W is dense in $L^2(\Omega)$ with a stronger norm determined by $\mathcal{A}_1$.

The Riesz map $\mathcal{A}_1 : W \to W'$ represents the *rate of dissipation* of the fluid in the medium, and the Biot function $\mathbf{B} : L^2(\Omega) \to L^2(\Omega)'$ represents the *local fluid content* in the medium, i.e., $\mathbf{B}$ is the *pressure-content operator*. We shall develop their combined properties that will be used below.

Lemma 3.1. *The operator* $\mathbf{B}$ *is maximal monotone from* $L^2(\Omega)$ *to* $L^2(\Omega)'$.

Proof. First, we show that $\mathbf{B}$ is *monotone.* Let the pairs $(p_1, \mathbf{u}_1)$ and $(p_2, \mathbf{u}_2)$ in $L^2(\Omega) \times \mathbf{V}$ be corresponding solutions of (3.8). Since $\mathbf{u}_i = \mathcal{A}_2^{-1}(\mathcal{B}p_i + \mathcal{F}_2)$, subtracting respective components of the equivalent systems (3.8) gives

$$(\mathcal{C}(p_1) - \mathcal{C}(p_2))(p_1 - p_2) + \mathcal{B}'(\mathbf{u}_1 - \mathbf{u}_2)(p_1 - p_2) = (\mathbf{B}(p_1) - \mathbf{B}(p_2))(p_1 - p_2),$$
$$(\mathcal{A}_2(\mathbf{u}_1) - \mathcal{A}_2(\mathbf{u}_2))(\mathbf{u}_1 - \mathbf{u}_2) - \mathcal{B}(p_1 - p_2)(\mathbf{u}_1 - \mathbf{u}_2) \ni 0,$$

and by adding these, we get

$$(\mathcal{A}_2(\mathbf{u}_1) - \mathcal{A}_2(\mathbf{u}_2))(\mathbf{u}_1 - \mathbf{u}_2) + (\mathcal{C}(p_1) - \mathcal{C}(p_2))(p_1 - p_2) \ni (\mathbf{B}(p_1) - \mathbf{B}(p_2))(p_1 - p_2). \tag{3.10}$$

That is, $\mathbf{B}(p_j) + \mathcal{F}_2$ is the selection from $\mathcal{A}_2(\mathbf{u}_j),\ j = 1, 2$, for which we have equality in (3.10). Both $\mathcal{A}_2$ and $\mathcal{C}$ are monotone, so the function $\mathbf{B}$ is monotone. Since $\mathcal{A}_2^{-1}$ is demi-continuous and $\mathcal{C}$ is continuous, also $\mathbf{B}$ is demi-continuous, and it follows from Minty's Theorem ([48], p.39) that it is maximal monotone. $\square$

Corollary 3.2. *The Biot function* $\mathbf{B}$ *is maximal monotone from* W *to* W'*: for every* $\beta > 0$*, the Riesz mapping* $\mathcal{A}_1 : W \to W'$ *and the monotone demi-continuous function* $\mathbf{B}$ *satisfy* $Rg(\mathcal{A}_1 + \beta\mathbf{B}) = W'$*. Moreover, if* $\mathbf{B}(p_1) = \mathbf{B}(p_2)$ *for* $p_1, p_2 \in W$*, then* $\mathbf{u}_1 = \mathbf{u}_2$ *and* $\mathcal{C}(p_1) = \mathcal{C}(p_2)$*.*

Proposition 3.3. *In the situation of Corollary 2.4, the operators* $\mathcal{A}_2 : \mathbf{V} \to \mathbf{V}'$ *and* $\mathbf{B} : W \to W'$ *are subdifferentials.*

Proof. Since $\mathcal{A}_2$ is the sum of a continuous Fréchet derivative and a subdifferential, it is a subdifferential. To see that $\mathbf{B}$ is a subdifferential, we use a theorem of Rockafellar [42] that a maximal monotone operator is a subdifferential if and only if it is *cyclic monotone.* (See II 2.5 in [17] or II Thm 2.3 in [7].) That $\mathcal{A}_2$ is cyclic monotone means that for every choice of $v_j^* \in \mathcal{A}_2(v_j),\ j = 1, ...n$ and $v_0 = v_n, v_{n+1}^* = v_1^*$, we have

$$v_1^*(v_1 - v_0) + v_2^*(v_2 - v_1) + ... + v_{n-1}^*(v_{n-1} - v_{n-2}) + v_n^*(v_n - v_{n-1}) \geq 0.$$

This is equivalent to

$$(v_1^* - v_2^*)v_1 + (v_2^* - v_3^*)v_2 + ... + (v_{n-1}^* - v_n^*)v_{n-1} + (v_n^* - v_1^*)v_n \geq 0. \tag{3.11}$$

If we choose each $v_j^* \equiv \mathcal{B}p_j + \mathcal{F}_2$ and $p_{n+1} = p_1$, then $(v_j^* - v_{j+1}^*)v_j = \mathcal{B}(p_j - p_{j+1})v_j = \mathcal{B}'v_j(p_j - p_{j+1}) = \mathcal{B}'\mathcal{A}_2^{-1}(\mathcal{B}p_j + \mathcal{F}_2)(p_j - p_{j+1})$, so (3.11) is

$$\begin{aligned} &\mathcal{B}'\mathcal{A}_2^{-1}(\mathcal{B}p_1 + \mathcal{F}_2)(p_1 - p_2) + \mathcal{B}'\mathcal{A}_2^{-1}(\mathcal{B}p_2 + \mathcal{F}_2)(p_2 - p_3) + \dots \\ &\qquad + \mathcal{B}'\mathcal{A}_2^{-1}(\mathcal{B}p_{n-1} + \mathcal{F}_2)(p_{n-1} - p_n) + \mathcal{B}'\mathcal{A}_2^{-1}(\mathcal{B}p_n + \mathcal{F}_2)(p_n - p_1) \geq 0. \end{aligned}$$

This shows the maximal monotone function $p \mapsto \mathcal{B}'\mathcal{A}_2^{-1}(\mathcal{B}p + \mathcal{F}_2)$ is cyclic monotone. Then its sum with $\mathcal{C}$, namely, $p \mapsto \mathbf{B}p = \mathcal{C}p + \mathcal{B}'\mathcal{A}_2^{-1}(\mathcal{B}p + \mathcal{F}_2) : W \to W'$, is a subdifferential. ☐

The definition of $\mathbf{B}$ in (3.9) implies that the variational formulation (2.7) in $W' \times \mathbf{V}'$ is equivalent to the semilinear evolution equation

$$p(t) \in W : \quad \tfrac{d}{dt}\mathbf{B}(p(t)) + \mathcal{A}_1 p(t) = \mathcal{F}_1(t) \text{ in } W',\ 0 < t < T. \tag{3.12}$$

Definition 3.4. *A strong solution of* (3.12) *with* $\mathcal{F}_1 \in L^1(0, T; W')$ *is defined to be a function* $p : (0, T) \to W$ *for which* $\mathbf{B}(p) \in C([0, T], W')$ *with* $\frac{d}{dt}\mathbf{B}(p) \in L^1(\delta, T; W')$ *for each* $0 < \delta < T$*, and* (3.12) *is satisfied at a.e.* $t \in [0, T]$.

Since $\mathcal{A}_1$ is an isomorphism, a strong solution necessarily satisfies $p \in L^1(\delta, T; W)$ for $0 < \delta < T$ and

$$\mathbf{B}(p(t)) + \int_\delta^t \mathcal{A}_1 p(\tau)\, d\tau = \mathbf{B}(p(\delta)) + \int_\delta^t \mathcal{F}_1(\tau)\, d\tau \text{ in } W',\ 0 < \delta < t \leq T.$$

Taking the limit $\delta \to 0^+$ above shows the resulting improper integral in the second term satisfies

$$\mathbf{B}(p(t)) + \mathcal{A}_1 \int_{0^+}^{t} p(\tau)\, d\tau = \mathbf{B}(p(0^+)) + \int_0^t \mathcal{F}_1(\tau)\, d\tau \text{ in } W',\ 0 < t \leq T, \tag{3.13}$$

where we denote

$$\mathbf{B}(p(0^+)) = \lim_{\delta \to 0^+} \mathbf{B}(p(\delta)).$$

Lemma 3.5. *Any two strong solutions p_1, p_2 of* (3.12) *with corresponding right sides $\mathcal{F}_1^1(t), \mathcal{F}_1^2(t) \in L^1(0,T;W')$ satisfy for each $t \in (0,T]$ the estimate*

$$\|\mathbf{B}(p_1(t)) - \mathbf{B}(p_2(t))\|_{W'} \leq \|\mathbf{B}(p_1(0^+)) - \mathbf{B}(p_2(0^+))\|_{W'} + \int_0^t \|\mathcal{F}_1^1(\tau) - \mathcal{F}_1^2(\tau)\|_{W'}\, d\tau. \tag{3.14}$$

Proof. Set $z(t) \equiv \mathbf{B}(p(t))$ in (3.12) so that $p(t) \in \mathbf{B}^{-1}(z(t))$. Define the composite operator $\mathbf{A} \equiv \mathcal{A}_1\mathbf{B}^{-1} : W' \to W'$. Then we can rewrite (3.12) as

$$z(t) \in W' : \ \frac{dz}{dt} + \mathbf{A}(z) \ni \mathcal{F}_1(t) \text{ in } W',\ 0 < t < T. \tag{3.15}$$

Note that $\mathrm{Dom}(\mathbf{A}) = \mathrm{Rg}(\mathbf{B})$, $\mathbf{A}(z) = \{\mathcal{A}_1 p : p \in W,\ \mathbf{B}(p) = z\}$, and $(I + \mathbf{A})(z) \ni (\mathbf{B} + \mathcal{A}_1)(p)$. Since $\mathcal{A}_1$ is the Riesz map on W and $\mathbf{B}^{-1} : W' \to W$ is monotone, the operator $\mathbf{A}$ is accretive on the Hilbert space W'. Similarly, since $\mathbf{B}$ is maximal monotone, $\mathbf{A}$ is m-accretive. The function z is a solution of (B.20) in the space $H = W'$, so (3.14) is the estimate (B.21) for (3.15). □

The correspondence between solutions of (3.12) and those of (3.15) is at the heart of the construction, and it is the reason for the definition of the operator $\mathbf{A}$. Note that $\mathbf{B}$ is a single-valued function which is not necessarily injective, so $\mathbf{A}$ can be multi-valued. Moreover, since $\mathbf{B}$ may be degenerate, the uniqueness of pressure p is delicate.

Definition 3.6. *A* generalized solution *of* (3.12) *is a function $p : (0,T) \to W$ for which $\mathbf{B}(p) \in C([0,T], W')$ and there is a sequence $\{p_n\}$ of strong solutions of* (3.12) *with corresponding right-sides $\mathcal{F}_n \in L^1(0,T;W')$ such* *that we have convergence* $\mathbf{B}(p_n) \to \mathbf{B}(p)$ *in* $C([0,T],W')$ *and* $\mathcal{F}_n \to \mathcal{F}_1$ *in* $L^1(0,T;W')$.

Thus, any two generalized solutions of (3.12) satisfy the estimate (3.14). A generalized solution of (3.12) is uniquely determined by its initial value

$$\mathbf{B}(p(0^+)) = (\mathcal{C}(p(t)) + \mathcal{B}'\mathbf{u}(t))|_{t=0^+} = \lim_{\delta\to 0^+} (\mathcal{C}(p(\delta)) + \mathcal{B}'\mathbf{u}(\delta)) \in W',$$

so the initial condition (1.4) is always meaningful. However, even if a prescribed initial-value $\zeta \in \mathrm{Rg}(\mathbf{B})$ is given, that is, there is a pair $(p_0, \mathbf{u}_0) \in L^2(\Omega)\times\mathbf{V}$ for which $\zeta = \mathcal{C}(p_0)+\mathcal{B}'\mathbf{u}_0$ with $\mathbf{u}_0 = \mathcal{A}_2^{-1}(\mathcal{B}p_0+\mathcal{F}_2)$, the initial condition (1.4) does not necessarily imply convergence of $p(\delta)$ to p_0 or of $\mathbf{u}(\delta)$ to $\mathbf{u}_0$ without additional assumptions. This is a consequence of the possible degeneracy of the Biot function $\mathbf{B}$.

Corollary 3.2 and Proposition 3.3 show we are in the situation of the following existence-uniqueness theorem for the semilinear evolution equation (3.12), which is equivalent to the variational formulation (2.7). See either of [47] or Ch.IV, Corollary 6.3 of [48], but we shall give an independent proof below. Specifically, the following theorem is a direct consequence of the theory for the nonlinear evolution equation (3.15) with the operator $\mathbf{A}$ defined in the proof of Lemma 3.5.

Theorem 3.7. *Let W be a Hilbert space with the scalar product given by the linear symmetric Riesz map $\mathcal{A}_1 : W \to W'$ and let $\mathcal{F}_1 \in L^1(0,T;W')$.*

- *(a) Assume $\mathbf{B} : W \to W'$ is a monotone function. For any generalized solution p of* (3.12)*, $\mathbf{B}(p(t))$ is uniquely determined by $\mathbf{B}(p(0^+))$ and $\{\mathcal{F}_1(s) :\ s \in (0,t)\}$ for each $t \in (0,T]$. If p_1 and p_2 are generalized solutions of* (3.12) *with $\mathcal{F}_1 = \mathcal{F}_1^1$ and $\mathcal{F}_1 = \mathcal{F}_1^2$, respectively, then they satisfy the estimate* (3.14)*.*
- *(b) Assume also that $Rg(\mathcal{A}_1 + \mathbf{B}) = W'$, i.e., $\mathbf{B}$ is maximal monotone. Then for each $\zeta \in \mathbf{B}(W) \equiv \mathrm{Rg}(\mathbf{B})$ and each absolutely continuous $\mathcal{F}_1 \in W^{1,1}(0,T;W')$, there exists a unique strong solution of the initial-value problem for* (3.12) *with $\mathbf{B}(p(0^+)) = \zeta$, and it satisfies $p \in L^\infty(0,T;W)$. For each ζ in the W'-closure $\overline{\mathbf{B}(W)}$ and $\mathcal{F}_1 \in L^1(0,T;W')$, there is a generalized solution of this initial-value problem, and $\mathbf{B}(p(t))$ is uniquely determined as above for $t \in (0,T]$.*
- *(c) Assume $\mathbf{B}$ is the subdifferential of a proper, convex, and lower-semi-continuous function $\varphi : W \to \mathrm{I\!R}_\infty$ with conjugate $\varphi^* : W' \to \mathrm{I\!R}_\infty$. Then for each ζ in the W'-closure $\overline{\mathbf{B}(W)} = \overline{\mathrm{Dom}(\varphi^*)}$ and each*

$\mathcal{F}_1 \in L^2(0,T;W')$ the unique generalized solution of (3.12) with $\mathbf{B}(p(0^+)) = \zeta$ satisfies

$$\sqrt{t}p \in L^2(0,T;W),\ \varphi^*(\mathbf{B}(p)) \in L^1(0,T),\ \sqrt{t}\tfrac{d}{dt}\mathbf{B}(p) \in L^2(0,T;W'),$$
$$\text{and } p(t) \in \mathrm{Dom}(\mathbf{B}) \text{ for all } 0 < t \le T.$$

If also $\zeta \in \mathrm{Dom}(\varphi^*)$, then the generalized solution satisfies

$$p \in L^2(0,T;W),\ \varphi^*(\mathbf{B}(p)) \in L^\infty(0,T),\ \text{and } \tfrac{d}{dt}\mathbf{B}(p) \in L^2(0,T;W').$$

Proof. Define the operator $\mathbf{A}$ as in the proof of Lemma 3.5. Since $\mathcal{A}_1$ is the Riesz map on W and $\mathbf{B}$ is monotone on W, the operator $\mathbf{A}$ is accretive on W', so (a) follows from Corollary B.12. Moreover, $\mathbf{A}$ is m-accretive in (b) because $\mathbf{B}$ is maximal monotone, so it follows from Kato's Theorem B.10 that the initial-value problem for (3.15) with $z(0) = \zeta$ has a unique strong solution for each $\zeta \in \mathrm{Rg}\,(\mathbf{B}) = \mathrm{Dom}(\mathbf{A})$ and absolutely continuous $\mathcal{F}_1 : [0,T] \to W'$. This Lipschitz continuous solution is differentiable a.e. and $z(t) \in \mathrm{Dom}(\mathbf{A})$ for every $t \ge 0$, so we obtain the corresponding $p(t) \in W$ from (3.12) and the definition of $\mathbf{A}$, since $\mathcal{A}_1$ is invertible. Thus Corollary B.12 verifies (b). For part (c), we note that the inverse $\mathbf{B}^{-1} = \partial\varphi^* : W' \to W'' = W$ is the subdifferential of the corresponding proper, convex, and lower-semi-continuous conjugate function $\varphi^* : W' \to \mathbb{R}_\infty$. Then the composition with the Riesz map of W is its subgradient, $\mathbf{A} = \mathcal{A}_1\mathbf{B}^{-1} = \mathcal{D}\varphi^*$, and Theorem B.15 yields (c). □

The Initial-Boundary-Value Problem. It remains to apply Theorem 3.7 to the situation of Proposition 2.2 with the Biot function $\mathbf{B}$ defined by (3.9) and the operator $\mathbf{A}$ defined in the proof of Lemma 3.5. This yields results for the initial-value problem for the Biot system (2.7) in $W' \times \mathbf{V}'$ with initial-value $\lim_{\delta\to 0^+}\big(\mathcal{C}(p(\delta)) + \mathcal{B}'\mathbf{u}(\delta)\big) = \lim_{\delta\to 0^+}\mathbf{B}(p(\delta))$ given in W'. The first equation (2.7a) corresponds to (3.12) with $\mathbf{u}$ given by (2.7b).

Theorem 3.8. *Assume the conditions of Proposition 2.2, and define the Biot operator $\mathbf{B} : W \to W'$ by (3.9).*

(a) If $p_1, \mathbf{u}_1$ and $p_2, \mathbf{u}_2$ are generalized solutions of (2.7) with $\mathcal{F}_1 = \mathcal{F}_1^1$ and $\mathcal{F}_1 = \mathcal{F}_1^2$, respectively, then they satisfy (3.14). For any generalized solution $p, \mathbf{u}$ of (2.7), $\mathbf{B}(p) = \mathcal{C}(p) + \mathcal{B}'\mathbf{u} \in C([0,T], W')$, and both $\mathcal{C}(p(t))$ and $\mathbf{u}(t)$ are uniquely determined by $\mathbf{B}(p(0^+))$ and $\{\mathcal{F}_1(s) :\ s \in (0,t)\}$ for each $t \in (0,T]$.

(b) *For each* $\zeta \in \mathbf{B}(W) \equiv \mathrm{Rg}(\mathbf{B}) \subset L^2(\Omega)$ *and each absolutely continuous* $\mathcal{F}_1 \in W^{1,1}(0,T;W')$*, there exists a unique strong solution* $p, \mathbf{u}$ *of the initial-value problem for* (2.7) *with* $\mathbf{B}(p(0^+)) = \zeta$*, and we have*

$$p \in L^\infty(0,T;W),\ \tfrac{d}{dt}\mathbf{B}(p) \in L^\infty(0,T;W'),\ \textit{and } \mathbf{u} \in L^\infty(0,T;\mathbf{V}).$$

For each ζ *in the* W'*-closure* $\overline{\mathbf{B}(W)}$ *and* $\mathcal{F}_1 \in L^1(0,T;W')$*, there exists a generalized solution of this initial-value problem for which* $\mathcal{C}(p)$ *and* $\mathbf{u}$ *are uniquely determined.*

Assume also the conditions of Corollary 2.4.

(c) *Then* $\mathbf{B}$ *is the subdifferential of a proper, convex, and lower-semi-continuous function* $\varphi : W \to \mathbb{R}_\infty$*. Let* $\varphi^* : W' \to \mathbb{R}_\infty$ *be the corresponding conjugate function. For each* ζ *in the* W'*-closure* $\overline{\mathbf{B}(W)} = \overline{\mathrm{Dom}(\varphi^*)}$ *and each* $\mathcal{F}_1 \in L^2(0,T;W')$*, the unique generalized solution* $p, \mathbf{u}$ *of* (2.7) *with* $\mathbf{B}(p(0^+)) = \zeta$ *satisfies*

$$\sqrt{t}p \in L^2(0,T;W),\ \varphi^*(\mathbf{B}(p)) \in L^1(0,T),\ \sqrt{t}\tfrac{d}{dt}\mathbf{B}(p) \in L^2(0,T;W'),$$
$$\sqrt{t}\mathbf{u} \in L^2(0,T;\mathbf{V}),\ \textit{and } p(t) \in \mathrm{Dom}(\mathbf{B}) \textit{ for all } 0 < t \le T.$$

If also $\zeta \in \mathrm{Dom}(\varphi^*)$*, then the generalized solution satisfies*

$$p \in L^2(0,T;W),\ \varphi^*(\mathbf{B}(p)) \in L^\infty(0,T),\ \tfrac{d}{dt}\mathbf{B}(p) \in L^2(0,T;W'),$$
$$\mathbf{u} \in L^2(0,T;\mathbf{V}),\ \textit{and } p(t) \in \mathrm{Dom}(\mathbf{B}) \textit{ for all } 0 < t \le T.$$

Proof. Under the Assumptions 2.1, both (a) and (b) are true with $W, \mathcal{A}_1$ and $\mathbf{B}$ given in Proposition 2.2 and equation (3.9). The operator $\mathbf{A}$ is m-accretive in (b) because $\mathbf{B}$ is maximal monotone by Corollary 3.2. Estimates on $\mathbf{u}$ follow from those on p, since $\mathcal{E}$ is coercive and $\mathcal{A}_2^{-1}$ is bounded and demi-continuous. By Proposition 3.3, $\mathbf{B}$ is the subdifferential $\mathbf{B} = \partial\varphi$ of the proper, convex, and lower-semi-continuous function $\varphi : W \to \mathbb{R}_\infty$, so (c) follows. □

Finally, we show that even the generalized solution of (2.7) is the limit of solutions of the partial differential equations (1.2) and boundary conditions (1.3) with the time derivative replaced by a backward-difference. Following the equivalence of the variational formulation (2.7), the implicit evolution equation (3.12), and the standard evolution equation (3.15), we approximate (2.7) by corresponding ε-approximate solutions (Definition B.13).

Proposition 3.9. *Assume the conditions of Proposition 2.2. Let* $p : (0,T) \to W$, $\mathbf{u} : (0,T) \to \mathbf{V}$ *be a generalized solution of the initial-boundary-value problem for the system* (2.7) *with initial condition* (1.4) *and* $\zeta \in \overline{\mathrm{Rg}(\mathbf{B})} \subset W'$, $\mathrm{Rg}\,\mathbf{B} = \{\mathcal{C}(s) + \mathcal{B}'\mathbf{v} : s \in W,\ \mathbf{v} \in \mathbf{V}\}$.

(a) *For each* $\varepsilon > 0$ *there is a discretization of* $\mathcal{F}_1(t)$ *on* $(0,T)$

$$0 = t_0 < t_1 < \dots t_{n-1} < t_n = T,\ \mathcal{F}_1^1, \mathcal{F}_1^2, \dots \mathcal{F}_1^n \in W'$$

for which

$$\beta_k \equiv t_k - t_{k-1} < \varepsilon \text{ for } 1 \le k \le n,\ \text{ and } \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} \|\mathcal{F}_1(t) - \mathcal{F}_1^k\|\, dt < \varepsilon,$$

and a pair of sequences $\{p_k\}, \{\mathbf{u}_k\}, 1 \le k \le n$, *defined recursively from* $\zeta_0 = \mathcal{C}(p_0) + \mathcal{B}'\mathbf{u}_0 \in \mathrm{Rg}(\mathbf{B})$ *with* $\|\zeta_0 - \zeta\|_{W'} < \varepsilon$ *by the backward-difference equations for* $p_k \in W, \mathbf{u}_k \in \mathbf{V}$,

$$(\mathcal{C}(p_k) + \mathcal{B}'\mathbf{u}_k) + \beta_k \mathcal{A}_1 p_k = \mathcal{C}(p_{k-1}) + \mathcal{B}'\mathbf{u}_{k-1} + \beta_k \mathcal{F}_1^k \text{ in } W', \tag{3.16a}$$

$$\mathcal{E}(\mathbf{u}_k) + \partial I_K(\mathbf{u}_k) - \mathcal{B} p_k \ni \mathcal{F}_2 \text{ in } \mathbf{V}',\ 1 \le k \le n, \tag{3.16b}$$

such that the functions $p_\varepsilon : [0,T] \to W$ *and* $\mathbf{u}_\varepsilon : [0,T] \to \mathbf{V}$ *defined by* $p_\varepsilon(0) = p_0$, $\mathbf{u}_\varepsilon(0) = \mathbf{u}_0$, $p_\varepsilon(t) = p_k$ *and* $\mathbf{u}_\varepsilon(t) = \mathbf{u}_k$ *for* $t \in (t_{k-1}, t_k]$, $1 \le k \le n$, *are an* ε-approximate solution *of* (2.7). *That is,* $\|\mathcal{C}p_\varepsilon(t) + \mathcal{B}'\mathbf{u}_\varepsilon(t) - (\mathcal{C}p(t) + \mathcal{B}'\mathbf{u}(t))\|_{W'} < \varepsilon$ *for* $0 \le t \le T$.

(b) *The sequences* $\{p_k\}, \{\mathbf{u}_k\}, 1 \le k \le n$, *satisfy the partial differential equations*

$$\begin{aligned}(c(p_k) + \alpha \boldsymbol{\nabla} \cdot \mathbf{u}_k) - \beta_k \boldsymbol{\nabla} \cdot \kappa(\boldsymbol{\nabla} p_k - \mathbf{g})&\\ = (c(p_{k-1}) + \alpha \boldsymbol{\nabla} \cdot \mathbf{u}_{k-1}) + \beta_k F_k\,,&\ \text{ and}\end{aligned} \tag{3.17a}$$

$$-\boldsymbol{\nabla} \cdot E(\varepsilon(\mathbf{u}_k)) + \alpha \boldsymbol{\nabla} p_k = \mathbf{f}_k \text{ in } \Omega,\ 0 < t \le T. \tag{3.17b}$$

The boundary conditions on the side are

$$q_n^k = 0,\ \text{ and } \mathbf{u}_k = \mathbf{0} \text{ on } \Gamma_S, \tag{3.18a}$$

those at the bottom are

$$\begin{aligned} q_n^k = \kappa_0(p_k - P_0),\ \sigma^k(\mathbf{n})_T = \mathbf{0},\ \text{ and}&\\ \sigma_n^k - \alpha p_k + P_0 = 0 \text{ on } \Gamma_0,&\end{aligned} \tag{3.18b}$$

and at the top, they are

$$q_n^k = \kappa_1(p_k - P_1),\ \sigma^k(\mathbf{n})_T = \mathbf{0},\ \textit{and} \tag{3.18c}$$
$$\begin{cases} \mathbf{u}_k \cdot \mathbf{n} \leq h,\ \sigma_n^k - \alpha p_k + P_1 \leq 0, \\ (\mathbf{u}_k \cdot \mathbf{n} - h)(\sigma_n^k - \alpha p_k + P_1) = 0 \ \textit{on}\ \Gamma_1. \end{cases}$$

Proof. Part (a) is a direct consequence of Theorem B.14, Lemma 3.5, and the definition of $\mathbf{B}$, (3.9). For part (b), we note that $\mathcal{C}(p_k) + \mathcal{B}'\mathbf{u}_k \in L^2(\Omega)$ for each $k = 1, 2, \dots n$, so restriction of (3.16a) to $C_0^\infty(\Omega)$ shows $\mathcal{A}_1 p_k \in L^2(\Omega)$. The restriction of (3.16b) to $\mathbf{C}_0^\infty(\Omega)$ shows $\boldsymbol{\nabla} \cdot \sigma$ belong to $\mathbf{L}^2(\Omega)$. We can take test functions $s \in C_0^\infty(\Omega)$ in (3.16a) and $\mathbf{v} = \mathbf{u}(t) \pm \varphi$ with $\varphi \in \mathbf{C}_0^\infty(\Omega)$ in (3.16b) to get the equations (3.17), and from these we obtain $\mathbf{q}^k \in \mathbf{H}(\mathrm{div}, \Omega)$ and (the columns of) $\sigma \in \mathbf{H}(\mathrm{div}, \Omega)$. With this added regularity and the use of (A.19) on $\mathbf{q}^k$ and on each column of σ, we are able to reverse the calculations, which led to (2.6) and obtain the boundary conditions (3.18). This shows that the variational formulation (3.16) of the ε-approximations has meaningful boundary conditions that satisfy (3.18) as well. $\square$

Concluding Remarks. The initial-boundary-value problem discussed in this paper can be used as a prototype to resolve similar problems that arise in applications. The Signorini unilateral boundary condition is a classical example of a variational inequality used to formulate a challenging contact problem in mechanics. It is a monotone multi-valued stress-strain relation that models the non-penetration of the solid into a prescribed exterior rigid constraint. Related nonlinear boundary conditions include contact with normal elastic compliance with a deformable constraint or truncated stress at a prescribed maximum level due to damage or wear. These can be modeled with a monotone stress-strain function on the boundary and described similarly by Theorem 3.7. It is necessary only to replace the subdifferential $\partial I_{\mathbf{K}}$ with an appropriate monotone Nemytskii operator on the relevant part of the boundary. Alternatively, the indicator function $I_{\mathbf{K}}$ could be replaced with a Lipschitz and convex function. Moreover, the fixed displacement condition in (1.3a) can be replaced with the bilateral contact condition $u_n = 0,\ \sigma(\mathbf{n})_T = 0$ on a part of the side Γ_S where the medium is pressed against the sides but free to slide tangentially. For a well-posed problem, it would be sufficient to have constraints both above and below the medium Ω. Dirichlet conditions on pressure p could replace the flux restrictions on the top or bottom. See [29, 34] for additional examples and discussion.

Appendix A. Sobolev Spaces and Trace

We use spaces of square-summable Lebesgue measurable functions $L^2(\Omega)$; corresponding spaces of vector-valued functions are distinguished by boldface $\mathbf{L}^2(\Omega)$. From these one constructs the usual Sobolev spaces $H^1(\Omega) \equiv \{s \in L^2(\Omega) : \boldsymbol{\nabla} s \in \mathbf{L}^2(\Omega)\}$ and $\mathbf{H}(\mathrm{div}, \Omega) \equiv \{\mathbf{v} \in \mathbf{L}^2(\Omega) : \boldsymbol{\nabla} \cdot \mathbf{v} \in L^2(\Omega)\}$. The derivatives are taken in the sense of distributions on Ω, and we recall that a function f on Ω is identified with the distribution $\tilde{f}$ given by $\tilde{f}(\phi) = \int_\Omega f\phi\, dx$ for $\phi \in C_0^\infty(\Omega)$. (Here dx denotes integration with respect to Lebesgue measure on Ω.) This implies that L^2 spaces are the only Hilbert spaces for which the Riesz map $\mathcal{R}_{L^2}$ is the identity. Specifically, this means we must distinguish subgradients from subdifferentials in most Hilbert spaces, namely, those which are not L^2 over some set.

The continuous linear *boundary trace* operator $\gamma : H^1(\Omega) \to L^2(\Gamma)$ denotes restriction to the boundary, $\gamma(s) = s|_\Gamma$. The kernel of γ is $H_0^1(\Omega)$, the closure of $C_0^\infty(\Omega)$ in $H^1(\Omega)$. The range of γ is the space $H^{1/2}(\Gamma)$ with the norm induced by the quotient map from $H^1(\Omega)/H_0^1(\Omega)$ onto $H^{1/2}(\Gamma)$; the inclusion map $H^{1/2}(\Gamma) \hookrightarrow L^2(\Gamma)$ is compact.

The dual of the range space is denoted by $H^{-1/2}(\Gamma)$. From the gradient, divergence, and boundary trace operators, the continuous linear *normal trace* operator $\gamma_n : \mathbf{H}(\mathrm{div}, \Omega) \to H^{-1/2}(\Gamma)$ is constructed by

$$\gamma_n \mathbf{v}(\gamma s) = \int_\Omega \boldsymbol{\nabla} \cdot \mathbf{v} s\, dx + \int_\Omega \mathbf{v} \cdot \boldsymbol{\nabla} s\, dx, \ \mathbf{v} \in \mathbf{H}(\mathrm{div}, \Omega), \ s \in H^1(\Omega). \tag{A.19}$$

The normal trace maps onto $H^{-1/2}(\Gamma)$, and (A.19) extends the classical *Divergence Theorem* for which $\gamma_n \mathbf{v} = \gamma \mathbf{v} \cdot \mathbf{n} \in H^{1/2}(\Gamma)$ for smoother $\mathbf{v} \in \mathbf{H}^1(\Omega)$. That is, we identify $H^{1/2}(\Gamma) \subset L^2(\Gamma) = L^2(\Gamma)' \subset H^{-1/2}(\Gamma)$, so $\gamma_n \mathbf{v}(\gamma s) = \int_\Gamma v_n \gamma s\, dS$ with dS denoting surface measure on Γ. See [7, 25, 29, 48] for additional results or references.

Appendix B. Initial Value Problem

Here we review preliminary material on nonlinear operators and evolution equations in Hilbert space that will be applied to the initial-boundary-value problem (1.2), (1.3), (1.4). Let's recall from [7, 17, 29, 48] some definitions that we use throughout this paper. Suppose that H is a real Hilbert space and denote by H' its dual space.

Definition B.1. *The* Riesz map *of H is the isomorphism $\mathcal{R}_H : H \to H'$ determined by the scalar product, $\mathcal{R}_H x(y) = (x, y)_H,\ x, y \in H$.*

Monotone and Accretive Operators. We give the definitions of multi-valued operators that are used to describe various boundary-value problems, particularly those containing variational inequalities.

Definition B.2. *An* operator *$A : H \to H'$ is a subset of $H \times H'$. Its domain is* $\mathrm{Dom}(A) = \{x \in H : [x, y] \in A$ *for some* $y \in H'\}$ *and its range is* $\mathrm{Rg}(A) = \{y \in H' : [x, y] \in A$ *for some* $x \in H\}$.

A function is identified with its graph, and operators can be regarded as set-valued functions, *i.e.*, $A(x) = \{y \in H' : [x, y] \in A\}$ with inverse $A^{-1} = \{[y, x] : [x, y] \in A\}$.

Definition B.3. *The operator A is called*

(1) monotone *if $(w_1 - w_2)(u_1 - u_2) \geq 0$ for all $[u_1, w_1],\ [u_2, w_2] \in A$,*
(2) strictly monotone *if $(w_1 - w_2)(u_1 - u_2) > 0$ for all $[u_1, w_1],\ [u_2, w_2] \in A$ with $u_1 \neq u_2$,*
(3) strongly monotone *if for some constant $k_0 > 0$ we have $(w_1 - w_2)(u_1 - u_2) \geq k_0\|u_1 - u_2\|^2$ for $w_1 \in A(u_1),\ w_2 \in A(u_2)$, and*
(4) maximal monotone *if there is no monotone operator that is a proper extension of A.*

The monotone operator A is maximal if and only if $\mathrm{Rg}(\mathcal{R}_H + A) = H'$, and in that case $\mathrm{Rg}(\mathcal{R}_H + \beta A) = H'$ for every $\beta > 0$.

Definition B.4. *The operator A is called* coercive *if*

$$\lim_{\|u\| \to +\infty} \{w(u)/\|u\| : w \in A(u)\} = +\infty.$$

Let $\varphi : H \to \mathbb{R}_\infty \equiv (-\infty, +\infty]$ be a proper lower-semi-continuous convex extended-real-valued function. Its *effective domain* is denoted by $\mathrm{Dom}(\varphi) = \{v \in H : \varphi(v) < +\infty\}$. For example, the *indicator function* of the set $S \subset H$, given by $I_S(u) = 0$ if $u \in S$ and $I_S(u) = +\infty$ if $u \notin S$, is such a function when the set S is nonempty, closed, and convex.

Definition B.5. *The* subdifferential *of φ is the operator $\partial\varphi : H \to H'$ defined by*

$$\partial\varphi(u) = \{w \in H' : w(v - u) \leq \varphi(v) - \varphi(u) \text{ for all } v \in H\}.$$

These are a special class of maximal monotone operators.

Definition B.6. *The* conjugate *of* φ *is the convex function* $\varphi^* : H' \to \mathbb{R}_\infty$ *defined by* $\varphi^*(w) = \sup\{\big(w(v) - \varphi(v)\big) : v \in H\}$.

This function is constructed so that the inverse of $\partial\varphi$ is $\partial\varphi^*$, and the three conditions $w \in \partial\varphi(v)$, $v \in \partial\varphi^*(w)$, and $\varphi(v) + \varphi^*(w) = \ell(v)$ are equivalent. If φ is Fréchet differentiable, then $\partial\varphi$ is the Fréchet derivative. For the particular case of indicator functions, $w \in \partial I_S(u)$ exactly when $u \in S$ and $w(v-u) \le 0$ for all $v \in S$, and this gives a useful characterization of constraints and variational inequalities.

Definition B.7. *The* subgradient *of* φ *is the corresponding operator* $\mathcal{D}\varphi : H \to H$*, which uses the scalar product to characterize its values by*

$$\mathcal{D}\varphi(u) = \{u^* \in H : (u^*, v-u)_H \le \varphi(v) - \varphi(u) \textit{ for all } v \in H\}.$$

These equivalent notions are related by the composition $\partial\varphi = \mathcal{R}_H\mathcal{D}\varphi$ with the Riesz map of H. Note that the terms *subgradient* and *subdifferential* are used inconsistently in the literature. Here we are following the usage in [48] as described above.

Evolution Equations in Hilbert Space. Finally, we summarize fundamental results on the solvability of the initial value problem for an operator A in Hilbert space H. Let $C([0,T],H)$ be the uniformly continuous H-valued functions on the real interval $[0,T]$ with the usual sup norm. Denote by $L^p(0,T;H)$ the space of p^{th}-power Bochner integrable H-valued functions on the real interval $(0,T)$ and by $W^{1,p}(0,T;H)$ such functions which are absolutely continuous with derivative in $L^p(0,T;H)$. In particular, we recall that a function $f : [0,T] \to H$ is absolutely continuous if and only if $f \in W^{1,1}(0,T;H)$.

Definition B.8. *The operator* $A : H \to H$ *is called* accretive *if* $(w_1 - w_2, u_1 - u_2)_{\mathcal{H}} \ge 0$ *for all* $[u_1, w_1],\ [u_2, w_2] \in A$ *and* m-accretive *if additionally* $\mathrm{Rg}(I + A) = H$.

That is, A is accretive if and only if the composition $\mathcal{R}_H A : H \to H'$ is monotone, and A is m-accretive if and only if $\mathcal{R}_H A$ is maximal monotone.

Definition B.9. *A* strong solution *on* $[0,T]$ *of the evolution equation*

$$\frac{dz}{dt} + A(z(t)) \ni f(t) \textit{ in } H,\ 0 < t < T, \tag{B.20}$$

with $f \in L^1(0,T;H)$ *is a continuous function* $z \in C([0,T],H)$ *with* $z \in W^{1,1}(\delta,T;H)$ *for each* $0<\delta<T$ *and which satisfies* $z(t) \in \mathrm{Dom}(A)$ *and* (B.20) *at a.e.* $t \in (0,T)$*.*

The *initial value problem* is to find a solution of (B.20) with a specified initial value $z(0)=z_0 \in H$. If A is accretive, then any two strong solutions $z_1,\ z_2$ of (B.20) with $f=f_1$ and $f=f_2$, respectively, will satisfy the basic estimate

$$\|z_1(t)-z_2(t)\|_H \le \|z_1(0)-z_2(0)\|_H + \int_0^t \|f_1(s)-f_2(s)\|_H \, ds \tag{B.21}$$

for $t \in [0,T]$, and this implies that the initial value problem has at most one strong solution. For the existence of strong solutions, we have the fundamental result of Kato [33]. (See Chapter III.2 of [7] or [17], or Theorem 4.1 of [48].)

Theorem B.10. *If* A *is m-accretive, then the initial value problem of* (B.20) *with* $z(0)=z_0$ *has a unique Lipschitz continuous strong solution* $z \in W^{1,\infty}(0,T;H)$ *for each* z_0 *in* $\mathrm{Dom}(A)$ *and absolutely continuous* $f \in W^{1,1}(0,T;H)$*.*

The estimate (B.21) leads to the notion of a *generalized solution* of the evolution equation (B.20).

Definition B.11. *A* generalized solution *on* $[0,T]$ *of* (B.20) *is a continuous function* $z \in C([0,T],H)$ *for which there is a sequence* $\{f_n\}$ *in* $L^1(0,T;H)$ *with* $f_n \to f$ *in* $L^1(0,T;H)$ *and corresponding solutions* $\{z_n\}$ *on* $[0,T]$ *of* $\frac{dz_n}{dt} + A(z_n(t)) \ni f_n(t)$*, which converge (uniformly)* $z_n \to z$ *in* $C([0,T],H)$ *as* $n \to \infty$*.*

If A is accretive, any two generalized solutions $z_1,\ z_2$ of (B.20) with $f=f_1$ and $f=f_2$, respectively, will satisfy (B.21).

Corollary B.12. *If* A *is accretive, then the initial value problem for* (B.20) *has at most one generalized solution. If* A *is m-accretive, then the initial value problem has a unique generalized solution for each* $z(0)=z_0$ *in the closure* $\overline{\mathrm{Dom}(A)}$ *and each function* $f \in L^1(0,T;H)$*.*

A considerably deeper result is that Corollary B.12 is true in a general Banach space. This remarkable result is obtained by use of the backward-difference approximation of (B.20).

Definition B.13. *Assume that A is m-accretive and $f \in L^1(0,T;H)$. Let $z_0 \in \overline{\mathrm{Dom}(A)}$, $\varepsilon > 0$ and a discretization of f on $(0,T)$*

$$0 = t_0 < t_1 < \dots t_{n-1} < t_n = T,\ f_1, f_2, \dots f_n \in H$$

be given for which

$$t_k - t_{k-1} < \varepsilon \text{ for } 1 \le k \le n, \text{ and } \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} \|f(t) - f_k\|\, dt < \varepsilon.$$

Define the sequence $\{z_k\}$ recursively by the backward-difference equations

$$\frac{z_k - z_{k-1}}{t_k - t_{k-1}} + A(z_k) \ni f_k,\ 1 \le k \le n. \tag{B.22}$$

*The function $z_\varepsilon : [0,T] \to H$ with $z_\varepsilon = z_k$ for $t \in (t_{k-1}, t_k]$, $1 \le k \le n$, is called an ε-*approximate solution *of* (B.20).

At each step of the construction, z_k is obtained from z_{k-1} by solving the *resolvent equation*

$$z_k + \beta_k A(z_k) \ni z_{k-1} + \beta_k f_k,\ 1 \le k \le n, \tag{B.23}$$

with $\beta_k \equiv t_k - t_{k-1} > 0$.

Theorem B.14. *If A is m-accretive, then the initial value problem for* (B.20) *has a unique generalized solution $z \in C([0,T],H)$ for each $z(0) = z_0$ in the closure $\overline{\mathrm{Dom}(A)}$ and each function $f \in L^1(0,T;H)$. For each $\varepsilon > 0$ there is an ε-approximate solution z_ε with $\|z(t) - z_\varepsilon(t)\|_H < \varepsilon$ for all $t \in [0,T]$.*

The generalized solution can be obtained as the uniform limit in $C([0,T],H)$ of piecewise-linear-in-time interpolants of a sequence of ε-approximate solutions z_ε, $\varepsilon > 0$. These necessarily satisfy $z_\varepsilon(t) \in \mathrm{Dom}(A)$ for $0 < t \le T$. (See Section IV.8 in [48] or [23].)

Additional regularity results for solutions of the evolution equation (B.20) were obtained by Brezis when the operator is a subgradient in Hilbert space. In that case, (B.20) is a *gradient flow*.

Theorem B.15. *If $A = \mathcal{D}\varphi$ is the subgradient of a proper, convex, and lower-semi-continuous function $\varphi : H \to \mathrm{I\!R}_\infty$, $f \in L^2(0,T;H)$, and $z_0 \in \overline{\mathrm{Dom}(A)} = \overline{\mathrm{Dom}(\varphi)}$, then the generalized solution of* (B.20) *with $z(0) = z_0$*

is a strong solution and satisfies

$$\sqrt{t}\tfrac{dz}{dt} \in L^2(0,T;H),\ \varphi(z) \in L^1(0,T), \\ \textit{and } z(t) \in \mathrm{Dom}(A) \textit{ for all } t \in (0,T]. \tag{B.24}$$

If also $z_0 \in \mathrm{Dom}(\varphi)$*, then the solution satisfies*

$$\tfrac{dz}{dt} \in L^2(0,T;H) \textit{ and } \varphi(z) \in L^\infty(0,T). \tag{B.25}$$

Theorem B.15 gives a strong solution that is more smooth than in Corollary B.12 or requires data less restricted than Theorem B.10. For the generalized solution of Corollary B.12 with an m-accretive operator, information on differentiability and belonging to $\mathrm{Dom}(A)$ may be lost. For thorough treatments of monotone operators and semigroup theory in Hilbert space, see [7, 17, 48].

References

[1] Ilona Ambartsumyan, Vincent J. Ervin, Truong Nguyen, and Ivan Yotov. A nonlinear Stokes-Biot model for the interaction of a non-Newtonian fluid with poroelastic media. *ESAIM Math. Model. Numer. Anal.*, 53(6):1915–1955, 2019.

[2] Kevin T. Andrews, Peter Shi, Meir Shillor, and Steve Wright. Thermoelastic contact with Barber's heat exchange condition. *Appl. Math. Optim.*, 28(1):11–48, 1993.

[3] J.-L. Auriault and E. Sánchez-Palencia. Étude du comportement macroscopique d'un milieu poreux saturé déformable. *J. Mécanique*, 16(4):575–603, 1977.

[4] L. C. Auton and C. W. MacMinn. From arteries to boreholes: transient response of a poroelastic cylinder to fluid injection. *Proc. A.*, 474(2216):20180284, 21, 2018.

[5] Santiago Badia, Annalisa Quaini, and Alfio Quarteroni. Coupling Biot and Navier-Stokes equations for modelling fluid-poroelastic media interaction. *J. Comput. Phys.*, 228(21):7986–8014, 2009.

[6] L. Banz and F. Bertrand. Contact problems in porous media. *arXiv:2302.02600v1 [math.NA]*, 2023.

[7] Viorel Barbu. *Nonlinear semigroups and differential equations in Banach spaces.* Editura Academiei Republicii Socialiste România, Bucharest; Noordhoff International Publishing, Leiden, 1976. Translated from the Romanian.

[8] Jacob Bear and Alexander H.-D. Cheng. *Modeling Groundwater Flow and Contaminant Transport*, volume 23 of *Theory and Applications of Transport in Porous Media.* Springer, 2010.

[9] M. A. Biot. Theory of elasticity and consolidation for a porous anisotropic solid. *J. Appl. Phys.*, 26:182–185, 1955.

[10] M. A. Biot. Theory of propagation of elastic waves in a fluid-saturated porous solid. I. Low-frequency range. *J. Acoust. Soc. Amer.*, 28:168–178, 1956.

[11] M. A. Biot. Theory of propagation of elastic waves in a fluid-saturated porous solid. II. Higher frequency range. *J. Acoust. Soc. Amer.*, 28:179–191, 1956.

[12] Lorena Bociu, Giovanna Guidoboni, Riccardo Sacco, and Justin T. Webster. Analysis of nonlinear poro-elastic and poro-visco-elastic models. *Arch. Ration. Mech. Anal.*, 222(3):1445–1519, 2016.

[13] Lorena Bociu, Boris Muha, and Justin T. Webster. Weak solutions in nonlinear poroelasticity with incompressible constituents. *Nonlinear Anal. Real World Appl.*, 67:Paper No. 103563, 22, 2022.

[14] Lorena Bociu and Justin T. Webster. Nonlinear quasi-static poroelasticity. *J. Differential Equations*, 296:242–278, 2021.

[15] Jakub Wiktor Both, Kundan Kumar, Jan Martin Nordbotten, and Florin Adrian Radu. The gradient flow structures of thermo-poro-visco-elastic processes in porous media. *arXiv preprint arXiv:1907.03134*, 2019.

[16] Jakub Wiktor Both, Iuliu Sorin Pop, and Ivan Yotov. Global existence of weak solutions to unsaturated poroelasticity. *ESAIM Math. Model. Numer. Anal.*, 55(6):2849–2897, 2021.

[17] H. Brézis. *Opérateurs maximaux monotones et semi-groupes de contractions dans les espaces de Hilbert.* North-Holland Publishing Co., Amsterdam, 1973. North-Holland Mathematics Studies, No. 5. Notas de Matemática (50).

[18] Martina Bukac, William Layton, Marina Moraiti, Hoang Tran, and Catalin Trenchea. Analysis of partitioned methods for the Biot system. *Numer. Methods Partial Differential Equations*, 31(6):1769–1813, 2015.

[19] Robert Burridge and Joseph B. Keller. Poroelasticity equations derived from microstructure. *J. Acoust. Soc. Am.*, 70:1140–1146, 1981.

[20] Yanzhao Cao, Song Chen, and A. J. Meir. Analysis and numerical approximations of equations of nonlinear poroelasticity. *Discrete Contin. Dyn. Syst. Ser. B*, 18(5):1253–1273, 2013.

[21] Yanzhao Cao, Song Chen, and A. J. Meir. Quasilinear poroelasticity: analysis and hybrid finite element approximation. *Numer. Methods Partial Differential Equations*, 31(4):1174–1189, 2015.

[22] Olivier Coussy. *Mechanics of Porous Continua.* John Wiley and Sons, New York, 1995. Translated from the French edition, 1991.

[23] Michael G. Crandall and L. C. Evans. On the relation of the operator $\partial/\partial s + \partial/\partial \tau$ to evolution governed by accretive operators. *Israel J. Math.*, 21(4):261–278, 1975.

[24] Constantine M. Dafermos. On the existence and the asymptotic stability of solutions to the equations of linear thermoelasticity. *Arch. Rational Mech. Anal.*, 29:241–271, 1968.

[25] G. Duvaut and J.-L. Lions. *Inequalities in mechanics and physics.* Springer-Verlag, Berlin, 1976. Translated from the French by C. W. John, Grundlehren der Mathematischen Wissenschaften, 219.

[26] Angiolo Farina and Luigi Preziosi. Deformable porous media and composites manufacturing. In *Heterogeneous media*, Model. Simul. Sci. Eng. Technol., pages 321–410. Birkhäuser Boston, Boston, MA, 2000.

[27] Robert P. Gilbert, Alexander Panchenko, and Ana Vasilic. Acoustic propagation in a random saturated medium: the biphasic case. *Appl. Anal.*, 93(4):676–697, 2014.

[28] Boris Gurevich and Michael Schoenberg. Interface conditions for biot's equations of poroelasticity. *J. Acoustical Society of America*, 105(5):2585–2589, 1999.

[29] Weimin Han and Mircea Sofonea. *Quasistatic contact problems in viscoelasticity and viscoplasticity*, volume 30 of *AMS/IP Studies in Advanced Mathematics.* American Mathematical Society, Providence, RI; International Press, Somerville, MA, 2002.

[30] Daniel B. Henry, Anisio Perissinitto, Jr., and Orlando Lopes. On the essential spectrum of a semigroup of thermoelasticity. *Nonlinear Anal.*, 21(1):65–75, 1993.

[31] Alireza Hosseinkhan and Ralph E. Showalter. Biot-pressure system with unilateral displacement constraints. *J. Math. Anal. Appl.*, 497(1):Paper No. 124882, 18, 2021.

[32] Yuhang Hu, Xuanhe Zhao, Joost J Vlassak, and Zhigang Suo. Using indentation to characterize the poroelasticity of gels. *Applied Physics Letters*, 96(12):121904, 2010.

[33] Tosio Kato. Nonlinear semigroups and evolution equations. *J. Math. Soc. Japan*, 19:508–520, 1967.

[34] N. Kikuchi and J. T. Oden. *Contact problems in elasticity: a study of variational inequalities and finite element methods*, volume 8 of *SIAM Studies in Applied Mathematics.* Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1988.

[35] Irena Lasiecka and Roberto Triggiani. *Control theory for partial differential equations: continuous and approximation theories. I*, volume 74 of *Encyclopedia of Mathematics and its Applications.* Cambridge University Press, Cambridge, 2000. Abstract parabolic systems.

[36] Chiang C. Mei and Bogdan Vernescu. *Homogenization methods for multiscale mechanics.* World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2010.

[37] Márcio A. Murad, Vidar Thomée, and Abimael F. D. Loula. Asymptotic behavior of semidiscrete finite-element approximations of Biot's consolidation problem. *SIAM J. Numer. Anal.*, 33(3):1065–1083, 1996.

[38] Sebastian Owczarek. A Galerkin method for Biot consolidation model. *Math. Mech. Solids*, 15(1):42–56, 2010.

[39] Ricardo Oyarzúa and Ricardo Ruiz-Baier. Locking-free finite element methods for poroelasticity. *SIAM J. Numer. Anal.*, 54(5):2951–2973, 2016.

[40] BNJ Persson. Contact mechanics for poroelastic, fluid-filled media, with application to cartilage. *The Journal of Chemical Physics*, 145(23):234703, 2016.

[41] Phillip Joseph Phillips and Mary F. Wheeler. A coupling of mixed and discontinuous Galerkin finite-element methods for poroelasticity. *Comput. Geosci.*, 12(4):417–435, 2008.

[42] R. T. Rockafellar. Characterization of the subdifferentials of convex functions. *Pacific J. Math.*, 17:497–510, 1966.

[43] B. A. Schrefler. Numerical methods in theories of porous materials. In *Kinetic and continuum theories of granular and porous media*, volume 400 of *CISM Courses and Lectures*, pages 187–244. Springer, Vienna, 1999.

[44] Brian Seguin and Noel J. Walkington. Multi-component multiphase flow through a poroelastic medium. *J. Elasticity*, 135(1-2):485–507, 2019.

[45] Peter Shi and Meir Shillor. Existence of a solution to the N-dimensional problem of thermoelastic contact. *Comm. Partial Differential Equations*, 17(9-10):1597–1618, 1992.

[46] R. E. Showalter. Degenerate evolution equations and applications. *Indiana Univ. Math. J.*, 23:655–677, 1973/74.

[47] R. E. Showalter. Nonlinear degenerate evolution equations and partial differential equationsof mixed type. *SIAM J. Math. Anal.*, 6:25–42, 1975.

[48] R. E. Showalter. *Monotone Operators in Banach Space and Nonlinear Partial Differential Equations*, volume 49 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 1997.

[49] R. E. Showalter. Diffusion in poro-elastic media. *J. Math. Anal. Appl.*, 251(1):310–340, 2000.

[50] R. E. Showalter. Nonlinear degenerate evolution equations in mixed formulation. *SIAM J. Math. Anal.*, 42(5):2114–2131, 2010.

[51] R. E. Showalter and Ulisse Stefanelli. Diffusion in poro-plastic media. *Math. Methods Appl. Sci.*, 27(18):2131–2151, 2004.

[52] R. E. Showalter and Ning Su. Partially saturated flow in a poroelastic medium. *Discrete Contin. Dyn. Syst. Ser. B*, 1(4):403–420, 2001.

[53] J. C. Simo and T. J. R. Hughes. *Computational inelasticity*, volume 7 of *Interdisciplinary Applied Mathematics*. Springer-Verlag, New York, 1998.

[54] Erlend Storvik, Jakub Wiktor Both, Jan Martin Nordbotten, and Florin Adrian Radu. A Cahn-Hilliard-Biot system and its generalized gradient flow structure. *Appl. Math. Lett.*, 126:Paper No. 107799, 8, 2022.

[55] M. M. Vainberg. *Variational method and method of monotone operators in the theory of nonlinear equations*. Halsted Press (A division of John Wiley & Sons), New York-Toronto, Ont.; Israel Program for Scientific Translations, Jerusalem-London, 1973. Translated from the Russian by A. Libin, Translation edited by D. Louvish.

[56] C. J. van Duijn and Andro Mikelić. Mathematical theory of nonlinear single-phase poroelasticity. *J. Nonlinear Sci.*, 33(3):Paper No. 44, 41, 2023.

[57] Maurizio Verri, Giovanna Guidoboni, Lorena Bociu, and Riccardo Sacco. The role of structural viscoelasticity in deformable porous media with incompressible constituents: applications in biomechanics. *Math. Biosci. Eng.*, 15(4):933–959, 2018.

[58] K. von Terzaghi. *Theoretical Soil Mechanics*. Wiley, New York, 1943.

[59] Tessa Weinstein and Lynn S. Bennethum. On the derivation of the transport equation for swelling porous materials with finite deformation. *Internat. J. Engrg. Sci.*, 44(18-19):1408–1422, 2006.

[60] Xiangsheng Xu. The N-dimensional quasistatic problem of thermoelastic contact with Barber's heat exchange conditions. *Adv. Math. Sci. Appl.*, 6(2):559–587, 1996.

[61] Son-Young Yi. A study of two modes of locking in poroelasticity. *SIAM J. Numer. Anal.*, 55(4):1915–1936, 2017.

[62] Alexander Zenísek. The existence and uniqueness theorem in Biot's consolidation theory. *Apl. Mat.*, 29(3):194–211, 1984.

[63] Jingyuan Zhang, Chenguang Zhou, Yanzhao Cao, and A. J. Meir. A locking free numerical approximation for quasilinear poroelasticity problems. *Comput. Math. Appl.*, 80(6):1538–1554, 2020.