BIOT-PRESSURE SYSTEM WITH UNILATERAL DISPLACEMENT CONSTRAINTS

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Abstract. A fully saturated poroelastic medium is confined by the sides of a cylinder, and the regions below and above the medium are filled with fluid at respective constant pressures. The filtration flow of fluid through the poroelastic medium and the small deformations of the medium are described by a quasi-static Biot system of partial differential equations. Our objective here is to establish the well-posedness of an initial-boundary-value problem for this system in which the poroelastic medium is fixed and sealed on the sides, free and in contact with the exterior fluid on the top and bottom, and displacement of the medium is unilaterally constrained on the top by a Signorini-type free boundary condition.

Contents

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1. Introduction

The Biot model of poroelasticity arose from classical consolidation [6, 58] and acoustics problems [18, 24, 30] in soil science and has been developed for increasingly more demanding modeling needs of geophysics. Recently developed sensor technologies continue to deliver large amounts of detailed data which will be useful with appropriate models, but there is a corresponding need to determine coefficients for these models from laboratory experiments, many of which involve contact or indentation problems [5]. Similar problems arise in the manufacture of composite materials by injection molding [46]. These involve injection of a liquid into a porous matrix of reinforcing elements and depend on the rheological properties of the liquid as well as the mechanical properties of the solid matrix; this is constrained by a form to determine its final shape. The case of incompressible components arises in soft hydrated biological tissue modeled as a solid-fluid aggregate [38]. Biomechanical experiments on soft tissue have shown that it is often described well by the Biot system of poroelasticity. The best example is articular cartilage, a particularly simple tissue to model because it is mostly water by weight, and it follows a remarkably linear relationship between stress and strain up to about 20% of strain [37, 40]. In addition to a myriad of forward contact problems of interest for cartilage behavior in joints, obstacle problems for this system arise naturally in indentation experiments to calibrate the model by determining the coefficients in the system [5]. Fractures in deformable porous media and debris-filled fractures in rigid porous media have been studied [39]; their response to increased fluid pressure can be modeled as contact problems. In each of these areas, computational modeling has made substantial strides, but the theory of contact problems for the Biot system is very limited.

The paper [59] proves existence for an analogous problem of quasistatic unilateral frictionless contact of a thermoelastic body with a rigid constraint. The contact is modeled by Signorini's condition, but the thermal coupling depends on the distance between the medium and the rigid foundation. Similar problems were developed earlier in [2, 4, 48]. Existence theory was presented in [56] for a highly nonlinear Biot system modeling diffusion of a slightly compressible fluid through a partially saturated poroelastic medium, and the seepage surface was determined there by a variational inequality on the boundary for a unilateral constraint on the pressure.

We develop here an initial-boundary-value problem for the Biot system with a unilateral constraint on the normal displacement at the boundary, and it includes additional aspects of the applications that are frequently omitted in the theory. The poroelastic medium may have a resistance to normal fluid flow across the boundary [31]; this reflects an additional concentrated drag due to a filter cake or a semi-porous membrane. It is manifested as a jump in fluid pressure or a steep pressure gradient near the interface, and we shall see that it determines the appropriate space for the pressure or flux. In many applications both the fluid and the material of the porous medium may be incompressible, and this degenerate condition is permitted below. It may lead to special behavior of solutions.

The Plan. The notation to be used together with relevant classical material for the development constitute the remainder of this Introduction. The Biot system of partial differential equations and the boundary conditions for our problem are presented in Section 2, and these lead to a weak formulation of the problem. In Section 3 we write the weak problem in an abstract mixed form and construct a nonlinear operator that contains the elasticity problem with a unilateral boundary constraint. This operator gives the local fluid content as a function of pressure. Section 4 begins with the construction of the linear symmetric flux-pressure operator and the corresponding Hilbert space in which the weak problem is then shown to be well-posed. This is done by reducing it to an initial-value problem, first for a semilinear implicit evolution equation and then by using techniques from [54] for an evolution equation with a single m-accretive operator in the Hilbert space. Section 5 contains a summary of the results for the weak problem.

Notation. We assume the mechanical behavior of the porous medium is determined by classical small-strain elasticity. In order to describe this, we denote hereafter by Σ the space of *symmetric second-order tensors*. Boldface letters will be used to indicate vectors in $\mathbb{R}^3$ and Greek letters to indicate second-order tensors in Σ. With $\delta = \{\delta_{ij}\}$ we denote the identity tensor consisting of ones on the diagonal and zeros elsewhere. We adopt the convention that repeated indices are summed. In particular, the scalar product of two vectors is $\mathbf{v}\cdot\mathbf{w} = v_i w_i$, and that of two second-order tensors is $\sigma : \tau = \sigma_{ij}\tau_{ij}$.

For any piecewise smoothly bounded region Ω in $\mathbb{R}^3$, we denote its boundary by $\Gamma \equiv \partial\Omega$. A unit normal vector on a surface is denoted by $\mathbf{n} = \{n_i\}$ and on Γ it is always oriented outward. For a vector $\mathbf{w}$, we denote the *normal coordinate* $w_n = \mathbf{w}\cdot\mathbf{n}$ and the *tangential component* $\mathbf{w}_T = \mathbf{w} - w_n\mathbf{n}$. Likewise for the tensor τ in Σ, we have the value at $\mathbf{w}$, $\tau(\mathbf{w}) = \{\tau_{ij}w_i\} \in \mathbb{R}^3$, its normal coordinate $\tau(\mathbf{w})_n = \tau(\mathbf{w})\cdot\mathbf{n} = \tau_{ij}w_i n_j$, and its tangential component $\tau(\mathbf{w})_T = \tau(\mathbf{w}) - \tau(\mathbf{w})_n\mathbf{n}$. We also set $\tau_n = \tau(\mathbf{n})\cdot\mathbf{n}$.

For any Banach space B we denote its dual space of continuous linear functionals on B by B'. Standard function spaces will be used [1, 20, 57]. Let $H^1(\Omega)$ be the *Sobolev space* consisting of those functions in $L^2(\Omega)$ having each of their partial derivatives also in $L^2(\Omega)$. The *trace* map or restriction to the boundary is the continuous linear map $\gamma : H^1(\Omega) \to L^2(\Gamma)$ defined by $\gamma(w) = w|_\Gamma$; its range is $H^{\frac{1}{2}}(\partial\Omega)$ with the scalar-product inherited from the quotient map, and the dual of the range is $H^{\frac{1}{2}}(\partial\Omega)' \equiv H^{-\frac{1}{2}}(\partial\Omega)$. Corresponding spaces of vector-valued functions will be denoted by boldface symbols. For example, we denote the product space $L^2(\Omega)^3$ by $\mathbf{L}^2(\Omega)$ and the corresponding triple of Sobolev spaces by $\mathbf{H}^1(\Omega) \equiv H^1(\Omega)^3$. We shall also use an intermediate space $\mathbf{L}^2_{\mathrm{div}}(\Omega)$ of vector functions in $\mathbf{L}^2(\Omega)$ for which the divergence belongs to $L^2(\Omega)$. Recall that for the functions $\mathbf{r} \in \mathbf{L}^2_{\mathrm{div}}(\Omega)$ there is a *normal trace* on the interface, and this is denoted by $r_n \in H^{-\frac{1}{2}}(\partial\Omega)$ since it takes the value $\gamma(\mathbf{r})\cdot\mathbf{n}$ on the smooth functions $\mathbf{r}$ in $\mathbf{L}^2_{\mathrm{div}}(\Omega)$. Then we have the *Stokes formula* [20, 57]

$$\int_\Omega (\boldsymbol{\nabla}\cdot\mathbf{r}\, v + \mathbf{r}\cdot\boldsymbol{\nabla} v)\, dx = r_n(\gamma(v)) \text{ for } \mathbf{r} \in \mathbf{L}^2_{div}(\Omega),\ v \in H^1(\Omega).$$

Here $\boldsymbol{\nabla}\cdot$ denotes the *divergence* and $\boldsymbol{\nabla}$ is the *gradient* differential operator. If $\mathbf{r} = \boldsymbol{\nabla} u$, then $\boldsymbol{\nabla}\cdot\mathbf{r} = \Delta u$ is the *Laplacian* of u, and $r_n = \frac{\partial u}{\partial n}$ is the normal derivative on the boundary. We denote by $\mathbb{L}^2(\Omega)$ the indicated space of Σ-valued functions on Ω.

Corresponding spaces of vector-valued functions of time will be used. If H is a Hilbert space and $1 \le p < \infty$, then $L^p(0,T;H)$ is the Banach space of (equivalence classes of) H-valued functions v for which the Bochner integral $\int_0^T \|v(t)\|_H^p\, dt$ is finite, and we define its norm by $\|v\|_{L^p(0,T;H)} =$

$(\int_0^T \|v(t)\|_H^p \, dt)^{1/p}$. For $p = \infty$, $L^\infty(0,T;H)$ denotes the bounded measurable functions with the essential supremum norm. $W^{1,p}(0,T;H)$ is the Banach space of antiderivatives, $u(t) = u(0) + \int_0^t v(s)\, ds$, with $v \in L^p(0,T;H)$. A superscript dot will denote the time derivative $\frac{d}{dt}$, so $u \in W^{1,p}(0,T;H)$ is absolutely continuous and $\dot{u} = v$. If H is a space of real-valued functions on Ω, then for $u \in W^{1,p}(0,T;H)$ superscript dot corresponds to the time derivative $\frac{\partial}{\partial t}$.

We will use some concepts and constructions from monotone operators and convex analysis in Hilbert space [10, 21, 49]. Let V be a Hilbert space and $\mathcal{A} : \mathrm{Dom}(\mathcal{A}) \to V'$ a function with domain $\mathrm{Dom}(\mathcal{A}) \subset V$. Then $\mathcal{A}$ is *monotone* if $(\mathcal{A}(v) - \mathcal{A}(w))(v - w) \geq 0$ for all $v, w \in \mathrm{Dom}(\mathcal{A})$. More generally, let $\mathcal{A} \subset V \times V'$ be a multi-valued *relation* on $V \times V'$ with domain $\mathrm{Dom}(\mathcal{A}) = \{v \in V : (v,f) \in \mathcal{A}\}$ and range $\mathrm{Rg}(\mathcal{A}) = \{f \in V' : (v,f) \in \mathcal{A}\}$. Then the domain $\mathrm{Dom}(\mathcal{A})$ is the set of all $v \in V$ with $\mathcal{A}(v) \equiv \{f \in V' : (v,f) \in \mathcal{A}\} \neq \emptyset$. The relation $\mathcal{A}$ is *monotone* if $(f - g)(v - w) \geq 0$ for all $(v,f),\ (w,g) \in \mathcal{A}$, that is, for all $f \in \mathcal{A}(v),\ g \in \mathcal{A}(w)$. It is *strictly monotone* if strict inequality holds whenever $v \neq w$.

Suppose $j : V \to \mathbb{R} \cup \{+\infty\}$ is an extended real-valued function. It is *proper* if $\mathrm{Dom}(j) \equiv \{x \in V : j(x) < +\infty\}$ is non-empty and *convex* if $j(tx + (1-t)y) \leq tj(x) + (1-t)j(y)$ for $x, y \in V$ and $0 \leq t \leq 1$. Assume it is also lower semi-continuous. The *subdifferential* of j is the relation $\partial j : V \to V'$ defined by $f \in \partial j(u)$ if

$$u \in V,\ f \in V',\ f(v - u) \leq j(v) - j(u) \text{ for all } v \in V.$$

Then ∂j is a monotone operator, possibly multi-valued. If j has a Gateaux differential $j'(u) \in V'$ at $u \in V$, then $\partial j(u) = \{j'(u)\}$. If a set $K \subset V$ is non-empty, convex and closed, its *indicator function* is defined by $I_K(v) = 0$ if $v \in K$ and $I_K(x) = +\infty$ otherwise. It is proper, convex and lower-semicontinuous, and $f \in \partial I_K(u)$ if and only if

$$u \in K,\ f \in V',\ f(v - u) \geq 0 \text{ for all } v \in K.$$

This is an example of a *variational inequality*, and it means that f is minimum at u along every line segment $[u, v]$ in K.

If the dual space V' is replaced by V above and the duality $f(v)$ by the scalar product $(f, v)_V$, then monotone is replaced by *accretive*. In particular, if $\mathcal{R} : V \to V'$ denotes the Riesz isomorphism, then $\mathcal{A} : V \to V'$ is monotone if and only if $A = \mathcal{R}^{-1}\mathcal{A}$ is accretive, that is, $(u - v, x - y)_V \geq 0$

for all $u \in A(x), v \in A(y)$. Moreover, $\mathcal{A}$ is *maximal monotone* if it is monotone and $\mathcal{R} + \mathcal{A}$ is surjective. This is equivalent to $I + A$ being surjective, and we say that A is *m-accretive*.

2. The Biot System

Let $G \times \mathbb{R}$ denote the cylinder in $\mathbb{R}^3$ with connected and simply connected horizontal cross-section $G \subset \mathbb{R}^2$. The poroelastic medium initially occupies the region

$$\Omega \equiv \{\mathbf{x} = (x_1, x_2, x_3) : (x_1, x_2) \in G,\ \varphi_0(x_1, x_2) < x_3 < \varphi_1(x_1, x_2)\} \tag{2.1}$$

in the cylinder between the graphs of two prescribed smooth functions, $\varphi_j : \overline{G} \to \mathbb{R}$ for $j = 0, 1$ with $\varphi_0(x_1, x_2) < \varphi_1(x_1, x_2)$ for $(x_1, x_2) \in \overline{G}$. The boundary of Ω consists of the sides, bottom and top given respectively by

$$\begin{aligned} \Gamma_S &\equiv \{\mathbf{x} : (x_1, x_2) \in \partial G,\ \varphi_0(x_1, x_2) < x_3 < \varphi_1(x_1, x_2)\}, \\ \Gamma_0 &\equiv \{\mathbf{x} : (x_1, x_2) \in G,\ x_3 = \varphi_0(x_1, x_2)\}, \\ \Gamma_1 &\equiv \{\mathbf{x} : (x_1, x_2) \in G,\ x_3 = \varphi_1(x_1, x_2)\}. \end{aligned}$$

The regions in the cylinder below Γ_0 and above Γ_1 are filled with a slightly compressible viscous fluid at pressures P_0 and P_1, respectively, and the medium is fully saturated by that fluid. Another smooth non-negative function $h : \overline{G} \to [0, +\infty)$ determines the location of a *rigid drained constraint* above the medium at

$$\Gamma_h \equiv \{\mathbf{x} : (x_1, x_2) \in G,\ x_3 = \varphi_1(x_1, x_2) + h(x_1, x_2)\}.$$

By *drained* we mean that it constrains the porous solid but does not impede the flow of fluid. The function $h(\cdot)$ is the initial *vertical gap* between the top of the medium and the constraint.

We write the constitutive equations in the poroelastic medium Ω together with the equations for mass and momentum balance as a Biot system of partial differential equations. In particular, $\dot{\mathbf{u}}$ and $\dot{p}$ are the *velocity* corresponding to a *displacement* $\mathbf{u} = \mathbf{u}(\mathbf{x}, t)$ of the *porous structure* Ω and the rate of change of the *pressure* $p = p(\mathbf{x}, t)$ of the fluid in Ω, respectively. We work within the framework of the *infinitesimal theory* for deformations and define the *small strain tensor* or the *linearized strain tensor* $\varepsilon \in \Sigma$ as $\varepsilon(\mathbf{u}) \equiv \frac{1}{2}(\boldsymbol{\nabla}\mathbf{u} + (\boldsymbol{\nabla}\mathbf{u})^T)$. Its components are given by $\varepsilon_{ij}(\mathbf{u}) \equiv \frac{1}{2}(\partial_i u_j + \partial_j u_i)$. Then Hooke's law takes the form $\sigma_{ij}(\mathbf{u}) = E_{ijk\ell}\, \varepsilon_{k\ell}(\mathbf{u})$ in Ω for the *elastic stress* $\sigma(\mathbf{u})$ corresponding to the *strain* $\varepsilon(\mathbf{u})$ in the solid matrix of the porous medium. Here E is the *elasticity tensor*. In the case of a homogeneous and isotropic medium, this tensor is $E_{ijk\ell} = \lambda\delta_{ij}\delta_{k\ell} + \mu(\delta_{ik}\delta_{j\ell} + \delta_{i\ell}\delta_{jk})$, so the stress is given by $E_{ijk\ell}\,\varepsilon_{k\ell}(\mathbf{u}) \equiv \lambda\delta_{ij}\varepsilon_{kk}(\mathbf{u}) + 2\mu\varepsilon_{ij}(\mathbf{u})$, where the constants $\lambda > 0$ and $\mu > 0$ are the *Lamé coefficients*, the *dilation* and *shear* moduli of elasticity, respectively. See [28, 32, 34] for more details on the constitutive relations in elasticity.

The slow flow of fluid at pressure p through a fully-saturated poroelastic medium displaced by $\mathbf{u}$ is described by the *fully-dynamic Biot system*

$$c\dot{p} + \alpha\boldsymbol{\nabla}\cdot\dot{\mathbf{u}} - \boldsymbol{\nabla}\cdot\kappa(\boldsymbol{\nabla}p - \mathbf{g}) = F\,, \tag{2.2a}$$

$$\rho\ddot{\mathbf{u}} - \mu\Delta\mathbf{u} - (\lambda+\mu)\boldsymbol{\nabla}\boldsymbol{\nabla}\cdot\mathbf{u} + \alpha\boldsymbol{\nabla}p = \mathbf{f}\,. \tag{2.2b}$$

This system with $\rho > 0$ was developed by Biot [16, 17, 18, 19] to describe (higher frequency) deformation in porous media. Here $\mathbf{f} = \mathbf{f}(\mathbf{x})$ denotes the volume distributed external forces acting on the structure, and $F = F(\mathbf{x}, t)$ denotes any volume distributed *fluid source density* within that medium. The coefficient $c(\mathbf{x}) \geq 0$ is a measure of the amount of fluid which is forced into a constant volume of the medium at $\mathbf{x}$ by pressure increments. When the fluid and solid are both *incompressible* we have $c(\mathbf{x}) = 0$; even then, the volume of the medium will vary as fluid is gained or lost or as displacements lead to consolidation of the solid. The Darcy (relative) velocity of fluid flow within the porous medium Ω is $\mathbf{q} = -\kappa(\boldsymbol{\nabla}p - \mathbf{g})$, where $\kappa = \kappa_{ij}$ is the permeability tensor divided by fluid viscosity and $\mathbf{g} = \mathbf{g}(\mathbf{x})$ is the gravitational force. The *total flux* of pore fluid is $\alpha\dot{\mathbf{u}} + \mathbf{q}$, where $\alpha \approx 1$ is the Biot-Willis constant; it depends on the mechanical properties of the medium. Note that $\sigma(\mathbf{u}) - \alpha\, p\, \delta$ is the *total stress* due to the combined elastic solid deformation and pore fluid pressure p within the structure. The term $\alpha\boldsymbol{\nabla}p$ arises from the additional stress of the fluid pressure within the structure, and $\alpha\boldsymbol{\nabla}\cdot\mathbf{u}$ accounts for the variation in fluid content due to the local change in pore volume, so $cp + \alpha\boldsymbol{\nabla}\cdot\mathbf{u}$ is the *fluid content* of the medium. For the theory of this system in the formally equivalent context of coupled *thermoelasticity*, see [47], the fundamental work of Dafermos [27], and the exhaustive and complementary accounts of Carleson [25] and Kupradze [36].

For deformations with sufficiently slow variations of the solid velocity the inertia effects of the solid are negligible, i.e., the first term in (2.2b) is negligible, so we formally set $\rho = 0$. In this case the solid matrix displacement responds instantly to variations of the pressure gradient of fluid, and the flow and deformation are described by the *quasi-static Biot system* [6, 7, 8, 13, 14, 15, 41, 42, 50, 51]

$$c\dot{p} + \alpha\boldsymbol{\nabla}\cdot\dot{\mathbf{u}} - \boldsymbol{\nabla}\cdot\kappa(\boldsymbol{\nabla}p - \mathbf{g}) = F\,, \tag{2.3a}$$

$$-\mu\Delta\mathbf{u} - (\lambda+\mu)\boldsymbol{\nabla}\boldsymbol{\nabla}\cdot\mathbf{u} + \alpha\boldsymbol{\nabla}p = \mathbf{f}\,. \tag{2.3b}$$

The incompressible case $c = 0$ is studied in [43]. The system (2.3) results from the elimination of the variables $\mathbf{q}$ and σ from the fundamental 4-field system [11, 52, 53]

$$c\dot{p} + \boldsymbol{\nabla}\cdot\mathbf{q} + \alpha\boldsymbol{\nabla}\cdot\dot{\mathbf{u}} = F\,, \tag{2.4a}$$

$$\mathcal{Q}\mathbf{q} + \boldsymbol{\nabla}p = \mathbf{g}\,, \tag{2.4b}$$

$$-\boldsymbol{\nabla}\cdot\sigma + \alpha\boldsymbol{\nabla}p = \mathbf{f}\,, \tag{2.4c}$$

$$\sigma - E\varepsilon(\mathbf{u}) = 0 \text{ in } \Omega\,. \tag{2.4d}$$

Conservation of fluid mass is (2.4a) and conservation of solid momentum is (2.4c). The equation (2.4b) is Darcy's law in which $\mathcal{Q} = \kappa^{-1}$ is the resistance tensor due to the medium drag and $\mathbf{g}$ is the gravitational force; the equation (2.4d) is Hooke's law. For both analytical and numerical purposes, it is useful to differentiate (2.4d) with respect to t and resolve for the stress variables p, σ and velocity variables $\mathbf{q}$, $\mathbf{v}$ where $\mathbf{v} = \dot{\mathbf{u}}$. This formulation extends easily to much more general cases, including the fully-dynamic system and visco-plastic porous media, and it identifies the physical quantities that need to be specified in boundary or interface conditions [31, 52, 55]. Moreover, the numerical implementation of such mixed models often leads to better accuracy of the variables of primary interest, namely, flux and stress [20]. See [35, 60] for equivalent systems written with total stress to avoid *locking*. Here we shall retain $\mathbf{u}$ as an unknown due to the unilateral boundary constraint on displacement, so we eliminate stress and write (2.4) as the 3-field system [12, 26, 45, 55, 56, 61]

$$c\dot{p} + \boldsymbol{\nabla}\cdot\mathbf{q} + \alpha\boldsymbol{\nabla}\cdot\dot{\mathbf{u}} = F\,, \tag{2.5a}$$

$$\mathcal{Q}\mathbf{q} + \boldsymbol{\nabla}p = \mathbf{g}\,, \tag{2.5b}$$

$$-\boldsymbol{\nabla}\cdot E\varepsilon(\mathbf{u}) + \alpha\boldsymbol{\nabla}p = \mathbf{f}\,. \tag{2.5c}$$

However we shall retain the symbol σ to be defined by (2.4d). See [3, 9, 23, 52, 53] for coupling of the Biot system to a free fluid and [22, 44, 47] for alternative 3-field formulations.

In the following section we introduce operators on appropriate function spaces for which the restrictions to Ω represent the terms in this system. These operators and the underlying spaces depend as well on the boundary conditions of solutions. After describing the boundary conditions for the poroelastic medium in a cylinder, we shall list the assumptions that will be followed thereafter and develop a *weak formulation* of the problem.

2.1. **Boundary Conditions.** The poroelastic medium is fixed and sealed along the sides. That is, we have null displacement and fluid flux determined respectively by

$$\mathbf{u} = \mathbf{0} \text{ and } q_n = 0 \text{ on } \Gamma_S. \tag{2.6a}$$

On the bottom there is a *fluid entry resistance* $\nu_0 \geq 0$ to fluid flux across the interface, and the total stress from the medium is balanced with the known *external fluid pressure* P_0 below. These lead to the conditions

$$p - \nu_0 q_n = P_0, \tag{2.6b}$$

$$\sigma(\mathbf{n})_T = \mathbf{0}, \text{ and} \tag{2.6c}$$

$$\sigma_n - \alpha p = -P_0 \text{ on } \Gamma_0. \tag{2.6d}$$

The absence of any tangential friction implies that $\sigma(\mathbf{n}) = \sigma_n \mathbf{n}$. From (2.6b) and (2.6d) it follows that $\sigma_n = \alpha\, \nu_0 q_n + (1-\alpha)(-P_0)$. We note that $1-\alpha \approx 0$. The permeability of the interface is commonly reduced by a *flux resistance* due to damage or clogging by fracturing fluids or their additives. The resulting pressure discontinuity or steep gradient is modeled by this interface resistance [41, 42].

On the top the conditions are similar, but the stress balance depends also on the rigid constraint. We have there

$$p - \nu_1 q_n = P_1, \tag{2.6e}$$

$$\sigma(\mathbf{n})_T = \mathbf{0}, \text{ and} \tag{2.6f}$$

$$\begin{gathered} u_n \leq h,\ \sigma_n - \alpha p + P_1 \leq 0, \\ (u_n - h)(\sigma_n - \alpha p + P_1) = 0 \text{ on } \Gamma_1, \end{gathered} \tag{2.6g}$$

where $\nu_1 \geq 0$ is the *fluid entry resistance* on this interface, and P_1 is the *external fluid pressure* above the medium. The constraints (2.6g) extend the classical *Signorini contact conditions* which model the contact of an elastic structure with a rigid constraint. Normal displacement u_n is bounded by the constraint, total normal stress $\sigma_n - \alpha p$ is bounded by the fluid pressure at the top, and the medium is either in contact with the constraint or not. If the medium is in contact with the constraint at a point $\mathbf{x} \in \Gamma_1$, then $u_n(\mathbf{x}) = h(\mathbf{x})$ and the constraint provides an additional negative normal stress $\sigma_n - \alpha p + P_1$ to the medium; otherwise, $u_n(\mathbf{x}) < h(\mathbf{x})$ and we will have $\sigma_n = \alpha p - P_1$. See [28, 32, 34] for more details.

2.2. **The Assumptions.** Hereafter we shall assume the following conditions hold.

- The domain G in $\mathbb{R}^2$ is open, bounded, connected and simply connected, and the boundary ∂G is Lipschitz continuous. The top and bottom functions $\varphi_j : \bar{G} \to \mathbb{R}$ are Lipschitz, $j = 0, 1$, $\varphi_0 < \varphi_1$, and the constraint gap $h \in H^{1/2}(\Gamma_1)$ satisfies $h \geq 0$. Define Ω by (2.1).
- The elasticity tensor E is symmetric and positive-definite, that is, $E\sigma : \tau = E\tau : \sigma$ for all $\sigma, \tau \in \Sigma$, and there is an $e_0 > 0$ for which $E\tau : \tau \geq e_0 \|\tau\|^2$ for $\tau \in \Sigma$.
- The permeability tensor κ and its inverse $\mathcal{Q} = \kappa^{-1}$ in Σ are positive definite.
- The compressibility and coupling coefficients, $c \in L^\infty(\Omega)$ and $\alpha \in \mathbb{R}$, satisfy $c(\mathbf{x}) \geq 0$ a.e. and $\alpha > 0$.
- For $j = 0, 1$, we have exterior pressure $P_j \in H^{1/2}(\Gamma_j)$ and interface resistance $\nu_j \in L^2(\Gamma_j)$ satisfying $\nu_j(\mathbf{x}) \geq 0$ a.e. in Γ_j. Let $\Gamma_j^\nu \subset \Gamma_j$ denote the closed and connected *support* of ν_j, and assume $\nu_j^{-1} \in L^1(\Gamma_j^\nu)$.

The Weak Formulation. Define linear spaces that are determined in part by boundary conditions (2.6a), namely, the spaces

$$\mathbf{V} = \{\mathbf{v} \in \mathbf{H}^1(\Omega) : \gamma(\mathbf{v}) = \mathbf{0} \text{ a.e. in } \mathbf{H}^{1/2}(\Gamma_S)\}, \quad H = L^2(\Omega),$$

$$\mathbf{W} = \{\mathbf{r} \in \mathbf{L}^2_{\mathrm{div}}(\Omega) : r_n = 0 \text{ in } H^{-1/2}(\Gamma_S) \text{ and } \int_{\Gamma_k} \nu_k r_n^2 \, dS < +\infty,\ k = 0, 1\},$$

for solid displacement, fluid pressure, and flux, respectively. Since the trace operator is continuous, the set $\mathbf{V}$ is a closed subspace of $\mathbf{H}^1(\Omega)$ with the same norm. The last boundary conditions on $\mathbf{W}$ mean that

$\nu_k^{1/2} r_n \in L^2(\Gamma_k) \subset H^{-1/2}(\Gamma_k)$, $k = 0, 1$. Wherever $\nu_k = 0$, the corresponding condition is not necessary. The space $\mathbf{W}$ equipped with the norm $\|\mathbf{r}\|^2_{\mathbf{W}} = \int_\Omega |\mathbf{r}|^2\,dx + \int_\Omega (\boldsymbol{\nabla}\cdot\mathbf{r})^2\,dx + \sum_{k=0,1} \int_{\Gamma_k} \nu_k r_n^2\,dS$ is complete.

Lemma 2.1. *We have inclusions* $\nu_k^{1/2} H^{1/2}(\Gamma_k) \subset L^2(\Gamma_k)$ *and* $\mathbf{V} \subset \mathbf{W}$.

Proof. Since Γ_k has dimension 2, the Sobolev imbedding theorems show that $H^{1/2}(\Gamma_k) \subset L^4(\Gamma_k)$. Then the result follows from the Hölder inequality. $\square$

We also define the set of *admissible displacements* $\mathbf{K}$ by

$$\mathbf{K} = \{\mathbf{v} \in \mathbf{V} : \gamma(v_n) \le h \text{ a.e. in } H^{1/2}(\Gamma_1)\}.$$

This is a closed convex subset of $\mathbf{V}$ with $\mathbf{0} \in \mathbf{K}$. It determines the first constraint in (2.6g); the remaining conditions are the complimentary constraints.

We seek a solution with $p(t) \in H$, $\mathbf{q}(t) \in \mathbf{W}$, $\mathbf{u}(t) \in \mathbf{K}$ for $t > 0$. Multiplying the fluid conservation equation (2.5a) by $s \in H$ and integrating give

$$\int_\Omega (c\,\dot{p}\,s + \delta : \varepsilon(\alpha\dot{\mathbf{u}} + \mathbf{q})s)\;dx = \int_\Omega F(t)\,s\,dx. \tag{2.7a}$$

Next multiply the Darcy law (2.5b) by $\mathbf{r} \in \mathbf{W}$, integrate the gradient term using Stokes' formula and use (2.6a), (2.6b), and (2.6e) to obtain

$$\int_\Omega (\mathcal{Q}\mathbf{q}\cdot\mathbf{r} - p\,\delta : \varepsilon(\mathbf{r}))\,dx + \int_{\Gamma_0} (\nu_0 \mathbf{n}\cdot\mathbf{q} + P_0)\mathbf{n}\cdot\mathbf{r}\,dS + \int_{\Gamma_1} (\nu_1 \mathbf{n}\cdot\mathbf{q} + P_1)\mathbf{n}\cdot\mathbf{r}\,dS = \int_\Omega \mathbf{g}\cdot\mathbf{r}\,dx. \tag{2.7b}$$

Finally, multiply the momentum equation (2.5c) by $\mathbf{v}-\mathbf{u}$ with $\mathbf{v} \in \mathbf{K}$ and make use of the Stokes formula $\int_\Omega (\boldsymbol{\nabla}\cdot\boldsymbol{\sigma})\cdot(\mathbf{v}-\mathbf{u})\,dx + \int_\Omega \sigma : \varepsilon(\mathbf{v}-\mathbf{u})\,dx = \int_{\partial\Omega} \sigma(\mathbf{n})\cdot(\mathbf{v}-\mathbf{u})\,dS$ and the boundary conditions (2.6c), (2.6d), (2.6f) and (2.6g) to get

$$\int_\Omega (E\varepsilon(\mathbf{u}) : \varepsilon(\mathbf{v}-\mathbf{u}) - \alpha\,p\delta : \varepsilon(\mathbf{v}-\mathbf{u}))\;dx + \int_{\Gamma_0} P_0(\mathbf{v}-\mathbf{u})\cdot\mathbf{n}\,dS + \int_{\Gamma_1} P_1(\mathbf{v}-\mathbf{u})\cdot\mathbf{n}\,dS \ge \int_\Omega \mathbf{f}\cdot(\mathbf{v}-\mathbf{u})\,dx\,. \tag{2.7c}$$

Define the functional $P \in \mathbf{W}'$ by

$$P(\mathbf{r}) \equiv r_n(P_0) + r_n(P_1) = \int_{\Gamma_0} P_0 r_n \, dS + \int_{\Gamma_1} P_1 r_n \, dS, \ \mathbf{r} \in \mathbf{W}, \tag{2.8}$$

where the first equality is the definition and the second equality holds for smoother r_n. In summary, the *weak formulation* of the problem will require $\big(p(t), \mathbf{q}(t), \mathbf{u}(t)\big) \in H \times \mathbf{W} \times \mathbf{K}$ for $t > 0$ such that

$$\int_\Omega (c\,\dot{p} s + \delta : \varepsilon(\alpha \dot{\mathbf{u}} + \mathbf{q}) s) \ dx = \int_\Omega F(t) \, s \, dx, \quad s \in H, \tag{2.9a}$$

$$\int_\Omega (\mathcal{Q}\mathbf{q} \cdot \mathbf{r} - p\delta : \varepsilon(\mathbf{r})) \ dx + \int_{\Gamma_0} \nu_0 q_n r_n \, dS + \int_{\Gamma_1} \nu_1 q_n r_n \, dS \\ = \int_\Omega \mathbf{g} \cdot \mathbf{r} \, dx - P(\mathbf{r}), \quad \mathbf{r} \in \mathbf{W}, \tag{2.9b}$$

$$\int_\Omega (E\varepsilon(\mathbf{u}) : \varepsilon(\mathbf{v} - \mathbf{u}) - \alpha \, p \, \delta : \varepsilon(\mathbf{v} - \mathbf{u})) \ dx \\ \geq \int_\Omega \mathbf{f} \cdot (\mathbf{v} - \mathbf{u}) \, dx - P(\mathbf{v} - \mathbf{u}), \quad \mathbf{v} \in \mathbf{K}, \tag{2.9c}$$

and the *initial condition*

$$\big(c\,p + \boldsymbol{\nabla} \cdot \alpha \mathbf{u}\big)(0) = b_0 \text{ in } \Omega \,. \tag{2.10}$$

We have shown that the equations (2.9a) and (2.5a) are equivalent in the space $L^2(\Omega)$. Also the equation (2.9b) implies (2.5b) in the space $H^{-1}(\Omega)$, so we get $p \in H^1(\Omega)$, $\gamma p \in H^{1/2}(\partial\Omega)$ and boundary conditions. Finally, the equation (2.9c) implies (2.5c) in the space $\mathbf{H}^{-1}(\Omega)$, which gives us $\boldsymbol{\nabla} \cdot \sigma \in \mathbf{L}^2(\Omega)$, hence, $\sigma(\mathbf{n}) \in \mathbf{H}^{-1/2}(\partial\Omega)$ and boundary conditions. In order to arrive at (2.9c), we made use of the boundary conditions (2.6c), (2.6d), (2.6f), (2.6g). The variational inequality in (2.9c) is equivalent to (2.5c) and $\gamma'(\sigma(\mathbf{n}) - \alpha \, p \, \mathbf{n} + P_1 \mathbf{n}) \in -\partial I_K(\mathbf{u})$.

3. Mixed Variational Formulations

The weak formulation (2.9) is in the mixed variational form of a coupled pair of saddle point problems [20, 29]. We shall construct operators in Hilbert spaces to represent this problem as a single semilinear implicit evolution equation for a corresponding pair of monotone operators in Hilbert space. Then it will follow from the theory for the corresponding Cauchy problem that the initial-boundary-value problem for this mixed formulation is well-posed in appropriate spaces.

We make use of the equations (2.9) to define the linear and continuous operators $\mathcal{A}_1 : \mathbf{W} \to \mathbf{W}'$, $\mathcal{B}_1 : \mathbf{W} \to H'$, and functional $\boldsymbol{\varphi} \in \mathbf{W}'$ as

$$\begin{aligned}
\mathcal{A}_1\mathbf{q}(\mathbf{r}) &= \int_\Omega \mathcal{Q}\mathbf{q}\cdot\mathbf{r}\,dx + \int_{\Gamma_0} \nu_0 q_n r_n\,dS + \int_{\Gamma_1} \nu_1 q_n r_n\,dS,\\
\mathcal{B}_1\mathbf{r}(s) &= -\int_\Omega \boldsymbol{\nabla}\cdot\mathbf{r}s\,dx,\\
\boldsymbol{\varphi}(\mathbf{r}) &= \int_\Omega \mathbf{g}\cdot\mathbf{r}\,dx - P(\mathbf{r}),
\end{aligned}$$

the maximal monotone operator $\mathcal{A}_2 : \mathbf{V} \to \mathbf{V}'$, the linear and continuous operators $\mathcal{B}_2 : \mathbf{V} \to H'$, $\mathcal{C} : H \to H'$, and functional $\boldsymbol{\psi} \in \mathbf{V}'$ by

$$\begin{aligned}
\mathcal{A}_2(\mathbf{u})(\mathbf{v}) &= \int_\Omega E\varepsilon(\mathbf{u}) : \varepsilon(\mathbf{v})\,dx + \partial I_K(\mathbf{u})(\mathbf{v}),\\
\mathcal{B}_2\mathbf{v}(s) &= -\int_\Omega \alpha\,\boldsymbol{\nabla}\cdot\mathbf{v}s\,dx, \quad \mathcal{C}p(s) = \int_\Omega cps\,dx,\\
\boldsymbol{\psi}(\mathbf{v}) &= \int_\Omega \mathbf{f}\cdot\mathbf{v}\,dx - P(\mathbf{v}).
\end{aligned}$$

Note that the constraint is part of the nonlinear multi-valued operator $\mathcal{A}_2$. Then the *weak problem* (2.9) can be written in the mixed form

$$\mathbf{q}(t) \in \mathbf{W} :\ \mathcal{A}_1\mathbf{q}(t) + \mathcal{B}_1'p(t) = \boldsymbol{\varphi} \text{ in } \mathbf{W}', \tag{3.11a}$$

$$\mathbf{u}(t) \in \mathbf{V} :\ \mathcal{A}_2(\mathbf{u}(t)) + \mathcal{B}_2'p(t) \ni \boldsymbol{\psi} \text{ in } \mathbf{V}', \tag{3.11b}$$

$$p(t) \in H :\ -\mathcal{B}_1\mathbf{q}(t) - \mathcal{B}_2\dot{\mathbf{u}}(t) + \mathcal{C}\dot{p}(t) = F(t) \text{ in } H'. \tag{3.11c}$$

The system (3.11) consists of two standard mixed problems which are coupled in (3.11c). The flux-pressure pair $\mathcal{A}_1, \mathcal{B}_1$ will be used to construct a linear operator $\mathbf{A}$ and the displacement-pressure triple $\mathcal{A}_2, \mathcal{B}_2, \mathcal{C}$ will be used to construct a second operator $\mathbf{B}$. The pair $\mathbf{A}, \mathbf{B}$ will be used to express the system as a single implicit evolution equation for pressure in the space H'. The second operator contains the variational inequality and consequently is nonlinear. In order to resolve the system, we first develop elementary properties of the operators in the system (3.11). Then we construct from these the corresponding pair of operators for the evolution equation. First we recall some range conditions on the operators $\mathcal{B}_1$ and $\mathcal{B}_2$. In fact we will confirm that they both have full range.

Lemma 3.1. $Rg(\mathcal{B}_1) = L^2(\Omega)$.

Proof. If $F \in L^2(\Omega)$, then there exists a unique $p \in H^1(\Omega)$ such that

$$\int_\Omega \boldsymbol{\nabla} p \cdot \boldsymbol{\nabla} s \, dx + \int_{\Gamma_k} \gamma(p)\gamma(s) \, dS = \int_\Omega F s \, dx, \, \forall s \in H^1(\Omega).$$

Set $\mathbf{q} = \boldsymbol{\nabla} p$. Since $p \in H^1(\Omega)$, we have $\mathbf{q} = \boldsymbol{\nabla} p \in \mathbf{L}^2(\Omega)$, and $-\boldsymbol{\nabla} \cdot \mathbf{q} = -\Delta p = F \in L^2(\Omega)$ which implies $\mathbf{q} \in \mathbf{L}^2_{\text{div}}(\Omega)$. Moreover, by Stokes' formula we have

$$q_n(\gamma(s)) + \int_{\Gamma_0 \cup \Gamma_1} \gamma(p)\gamma(s) \, dS = 0, \, \forall s \in H^1(\Omega).$$

This gives $q_n = 0$ on Γ_S and $q_n = -\gamma(p)$ on $\Gamma_0 \cup \Gamma_1$. So from Lemma 2.1 we obtain $\nu_k^{\frac{1}{2}} q_n = -\nu_k^{\frac{1}{2}} p \in \nu_k^{\frac{1}{2}} H^{\frac{1}{2}}(\Gamma_k) \subset L^2(\Gamma_k)$. Hence $\mathbf{q} \in \mathbf{W}$ and $\mathcal{B}_1 \mathbf{q} = F$. □

Corollary 3.2. $\mathcal{B}_1' : H \to \mathbf{W}'$ *is bounding.*

Lemma 3.3. $Rg(\mathcal{B}_2) = L^2(\Omega)$.

Proof. First we show that $\mathcal{B}_2'$ is injective: if $\mathcal{B}_2' s = 0$, then $\boldsymbol{\nabla} s = 0$ which gives $s = c$ for a constant c. Now notice that for all $\mathbf{v} \in \mathbf{V}$ we have $\mathcal{B}_2' c(v) = \mathcal{B}_2 v(c) = -c \int_\Omega \alpha \boldsymbol{\nabla} \cdot \mathbf{v} \, dx = -c \int_{\Gamma_1 \cup \Gamma_2} \alpha v_n \, dS = 0$, which in turn implies that $c = 0$. Hence $\text{Ker}(\mathcal{B}_2') = \{\mathbf{0}\}$.

Next we show that $\mathcal{B}_2$ is surjective. Since Ω is bounded, we have the gradient estimate for $s \in L^2(\Omega)$,

$$\|s\|_{L^2} \leq c(\Omega) \left(|\int_\Omega s \, dx| + \|\boldsymbol{\nabla} s\|_{H^{-1}} \right). \tag{3.12}$$

For a proof see [57]. We can regard $\boldsymbol{\nabla} s$ as the restriction of $\mathcal{B}_2' s$ to $\mathbf{H}^1_0(\Omega)$, i.e. $\alpha \boldsymbol{\nabla} s = \mathcal{B}_2' s|_{\mathbf{H}^1_0(\Omega)}$. Note that $\mathbf{H}^1_0(\Omega) \subset \mathbf{V}$, and for $\mathbf{v} \in \mathbf{H}^1_0(\Omega)$ we have $\|\mathbf{v}\|_{\mathbf{V}} \leq c_1 \|\mathbf{v}\|_{\mathbf{H}^1_0(\Omega)}$. Then we get

$$\alpha \|\boldsymbol{\nabla} s\|_{H^{-1}} = \sup_{\mathbf{v} \in \mathbf{H}^1_0} \frac{|\mathcal{B}_2' s(\mathbf{v})|}{\|\mathbf{v}\|_{\mathbf{H}^1_0}} \leq c_1 \sup_{\mathbf{v} \in \mathbf{V}} \frac{|\mathcal{B}_2' s(\mathbf{v})|}{\|\mathbf{v}\|_{\mathbf{V}}} = c_1 \|\mathcal{B}_2' s\|_{\mathbf{V}'} . \tag{3.13}$$

We claim that $\mathcal{B}_2'$ is bounding, i.e. there is a constant $\tilde{c} > 0$ such that

$$\|\mathcal{B}_2' s\|_{\mathbf{V}'} \geq \tilde{c}\, \|s\|_{L^2}\,,\ \forall s \in L^2(\Omega).$$

If it is not the case that $\mathcal{B}_2'$ is bounding, then there exists a sequence $s_k \in L^2(\Omega)$ with $\|s_k\|_{L^2} = 1$ and $s \in L^2(\Omega)$ such that $s_k \rightharpoonup s$, i.e. s_k converges to s weakly in L^2, and $\|\mathcal{B}_2' s_k\|_{\mathbf{V}'} \to 0$. The weak convergence implies that $\int_\Omega s_k\, dx \to \int_\Omega s\, dx$. Then using (3.12) and (3.13) we see that s_k is a Cauchy sequence, hence the convergence is strong, $s_k \to s$ in $L^2(\Omega) = H$ and $\|s\|_{L^2} = 1$. Continuity of $\mathcal{B}_2'$ leads to $\mathcal{B}_2' s_k \to \mathcal{B}_2' s = 0$, but by injectivity of $\mathbf{B}_2'$ it follows that $s = 0$, a contradiction. $\square$

A Reduced Weak Problem. By means of a translation we can assume without loss of generality that $\boldsymbol{\varphi} \equiv 0$ in (3.11a). To see this, we use the linear first standard mixed problem for flux and pressure in the system (3.11).

Lemma 3.4. *For $\boldsymbol{\varphi} \in \mathbf{W}'$, there is a unique pair $(\mathbf{q}_{\boldsymbol{\varphi}}, p_{\boldsymbol{\varphi}}) \in \mathbf{W} \times H$ such that*

$$\begin{aligned} \mathcal{A}_1 \mathbf{q}_{\boldsymbol{\varphi}} + \mathcal{B}_1' p_{\boldsymbol{\varphi}} &= \boldsymbol{\varphi} \ \text{ in } \mathbf{W}', \\ -\mathcal{B}_1 \mathbf{q}_{\boldsymbol{\varphi}} &= 0 \ \text{ in } H', \end{aligned}$$

and the map $\boldsymbol{\varphi} \mapsto (\mathbf{q}_{\boldsymbol{\varphi}}, p_{\boldsymbol{\varphi}})$ from $\mathbf{W}'$ to $\mathbf{W} \times H$ is linear and continuous.

Proof. The operator $\mathcal{A}_1$ is symmetric, non-negative, and $\mathbf{W}$-coercive on $\operatorname{Ker} \mathcal{B}_1$. Since by Corollary 3.2 the operator $\mathcal{B}_1'$ is injective and bounding, there exists a unique solution $(\mathbf{q}_{\boldsymbol{\varphi}}, p_{\boldsymbol{\varphi}})$ of the above mixed problem [20]. $\square$

Let $\mathbf{u}$, $\mathbf{q}$, p be a solution to (3.11). Then $\mathbf{u}$, $\mathbf{q} - \mathbf{q}_{\boldsymbol{\varphi}}$, $p - p_{\boldsymbol{\varphi}}$ is a solution of (3.11) with $\boldsymbol{\varphi} = 0$ and $\boldsymbol{\psi}$ replaced by $\boldsymbol{\psi} - \mathcal{B}_2' p_{\boldsymbol{\varphi}}$. The converse follows similarly, so solvability of the *reduced weak problem*

$$\mathbf{q}(t) \in \mathbf{W} :\ \mathcal{A}_1 \mathbf{q}(t) + \mathcal{B}_1' p(t) = 0 \ \text{ in } \mathbf{W}', \tag{3.14a}$$

$$\mathbf{u}(t) \in \mathbf{V} :\ \mathcal{A}_2(\mathbf{u}(t)) + \mathcal{B}_2' p(t) \ni \boldsymbol{\psi} \ \text{ in } \mathbf{V}', \tag{3.14b}$$

$$p(t) \in H :\ -\mathcal{B}_1 \mathbf{q}(t) - \mathcal{B}_2 \dot{\mathbf{u}}(t) + \mathcal{C} \dot{p}(t) = F(t) \ \text{ in } H'. \tag{3.14c}$$

is equivalent to that of (3.11).

The Fluid-Content Operator. Next we consider the complementary nonlinear mixed problem for displacement and pressure,

$$\mathcal{A}_2(\mathbf{u}) + \mathcal{B}_2' p \ni \boldsymbol{\psi} \ \text{ in } \mathbf{V}', \tag{3.15a}$$

$$-\mathcal{B}_2 \mathbf{u} + \mathcal{C} p = F \ \text{ in } \ H'. \tag{3.15b}$$

Since $\mathcal{A}_2$ is strongly monotone and $\mathbf{V}$-coercive, it is invertible and the inverse $\mathcal{A}_2^{-1} : \mathbf{V}' \to \mathbf{V}$ is monotone and Lipschitz. Define the nonlinear operator $\mathbf{B} : H \to H'$ for *fluid content* by

$$\mathbf{B}(p) \equiv \mathcal{C}p - \mathcal{B}_2 {\mathcal{A}_2}^{-1}(\boldsymbol{\psi} - \mathcal{B}_2' p).$$

The system (3.15) is equivalent to the single equation

$$p \in H : \ \mathbf{B}(p) = F \text{ in } H'.$$

We show that this operator is *monotone*. For a pair of solutions $(\mathbf{u}_1, p_1)$ and $(\mathbf{u}_2, p_2)$ of (3.15) with corresponding F_1 and F_2, we have

$$\begin{gathered} \mathcal{A}_2(\mathbf{u}_1) - \mathcal{A}_2(\mathbf{u}_2) + \mathcal{B}_2' p_1 - \mathcal{B}_2' p_2 \ni 0, \\ -\mathcal{B}_2 \mathbf{u}_1 + \mathcal{B}_2 \mathbf{u}_2 + \mathcal{C} p_1 - \mathcal{C} p_2 = F_1 - F_2. \end{gathered}$$

Evaluating the first equation at $\mathbf{u}_1 - \mathbf{u}_2$ and the second at $p_1 - p_2$, we obtain

$$\begin{aligned} (\mathcal{A}_2(\mathbf{u}_1) - \mathcal{A}_2(\mathbf{u}_2))(\mathbf{u}_1 - \mathbf{u}_2) + (\mathcal{B}_2' p_1 - \mathcal{B}_2' p_2)(\mathbf{u}_1 - \mathbf{u}_2) \ni 0, \\ -(\mathcal{B}_2 \mathbf{u}_1 - \mathcal{B}_2 \mathbf{u}_2)(p_1 - p_2) + (\mathcal{C} p_1 - \mathcal{C} p_2)(p_1 - p_2) = (F_1 - F_2)(p_1 - p_2). \end{aligned}$$

Add these equations together and use the duality of $\mathcal{B}_2$ and $\mathcal{B}_2'$ to get

$$(\mathcal{A}_2(\mathbf{u}_1) - \mathcal{A}_2(\mathbf{u}_2))(\mathbf{u}_1 - \mathbf{u}_2) + (\mathcal{C} p_1 - \mathcal{C} p_2)(p_1 - p_2) \ni (F_1 - F_2)(p_1 - p_2).$$

Since $\mathcal{A}_2$ and $\mathcal{C}$ are monotone, this shows the following.

Lemma 3.5. *The operator* $\mathbf{B}$ *is a monotone Lipschitz function from* H *into* H'*.*

It follows that $\mathbf{B}$ is a maximal monotone relation on $H \times H'$. Without additional assumptions, it is not necessarily injective, but $\mathbf{B}^{-1}$ is a maximal monotone and possibly multi-valued relation.

Commonly $c(\mathbf{x}) \geq c_2 > 0$, and then we are in the following situation.

Corollary 3.6. *Assume* $\mathcal{C}$ *is* H*-coercive. For* $\boldsymbol{\psi} \in \mathbf{V}'$ *and* $F \in H'$*, there is a unique pair* $\mathbf{u} \in \mathbf{V}$*,* $p \in H$ *such that* (3.15) *holds.*

Proof. Since $\mathcal{A}_2$ is strongly monotone and $\mathcal{C}$ is coercive, it follows from the preceding monotonicity estimate that there exist constants $c_1 > 0,\ c_2 > 0$ such that

$$c_1\|\mathbf{u}_1 - \mathbf{u}_2\|^2_{\mathbf{V}} + c_2\|p_1 - p_2\|^2_H \leq (F_1 - F_2)(p_1 - p_2),$$

and this shows that $\mathbf{B}$ is strongly monotone.

Next we prove that $\mathbf{B}$ is *coercive.* Proceeding as above, (3.15) implies

$$\begin{aligned} \mathcal{A}_2(\mathbf{u})(\mathbf{u}) + \mathcal{B}_2' p(\mathbf{u}) &\ni \boldsymbol{\psi}(\mathbf{u}), \\ -\mathcal{B}_2\mathbf{u}(p) + \mathcal{C}p(p) &= \mathbf{B}(p)(p), \end{aligned}$$

and adding these two equations yields

$$\mathcal{A}_2(\mathbf{u})(\mathbf{u}) + \mathcal{C}p(p) \ni \mathbf{B}(p)(p) + \boldsymbol{\psi}(\mathbf{u}).$$

Since $\mathcal{A}_2$ and $\mathcal{C}$ are coercive, we obtain

$$c_0\|\mathbf{u}\|^2_V + c_2\|p\|^2_H \leq \mathbf{B}(p)(p) + \|\boldsymbol{\psi}\|\,\|\mathbf{u}\|_V,$$

with positive constants c_0 and c_2, which in turn implies that

$$c_2\,\|p\|^2_H \leq \mathbf{B}(p)(p) + \frac{\|\boldsymbol{\psi}\|^2}{4{c_0}^2}.$$

This inequality shows that

$$\lim_{\|p\|_H\to\infty} \|p\|_H^{-1}\,\mathbf{B}(p)(p) = +\infty.$$

Thus the operator $\mathbf{B}$ is coercive, and the *Minty-Browder* Theorem ([49], Ch.II) asserts that for each $F \in H'$, the equation $\mathbf{B}(p) = F$ has a unique solution. Set $\mathbf{u} \equiv \mathcal{A}_2^{-1}(\boldsymbol{\psi} - \mathcal{B}_2' p)$ to obtain the corresponding solution $\mathbf{u}$, p of (3.15). $\square$

Note that in the situation of Corollary 3.6 the matrix operator $\begin{pmatrix} \mathcal{A}_2 & \mathcal{B}_2' \\ -\mathcal{B}_2 & \mathcal{C} \end{pmatrix}$: $\mathbf{V} \times H \to \mathbf{V}' \times H'$ is strongly monotone, so the *Minty-Browder Theorem* shows directly it is surjective. However, compressibility can be very small, so any estimates obtained from $c > 0$ could be delicate.

4. The Evolution Equation

The monotone Lipschitz continuous operator $\mathbf{B} : H \to H'$ constructed from the displacement-pressure system (3.15) is the first of the two that will be used to represent the 3-field system (3.14) as a single nonlinear evolution equation. For the second operator, we return to the flux-pressure system. Define a linear function $\mathbf{A} : \mathrm{Dom}(\mathbf{A}) \to H'$ by $p \in \mathrm{Dom}(\mathbf{A}) \subset H$ with $\mathbf{A}p = F$ if there exists a $\mathbf{q} \in \mathbf{W}$ for which the pair $(\mathbf{q}, p) \in \mathbf{W} \times H$ is the unique solution to the system

$$\mathcal{A}_1\mathbf{q} + \mathcal{B}_1' p = \mathbf{0} \text{ in } \mathbf{W}', \tag{4.16a}$$

$$-\mathcal{B}_1\mathbf{q} = F \text{ in } H'. \tag{4.16b}$$

Since $\mathcal{A}_1$ is invertible on $\mathrm{Ker}(\mathcal{B}_1)$, we have $\mathrm{Rg}(\mathbf{A}) = \mathrm{Rg}(\mathcal{B}_1) = H'$. The uniqueness of p follows from $\mathrm{Ker}(\mathcal{B}_1') = \{0\}$. Now we can rewrite the system (3.14) as the implicit nonlinear evolution equation

$$p(t) \in \mathrm{Dom}(\mathbf{A}) \subset H : \tfrac{d}{dt}\mathbf{B}(p(t)) + \mathbf{A}p(t) = F(t) \text{ in } H',\ 0 < t < T, \tag{4.17}$$

for fluid pressure. The first term represents the rate of increase of fluid content in each local cell, and the second is the outward flux across its boundary.

The 3-field Biot System. We shall show that the Cauchy problem is well-posed in a superspace of H' (with a weaker norm) for a formally weaker form of (4.17) determined by an extension of the operator $\mathbf{A}$. This will be a *weak solution* that takes values in the domain of a restriction of the operator $\mathbf{B}$. Note that $\mathbf{A}$ is not defined on all of H and that $\mathbf{B}$ need not be injective, *e.g.*, if $\mathcal{C} = 0$. That is, $\mathbf{A}$ is an unbounded linear operator and $\mathbf{B}$ is nonlinear and possibly *degenerate*, *i.e.*, not injective.

Lemma 4.1. *The operator* $\mathbf{A}$ *constructed from* (4.16) *is a symmetric monotone bijection of* $\mathrm{Dom}(\mathbf{A}) \subset H$ *onto* H'*.*

Proof. Let $\mathbf{A}p = -\mathcal{B}_1\mathbf{q}$ and $\mathbf{A}s = -\mathcal{B}_1\mathbf{r}$. Using the facts that $\mathcal{A}_1$ is linear continuous and symmetric, we have

$$\begin{aligned}\mathbf{A}p(s) = -\mathcal{B}_1\mathbf{q}(s) = -\mathcal{B}_1's(\mathbf{q}) = \mathcal{A}_1\mathbf{r}(\mathbf{q})& \\ = \mathcal{A}_1\mathbf{q}(\mathbf{r}) = -\mathcal{B}_1'p(\mathbf{r}) = -\mathcal{B}_1\mathbf{r}(p) = \mathbf{A}s(p).&\end{aligned}$$

Since $\mathcal{Q} = \kappa^{-1}$ is positive-definite and $\nu_0, \nu_1 \geq 0$, the operator $\mathcal{A}_1$ is monotone, so $\mathbf{A}p(p) = \mathcal{A}_1\mathbf{q}(\mathbf{q}) \geq 0$. Also $\mathcal{B}_1'$ is injective, and this gives the strict monotonicity of $\mathbf{A}$. Notice that $\mathbf{A}$ is onto H', and from the Poincaré inequality it follows that it is H-coercive, therefore the inverse $\mathbf{A}^{-1} : H' \to H$ exists and is continuous. $\square$

First we extend $\mathbf{A}$ from $\mathrm{Dom}(\mathbf{A})$ to a larger Hilbert space H_a. Since $\mathbf{A}$ is a closed linear bijection onto H', its domain $\mathrm{Dom}(\mathbf{A})$ is a Hilbert space with the scalar-product $(p, s)_{\mathrm{Dom}(\mathbf{A})} = (\mathbf{A}p, \mathbf{A}s)_{H'}$. A continuous scalar-product on $\mathrm{Dom}(\mathbf{A})$ is given by $(p, s)_{H_a} \equiv \mathbf{A}p(s),\ p, s \in \mathrm{Dom}(\mathbf{A})$. Define H_a to be the completion of $\mathrm{Dom}(\mathbf{A})$ with that scalar-product; H_a is a Hilbert space in which $\mathrm{Dom}(\mathbf{A})$ is dense and continuously imbedded. The dual space H_a' consists of those linear functionals in $\mathrm{Dom}(A)'$ that are continuous with the weaker norm of the H_a scalar-product. Then $\mathbf{A}$ has a unique extension by continuity from $\mathrm{Dom}(\mathbf{A})$ to H_a, and this extension, denoted also by $\mathbf{A} : H_a \to H_a'$, is the Riesz isomorphism of the Hilbert space H_a onto its dual. Since $\mathbf{A}$ is coercive over H, we have the second of the continuous inclusions $\mathrm{Dom}(\mathbf{A}) \subset H_a \subset H$ of the three Hilbert spaces. The identity $(u, v)_{H_a'} = v(\mathbf{A}^{-1}u)$ for the scalar product in H_a' will be crucial below. Next restrict the function $\mathbf{B} : H \to H'$ to $H_a \to \mathbf{B}(H_a)$. Then the inverse is a (possibly multi-valued) relation $\mathbf{B}^{-1} : \mathbf{B}(H_a) \to H_a$. Note that $\mathbf{B}(H_a) \subset H' \subset H_a'$ and that the relation $\mathbf{B}^{-1}$ is monotone. These are summarized in the following diagram.

$$
\begin{array}{ccccc}
H' & \xleftarrow{\mathbf{B}} & H & & \\
\bigcup & & \bigcup & & \\
\mathbf{B}(H_a) & \xleftarrow{\mathbf{B}} & H_a & \xrightarrow{\mathbf{A}} & H_a' \\
 & & \bigcup & & \bigcup \\
 & & \mathrm{Dom}(\mathbf{A}) & \xrightarrow{\mathbf{A}} & H'
\end{array}
$$

Finally, we define $\mathrm{Dom}(\mathbf{C}) \equiv \mathbf{B}(H_a)$ and set $\mathbf{C} \equiv \mathbf{A}\mathbf{B}^{-1}$ to obtain the composition $\mathbf{C} : \mathrm{Dom}(\mathbf{C}) \to H_a'$ with domain $\mathrm{Dom}(\mathbf{C}) \subset H' \subset H_a'$.

Lemma 4.2. *The relation $\mathbf{C} \subset H_a' \times H_a'$ is accretive and the range of the sum $\mathbf{C} + I = (\mathbf{A} + \mathbf{B})\mathbf{B}^{-1}$ is H_a'.*

Proof. Let $u_j \in \mathbf{C}(v_j)$ for $j = 1, 2$. That is, $u_j = \mathbf{A}s_j$ for some $s_j \in \mathbf{B}^{-1}(v_j)$, so $v_j = \mathbf{B}(s_j)$. Then we have

$$(u_1 - u_2, v_1 - v_2)_{H_a'} = (v_1 - v_2)(s_1 - s_2) = (\mathbf{B}(s_1) - \mathbf{B}(s_2))(s_1 - s_2) \geq 0$$

since $\mathbf{B}$ is monotone, and this shows $\mathbf{C}$ is accretive on H_a'. Moreover, we have $u \in (\mathbf{C} + I)(v)$ in H_a' if and only if there is an $s \in H_a$ with

$u = \mathbf{A}s + \mathbf{B}(s)$ and $v = \mathbf{B}(s)$. Since the range of $\mathbf{B}$ is equal to $\mathrm{Dom}(\mathbf{C})$, it suffices to check that $\mathbf{A}+\mathbf{B}$ maps H_a onto H_a'. But $\mathbf{B}$ is monotone and Lipschitz, hence, maximal monotone, and $\mathbf{A}$ is the Riesz isomorphism for H_a, so this follows from Minty's Theorem. $\square$

As a consequence of Lemma 4.2, the relation $\mathbf{C}$ is *m-accretive*, so its negative is the generator of a nonlinear semigroup of contractions in the Hilbert space H_a'. This implies that the initial-value problem for the corresponding abstract evolution equation is well-posed in H_a' [10, 21, 33, 49].

Theorem 4.3. *Let $\mathbf{C}$ be m-accretive in the Hilbert space H_a'. For each $b \in \mathrm{Dom}(\mathbf{C})$ and absolutely continuous $F \in W^{1,1}(0,T;H_a')$, there is a unique absolutely continuous solution $w \in W^{1,1}(0,T;H_a')$ of the initial-value problem*

$$\dot{w}(t) + \mathbf{C}(w(t)) \ni F(t) \text{ in } L^1(0,T;H_a'), \quad w(0) = b. \tag{4.18}$$

This solution is Lipschitz continuous ($w \in W^{1,\infty}(0,T;H_a')$) and $w(t) \in \mathrm{Dom}(\mathbf{C})$ for every $t \in [0,T]$.

Choose $p(t) \equiv \mathbf{A}^{-1}(F(t) - \dot{w}(t)) \in \mathbf{B}^{-1}(w(t))$. Then $p(t) \in H_a$ for almost every $t \in [0,T]$ and we have a *weak solution* of (4.17). Since $\mathbf{B}$ is a function, $\mathbf{B}(p(t)) \in W^{1,\infty}(0,T;H_a')$ and we have obtained the following result.

Corollary 4.4. *Assume the conditions of Section 2.2. Let the data $F \in W^{1,1}(0,T;H_a')$ and $b \in \mathbf{B}(H_a)$ be given. Then the reduced weak problem* (3.14) *is equivalent to* (4.17). *It has a unique* weak solution

$$p \in L^\infty(0,T;H_a) : \ \tfrac{d}{dt}\mathbf{B}(p(t)) + \mathbf{A}p(t) = F(t) \ \text{a.e. in } L^\infty(0,T;H_a'), \tag{4.19}$$

with $\mathbf{B}(p(0)) = b$.

It remains to characterize the space H_a and the corresponding extension of $\mathbf{A}$. Each $p \in \mathrm{Dom}(\mathbf{A})$ with $\mathbf{A}p = F \in H'$ satisfies

$$p \in H : \ \mathcal{Q}\mathbf{q} + \boldsymbol{\nabla} p = 0, \ \boldsymbol{\nabla}\cdot\mathbf{q} = F \text{ in } \Omega, \tag{4.20a}$$

$$q_n = 0 \text{ on } S, \ \nu_j q_n - p = 0 \text{ on } \Gamma_j, \ j = 0,1, \tag{4.20b}$$

for some $\mathbf{q} \in \mathbf{W}$. Thus $p \in H^1(\Omega)$ and

$$\begin{aligned}(4.21)\quad \mathbf{A}p(p) &= \int_\Omega \mathcal{Q}\mathbf{q}\cdot\mathbf{q}dx + \int_{\Gamma_0} \nu_0 q_n^2 dS + \int_{\Gamma_1} \nu_1 q_n^2 dS \\ &= \int_\Omega \kappa\boldsymbol{\nabla} p\cdot\boldsymbol{\nabla} p\,dx + \int_{\Gamma_0^\nu} \nu_0^{-1} p^2 dS + \int_{\Gamma_1^\nu} \nu_1^{-1} p^2 dS < +\infty.\end{aligned}$$

Recall that Γ_j^ν denotes the support of ν_j, $j = 0, 1$.

Suppose $\mathbf{A}p(p) = 0$. Then $q_n = 0$ and $p = p_0$, a constant; then (4.20b) shows $p = 0$, so $\mathbf{A}p(p)^{1/2}$ is a norm. Completion of $\mathrm{Dom}(\mathbf{A})$ in this norm yields the space

$$(4.22)\qquad H_a = \{s \in H^1(\Omega) : \gamma(s) \in \nu_0^{1/2} L^2(\Gamma_0) \oplus \nu_1^{1/2} L^2(\Gamma_1)\}.$$

Conversely, note that for every $p \in H^1(\Omega)$ the boundary integrals in (4.21) are finite by Lemma 2.1, so $\mathbf{A}p(p)$ is equivalent to the $H^1(\Omega)$ norm.

The extension $\mathbf{A} : H_a \to H_a'$ is defined by

$$(4.23)\quad \mathbf{A}p(s) = \int_\Omega \kappa\boldsymbol{\nabla} p\cdot\boldsymbol{\nabla} s\,dx + \int_{\Gamma_0^\nu} \nu_0^{-1} ps\,dS + \int_{\Gamma_1^\nu} \nu_1^{-1} ps\,dS,\ p, s \in H_a.$$

This is the Riesz map of H_a with the norm determined by (4.21). It is the classical minimization formulation of the elliptic equation $-\boldsymbol{\nabla}\cdot\kappa\boldsymbol{\nabla} p = F$ with Neumann and Robin boundary conditions (4.20b) for the reduced weak formulation (3.14). Note that for $s \in H_a$ we have $\gamma(s) = 0$ on $\Gamma_k \setminus \Gamma_k^\nu$. Also, the space H_a' has boundary values, unlike H', and these are part of an equation in H_a'.

The 2-field Biot System. Finally we show directly that the space H_a is the pressure-space for the weak formulation of the 2-field system (2.3). Recall that $\mathcal{Q}^{-1} = \kappa$. Define $\kappa_j = \nu_j^{-1}$ on Γ_j^ν for $j = 1, 2$ so that

$$H_a = \{s \in H^1(\Omega) : \kappa_j^{1/2}\gamma(s) \in L^2(\Gamma_j^\nu) \text{ for } j = 1, 2\}.$$

For a stationary solution of equations (2.5a), (2.5b) and boundary conditions (2.6a), (2.6b), (2.6e) we compute

$$\begin{aligned}(4.24)\quad \int_\Omega Fs\,dx &= \int_\Omega \boldsymbol{\nabla}\cdot\mathbf{q}s\,dx = -\int_\Omega \mathbf{q}\cdot\boldsymbol{\nabla} s\,dx + \int_{\partial\Omega} q_n s\,dS \\ &= \int_\Omega \kappa(\boldsymbol{\nabla} p - \mathbf{g})\boldsymbol{\nabla} s\,dx + \int_{\Gamma_0} \kappa_0(p - P_0)s\,dS + \int_{\Gamma_1} \kappa_1(p - P_1)s\,dS\end{aligned}$$

for $s \in V$. After translating to obtain homogeneous data $\mathbf{g} = \mathbf{0}$, $P_0 = P_1 = 0$, we obtain the characterization (4.23) of $\mathbf{A}$.

5. Summary: The Weak Solution

Theorem 5.1. *Assume the conditions of Section 2.2 and define H_a by (4.22). Let the data be given with $F \in W^{1,1}(0,T;H')$, $b \in \mathbf{B}(H_a)$. Then the reduced weak problem (3.14) has a unique weak solution for which $\mathbf{u} \in L^\infty(0,T;\mathbf{V})$, $p \in L^\infty(0,T;H_a)$,*

$$\mathcal{A}_2(\mathbf{u}(t)) + \mathcal{B}_2' p(t) \ni \boldsymbol{\psi} \text{ in } L^\infty(0,T;\mathbf{V}'), \text{ and} \tag{5.25a}$$

$$\mathbf{A}p(t) + \frac{d}{dt}\big(\mathcal{C}p(t) - \mathcal{B}_2\mathbf{u}(t)\big) = F(t) \text{ in } L^\infty(0,T;H_a'), \tag{5.25b}$$

and $\mathbf{A}p(t)$ is given by (4.23).

The weak solution obtained in Theorem 5.1 differs somewhat from the weak problem (3.14). The pressure $p(t)$ is obtained in the space H_a, so it is smoother than the required $p(t) \in H$ for (3.14). If additionally $p(t) \in \mathrm{Dom}(\mathbf{A})$, then

$$\mathcal{A}_1\mathbf{q}(t) + \mathcal{B}_1' p(t) = 0 \text{ in } L^\infty(0,T;\mathbf{W}'), \tag{5.26a}$$

$$\mathcal{A}_2(\mathbf{u}(t)) + \mathcal{B}_2' p(t) \ni \boldsymbol{\psi} \text{ in } L^\infty(0,T;\mathbf{V}'), \text{ and} \tag{5.26b}$$

$$-\mathcal{B}_1\mathbf{q}(t) + \frac{d}{dt}\big(\mathcal{C}p(t) - \mathcal{B}_2\mathbf{u}(t)\big) = F(t) \text{ in } L^\infty(0,T;H_a'), \tag{5.26c}$$

and the equation (5.26c) holds in the larger dual space H_a'.

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