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MATHEMATICAL FORMULATION OF THE STEFAN PROBLEM

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Abstract—The Stefan problem describes the conduction of heat in a medium involving a solid–liquid phase change at a prescribed melting temperature. Considerations of physical, mathematical and numerical experiences with such problems all imply that enthalpy (not temperature) is the natural dependent variable to specify the solution. Our discussion centers on the physical interpretation of the multi-valued Heaviside "function" which arises in the mathematical formulation as the fraction of water. We show that this permits the consideration of (possibly large) regions of mush at the melting temperature and of problems with internally distributed sources of heat. Moreover, in order for such problems to be well-posed, this fraction of water must necessarily be specified initially in the part of the region at the melting temperature.

1. INTRODUCTION

In formulating the Stefan problem one previously assumed that the region consisted of two parts, a region of water and a region of ice, and that these regions were separated initially and thereafter by an unknown surface, the *free boundary*. This restricts the data given for the problem. For example, the initial temperature distribution must be non-zero except possibly on the initial surface (of measure zero) and any distributed sources F must be positive in the water and negative in the ice. But these regions are not given so one assumes $F = 0$. Such situations are quite special. Indeed all geophysical applications will contain large regions at the melting temperature and a distributed source of (solar) heat. It is just such problems (as the melting of soil water, lake surfaces and exposed water pipes) which arise most often in applications.

Here we shall include in our formulation of the Stefan problem a region of *mush*, a mixture of water and ice coexisting in thermal equilibrium at the freezing temperature. This phenomenon is consistent with the physics of latent heat. The resulting classical formulation of the Stefan problem contains a new unknown, the fraction of water at each point. The corresponding weak formulation is exactly that of the previously considered special case with no mush: our considerations merely give a physical interpretation of the singularity or jump function that characterizes the Stefan problem.

It is worthwhile to recall the simple experiment in which one applies a uniform heat source of intensity F to a unit volume of ice at temperature $u < 0$. The temperature increases at the rate F/c_1 until it reaches $u = 0$; $c_1 > 0$ is the specific heat of ice. Then the temperature remains at zero until L units of heat have been added; $L > 0$ is the latent heat. During this period there is a fraction ξ of water coexisting with the ice and ξ increases at the constant rate F/L. When all the ice has melted, $\xi = 1$ and the temperature u begins to rise at the rate F/c_2; $c_2 > 0$ is the specific heat of water. We can summarize the above by stating that the rate of increase of heat energy or *enthalpy* $v \equiv C(u) + L\xi$ is given by F, where the specific heat is given by $C(u) = c_1 u$ for $u < 0$ and $C(u) = c_2 u$ for $u \geq 0$ and $\xi \epsilon H(u)$, the Heaviside graph being given by $H(u) = 1$ for $u > 0$, $H(0) = [0, 1]$ and $H(u) = 0$ for $u < 0$. Thus the thermodynamic state is determined by the enthalpy. The temperature is obtained from the function $u = (C + LH)^{-1}(v)$ and the fraction of water is given by $\xi = (v - C(u))/L$. We shall see below that the weak formulation of the problem also shows that enthalpy is the natural variable to determine the state of the process.

2. THE CLASSICAL PROBLEM

We begin with a mathematical description of heat conduction through a medium in which a change of phase occurs at a given temperature; a model problem is the melting of ice in a pipe or porous medium. Thus let the domain G be given in Euclidean space $\mathbf{R}^m$ and set $\Omega = G \times (0, \infty)$. Let $u(x, t)$ denote the temperature at the point $x \in G$ and time $t > 0$. The constants c_1, c_2 are specific heats and k_1, k_2 are conductivities of ice and water, respectively. The ice-water phase change occurs at the temperature $u = 0$; Ω is then separated into an ice region Ω_1 where $u < 0$, a water region Ω_2 where $u > 0$, and a mush region Ω_0 where $u = 0$. At each point (x, t) of Ω we introduce the fraction of water, $\xi(x, t)$; note that $\xi(x, t) \in H(u(x, t))$ in Ω where $H(\cdot)$ is the Heaviside graph given above. Let S_1 be the boundary of Ω_1 in Ω and S_2 be the boundary of Ω_2 in Ω. At each point of $S_1 \cup S_2$ we denote by $N = (N_1, N_2, \ldots, N_m, N_t)$ the unit normal oriented out of Ω_2 or into Ω_1. Note that this is consistent on $S_1 \cap S_2$, the interface between Ω_1 and Ω_2.

The Stefan problem is the following: find a pair of real-valued functions u and ξ on Ω for which

$$c_2 u_t - k_2 \Delta u = F(x, t) \text{ in } \Omega_2 \equiv \{(x, t) \in \Omega\colon u(x, t) > 0\} \tag{1a}$$

$$c_1 u_t - k_1 \Delta u = F(x, t) \text{ in } \Omega_1 \equiv \{(x, t) \in \Omega\colon u(x, t) < 0\} \tag{1b}$$

$$L\xi_t = F(x, t) \text{ in } \Omega_0 \equiv \{(x, t) \in \Omega\colon u(x, t) = 0\} \tag{1c}$$

and

$$\xi \in H(u) \text{ in } \Omega, \tag{2}$$

$$k_2 \nabla_x u \cdot (N_1, \ldots, N_m) = LN_t(1 - \xi) \text{ on } S_2 = (\partial\Omega_2) \cap \Omega \tag{3a}$$

$$k_1 \nabla_x u \cdot (N_1, \ldots, N_m) = -LN_t\xi \text{ on } S_1 = (\partial\Omega_1) \cap \Omega \tag{3b}$$

$$u(x, 0) = u_0(x), x \in G \tag{4a}$$

$$\xi(x, 0) = \xi_0(x) \in [0, 1] \text{ where } u_0(x) = 0, \tag{4b}$$

$$u(x, t) = g(s), s \in \partial G, t > 0. \tag{5}$$

The partial differential eqns (1a) and (1b) are the classical heat conduction equations in the water (Ω_2) and ice (Ω_1) regions, respectively. The fraction of water is determined by (1c) in Ω_0 (see the preceding remark) and from (2) we have $\xi = 1$ in Ω_2 and $\xi = 0$ in Ω_1. In order to interpret (3a) we let n be the normalized $(N_1, \ldots, N_m)$ and V be the velocity of S_2 at time t in the direction of n. Thus dividing by $(N_1^2 + \cdots + N_m^2)^{1/2}$ gives

$$k_2 \frac{\partial u}{\partial n} = -LV(1 - \xi) \text{ on } S_2.$$

The heat flux from Ω_2 determines the velocity V of the free boundary S_2 by melting the fraction of ice $(1 - \xi)$ with latent heat L. Similarly (3b) states that the heat flux drawn across S_1 into Ω_1 is obtained from the latent heat arising from the fraction of water ξ present on S_1. We note that on $S_1 \cap S_2$ we can substract (3a) and (3b) to obtain the usual classical Stefan free boundary constraint

$$(k_2 \nabla_x u^+ - k_1 \nabla_x u^-) \cdot (N_1, \ldots, N_m) = LN_t \text{ on } S_1 \cap S_2. \tag{3c}$$

The initial condition (4) specifies the temperature at all points of G and the fraction of water where the temperature is zero. Note that this is equivalent to specifying the initial enthalpy, $v_0(x) = C(u_0(x)) + L\xi_0(x)$ with $\xi_0(x) \in H(u_0(x))$. The Dirichlet boundary conditions (5) or any of the usual constraints can be specified on ∂G.

3. THE WEAK PROBLEM

A solution of even the classical form of the Stefan problem is known to exhibit discontinuities in first derivatives: such discontinuities comprise an essential ingredient of the solution on the surface $S_1 \cap S_2$. Thus we expect the solution to belong to the Sobolev space $H^1(\Omega)$ and to be smooth in each of Ω_1 and Ω_2. Such considerations of regularity of solutions together with experience with the usual "no-mush" Stefan problem suggest that we formulate an appropriate weak form of the problem.

In order to obtain such a weak formulation, we consider a solution u, ξ of the Stefan problem (1)–(5) for which Ω_1 and Ω_2 are sufficiently smooth. Define $K(s) = k_2 s$ for $s \geq 0$ and $K(s) = k_1 s$ for $s < 0$ and let $v = C(u) + L\xi$, the enthalpy of the solution. We shall compute $(\partial v/\partial t) - \Delta K(u)$ in the sense of distributions on Ω. Thus for any test function $\varphi \in C_0^\infty(\Omega)$ we have

$$\left\langle \frac{\partial v}{\partial t} - \Delta K(u), \varphi \right\rangle \equiv \int_\Omega (-v\varphi_t - K(u)\Delta\varphi)$$

$$= \int_{\Omega_1} (-c_1 u\varphi_t + k_1 \nabla u \cdot \nabla\varphi) - \int_{\Omega_0} L\xi\varphi_t + \int_{\Omega_2} (-(L + c_2 u)\varphi_t + k_2 \nabla u \cdot \nabla\varphi)$$

since $u \in H^1(\Omega)$. Since μ is regular on Ω_1 and Ω_2 and ξ is regular on Ω_0 we obtain by Gauss' theorem

$$\int_{\Omega_1} (c_1 u_t - k_1 \Delta u)\varphi - \int_{S_1} \{k_1 \nabla u \cdot (N_1, \ldots, N_m) + L\xi N_t\}\varphi + \int_{\Omega_0} L\xi_t\varphi$$

$$+ \int_{\Omega_2} (c_2 u_g - k_2 \Delta u)\varphi + \int_{S_2} \{k_2 \nabla u \cdot (N_1, \ldots, N_m) - L(1 - \xi)N_t\}\varphi.$$

Thus we have

$$\left\langle \frac{\partial v}{\partial t} - \Delta K(u), \varphi \right\rangle = \int_\Omega F\varphi = \langle F, \varphi \rangle$$

for every $\varphi \in C_0^\infty(\Omega)$ if and only if (1) and (3) hold. Since $v = C(u) + L\xi$ with (2) implies $u = (C + LH)^{-1}(v)$, i.e. temperature is the identicated function of enthalpy v, we have

$$\frac{\partial v}{\partial t} - \Delta K \cdot (C + LH)^{-1}(v) = F \text{ in } \mathscr{D}'(\Omega) \tag{6}$$

$$v(x, 0) = C(u_0(x)) + L\xi_0(x), x \in G, \tag{7}$$

$$K((C + LH)^{-1}(v(s, t))) = K(g(s, t)), s \in \partial G, t > 0. \tag{8}$$

Thus the abstract initial-boundary value problem (6)–(8) is the weak formulation of (1)–(5).

We have shown that the evolution eqn (6) includes solutions of the Stefan problem which begin with a region of mush as well as those in which a region of mush develops as a consequence of internal heat sources. These considerations are crucial in the corresponding "*one-phase*" problem in which $u \geq 0$ in Ω: the problem is interesting only if $u = 0$ in a substantial portion Ω_0 of Ω. In order to describe the one-phase solution of (6) by a variational inequality one introduces the new unknown *freezing index* $w(t) = \int_0^t u(s)\,ds$ and

$$P(w) = c_2 w'(t) - k_2 \Delta w(t) - \int_0^t F - c_2 u_0 + L(1 - \xi_0)$$

Thus an integration in time of (6) yields $P(w) = L(1 - \xi(t))$. Since $\xi(t) \in H(w'(t))$ we obtain the variational inequality

$$P(w) \geq 0, w'(t) \geq 0, P(w)(w'(t)) = 0, t \geq 0, \tag{9}$$

which characterizes the solution of (6). If $F \geq 0$ in Ω then the free-boundary S_2 is monotone so $u(t)$ and $w(t)$ have the same support. Thus $\xi(t) \in H(w(t))$ and we have

$$P(w) \geq 0, w(t) \geq 0, P(w)(w(t)) = 0, t \geq 0, \tag{10}$$

another variational inequality of evolution which characterizes the solution of (6).

4. REMARKS

The weak formulations we have described are equivalent to the corresponding weak forms of the classical Stefan problem without a region of mush. Thus we have immediately available a theory of existence, uniqueness and regularity of solutions as well as many techniques for the numerical computation of solutions. The discussion generalizes trivially to include nonlinear specific heats and conductivity functions in each phase as well as certain nonlinear boundary conditions. For existence-uniqueness theory one may see [1-3, 5, 7] for various approaches. These are based on the weak formulation of Oleinik; see [3] for a sketch of the history and development. For the formulation and resolution of associated variational inequalities one can see [4–6] and their references. See [8] for an extensive bibliography of work published from 1965 to 1978.

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