SIAM J. MATH. ANAL.
Vol. 7, No. 4, August 1976

REGULARIZATION AND APPROXIMATION OF SECOND ORDER EVOLUTION EQUATIONS*

R. E. SHOWALTER†

Abstract. We give a nonstandard method of integrating the equation $Bu''+Cu'+Au=f$ in Hilbert space by reducing it to a first order system in which the differentiated term corresponds to energy. Semigroup theory gives existence for hyperbolic and for parabolic cases. When $C=\varepsilon A$, $\varepsilon \geqq 0$, this method permits the use of Faedo–Galerkin projection techniques analogous to the simple case of a single first order equation; the appropriate error estimates in the energy norm are obtained. We also indicate certain singular perturbations which can be used to approximate the equation by one which is dissipative or by one to which the above projection techniques are applicable. Examples include initial-boundary value problems for vibrations (possibly) with inertia, dynamics of rotating fluids, and viscoelasticity.

1. Introduction. Let A and C be continuous linear operators from a Hilbert space V into its antidual V'. Let W be a Hilbert space, of which V is a dense subspace continuously imbedded, and let B be continuous and linear from W to W'. We naturally identify W' with a subspace of V' and use $\langle \cdot , \cdot \rangle$ to denote the various dualities.

Problem 1. Given $u_1 \in V$, $u_2 \in W$, $f \in C((0,\infty), W')$, find $u \in C([0,\infty), V) \cap C^1((0,\infty), V) \cap C^1([0,\infty), W) \cap C^2((0,\infty), W)$ such that $u(0)=u_1$, $u'(0)=u_2$, and

$$Bu''(t)+Cu'(t)+Au(t)=f(t), \qquad t>0. \tag{1.1}$$

We shall rewrite this as a first order system. Define the Hilbert product spaces $V_l = V\times V$, $V_m = V\times W$ and the operators

$$M(x_1, x_2)\equiv(Ax_1, Bx_2), \qquad L(x_1, x_2)\equiv(-Ax_2, Ax_1+Cx_2)$$

from V_m to V'_m and V_l to V'_l, respectively. If u is a solution of Problem 1, then $w\equiv(u, u')$ is a solution of the next problem.

Problem 2. Given $(u_1, u_2)\in V_m$, $f\in C((0,\infty), w')$, find $w\in C([0,\infty), V_m)\cap C^1((0,\infty), V_m)$ such that $w(0)=(u_1, u_2)$ and

$$Mw'(t)+Lw(t)=(0, f(t)), \qquad t>0. \tag{1.2}$$

Our plan is as follows. In § 2 we obtain existence and uniqueness results under hypotheses which imply that Problems 1 and 2 are equivalent. Examples of initial-boundary value problems to which our results apply are given in § 3. Approximate solutions are obtained in § 4 from standard Faedo–Galerkin projection techniques. When $C=\varepsilon A$, $\varepsilon\geqq 0$, the L-projection factors into the A-projection onto a subspace of V; then we can give energy norm error estimates for models of finite-element subspaces when A is an elliptic operator of order 2. Finally, in § 5 we examine the error resulting from certain perturbations of (1.1) into more regular models which are parabolic. In certain models these regularizations represent *artificial viscosity* or *artificial inertia.*

* Received by the editors July 15, 1974, and in final revised form June 5, 1975.
† Department of Mathematics, the University of Texas, Austin, Texas 78712.

2. Existence and uniqueness. We shall seek hypotheses for which Problem 1 is well-posed. Recall that the operator $A: V \to V'$ is *monotone* if $\mathrm{Re}\,\langle Ax, x\rangle \geqq 0$, $x \in V$, and *symmetric* if $\langle Ax, y\rangle = \overline{\langle Ay, x\rangle}$, $x, y \in V$. Such an operator induces a seminorm $\|x\|_a = \langle Ax, x\rangle^{1/2}$, $x \in V$, and we have a Cauchy–Schwartz inequality

$$|\langle Ax, y\rangle| \leqq \|x\|_a \|y\|_a, \qquad x, y \in V.$$

Let u be a solution of Problem 1. If M is symmetric, then $w \equiv (u, u')$ satisfies

$$D_t\langle Mw(t), w(t)\rangle + 2\,\mathrm{Re}\,\langle Lw(t), w(t)\rangle = 2\,\mathrm{Re}\,\langle f(t), u'(t)\rangle, \tag{2.1}$$

so we obtain

$$\langle Mw(t), w(t)\rangle + 2\,\mathrm{Re}\int_0^t \langle Lw, w\rangle = \langle Mw(0), w(0)\rangle + 2\,\mathrm{Re}\int_0^t \langle f, u'\rangle. \tag{2.2}$$

This is equivalent to the identity

$$\langle Au(t), u(t)\rangle + \langle Bu'(t), u'(t)\rangle + 2\,\mathrm{Re}\int_0^t \langle Cu', u'\rangle$$
$$= \langle Au(0), u(0)\rangle + \langle Bu'(0), u'(0)\rangle + 2\,\mathrm{Re}\int_0^t \langle f, u'\rangle.$$

Suppose B is also monotone, and denote by $\|\cdot\|_{W_b'}$ the norm on the Hilbert space W_b' which is the antidual of W with the seminorm $\|\cdot\|_b$ induced by B. The last term in (2.1) is bounded by

$$2\|f\|_{W_b'}\|u'\|_b \leqq T\|f\|_{W_b'}^2 + T^{-1}\|u'\|_b^2,$$

where $T > 0$ is arbitrary, so (2.1) gives

$$D_t(e^{-t/T}\langle Mw(t), w(t)\rangle) + e^{-t/T} 2\,\mathrm{Re}\,\langle Lw(t), w(t)\rangle \leqq Te^{-t/T}\|f\|_{W_b'}^2.$$

Integrating this inequality gives the a priori estimate

$$\langle Mw(t), w(t)\rangle + 2\,\mathrm{Re}\int_0^t \langle Lw, w\rangle \leqq e\langle Mw(0), w(0)\rangle + Te\int_0^t \|f\|_{W_b'}^2, \qquad 0 \leqq t \leqq T. \tag{2.3}$$

We summarize the above as the following proposition.

PROPOSITION 1. *Let u be a solution of Problem* 1 *on the interval* $[0, T]$ *and assume that A and B are symmetric and monotone. Then we have*

$$\langle Au(t), u(t)\rangle + \langle Bu'(t), u'(t)\rangle + 2\int_0^t \mathrm{Re}\,\langle Cu', u'\rangle$$
$$\leqq e\langle Au_1, u_1\rangle + e\langle Bu_2, u_2\rangle + Te\int_0^t \|f\|_{W_b'}^2, \qquad 0 \leqq t \leqq T.$$

From the representation $u(t) = u_1 + \int_0^t u'$ by the (strong) integral in W and the fact that $\|\cdot\|_b$ is a continuous seminorm on W, it follows from $\|u'\|_b = 0$ on $[0, T]$ that $\|u\|_b$ is constant on $[0, T]$. This gives the following proposition.

PROPOSITION 2. *Let A and B be symmetric and monotone and let C be monotone. If u is a solution of Problem* 1 *on* $[0, T]$ *with* $u_1 = u_2 = 0$ *and* $f(\cdot) = 0$, *then*

$$\|u(t)\|_a = \|u(t)\|_b = 0, \qquad 0 \leqq t \leqq T.$$

Thus, there is at most one solution of Problem 1 *if* $\ker(A) \cap \ker(B) = \{0\}$.

We could continue to permit B to be *degenerate* as in [15]; it would be necessary to modify the definitions above and work in dual spaces but nothing essential is changed. For the remainder of this section we shall assume B is W-*coercive*: there is a $c > 0$ such that

$$\langle Bx, x\rangle \geqq c\|x\|_W^2, \qquad x \in W.$$

This holds, for instance, if $B : V \to V'$ is given symmetric and (strictly) positive and if W is the completion of V with the norm $\|\cdot\|_b$.

We consider the question of existence. In addition to the hypotheses of Proposition 2, assume A is V-coercive and B is W-coercive. Define $D \equiv \{x \in V_l : Lx \in V'_m\}$. Since A and B are isomorphisms, M is also, and we can define an operator $N : D \to V_m$ by $N = M^{-1} \circ L$. Note that $(x, y)_m \equiv \langle Mx, y\rangle$ gives an (equivalent) inner product on V_m for which we have the identity

$$(Nx, y)_m = \langle Lx, y\rangle, \qquad x \in D, \quad y \in V_l.$$

It follows that N is *accretive*:

$$\operatorname{Re}(Nx, x)_m \geqq 0, \qquad x \in D.$$

To show that $-N$ generates a strongly continuous semigroup of contractions on V_m, it suffices to show that $\lambda + N$ is onto V_m for every $\lambda > 0$. But this is equivalent to the following lemma.

LEMMA 1. *$\lambda M + L$ maps D onto V'_m for every $\lambda > 0$.*

Proof. Let $f_1 \in V'$, $f_2 \in W'$. Since A is V-coercive, so also is $A + \lambda C + \lambda^2 B$, and each maps onto V', so there exist $x_1, x_2 \in V$ for which

$$(A + \lambda C + \lambda^2 B)x_2 = \lambda f_2 - f_1,$$
$$\lambda A x_1 = A x_2 + f_1.$$

It follows that $Ax_1 + Cx_2 = -\lambda B x_2 + f_2 \in W'$, hence $(x_1, x_2) \in D$, and that $(\lambda M + L)(x_1, x_2) = (f_1, f_2)$.

Our first existence result follows directly from the preceding discussion and standard results on the generation of semigroups [9].

PROPOSITION 3. *Let A be symmetric and V-coercive, B be symmetric and W-coercive, and C be monotone. If $u_1, u_2 \in V$ with $Au_1 + Cu_2 \in W'$ and if $f \in C^1([0, \infty), W')$ are given, then there is a (unique) solution of Problem* 1. *The equation* (1.1) *is satisfied up to the initial time: $(u, u') \in C^1([0, \infty), V \times W)$. From this it follows that $(u, u') \in C([0, \infty), V \times V)$.*

PROPOSITION 4. *In addition to the hypotheses of Proposition* 3, *suppose that $C + \lambda B$ is V-coercive for $\lambda > 0$. If $u_1 \in V$, $u_2 \in W$ and $f : [0, \infty) \to W'$ is Hölder continuous, then there is a (unique) solution of Problem* 1.

Proof. For each $\lambda > 0$ and $x = (x_1, x_2) \in D$ we have

$$\operatorname{Re}((\lambda + N)x, x)_m = \lambda\langle Ax_1, x_1\rangle + \langle(\lambda B + C)x_2, x_2\rangle,$$

so $\lambda+N$ is V_l-coercive and, hence, sectorial. Thus, $-N$ generates an analytic semigroup [9].

In the situation of Proposition 4, either the equation is *irreversible* or N is bounded, i.e., $V=W$ [7]. In particular, Proposition 4 applies to *parabolic* problems while Proposition 3 is appropriate in the *hyperbolic* situation.

3. Examples. We illustrate some of our preceding results with initial-boundary value problems which occur in various applications. These existence-uniqueness results are far from best possible, but will serve as models for our following work.

Let G be a nonempty open set in $\mathbb{R}^n$ lying on one side of its smooth $(n-1)$-dimensional boundary, ∂G. $H^1(G)$ is the Hilbert space of (equivalence classes of) functions in $L^2(G)$, all of whose (distribution) derivatives of first order belong to $L^2(G)$. The inner product is given by

$$(\varphi, \psi)_{H^1}=\sum_{j=0}^{n}(D_j\varphi, D_j\psi)_{L^2(G)},$$

where D_j, $1\leqq j\leqq n$, denotes a partial derivative and D_0 is the identity. Let Γ_0 be an open subset of ∂G and $\Gamma_1=\partial G\sim\Gamma_0$. Let V be that subspace of $H^1(G)$ consisting of those functions whose traces vanish on Γ_0. We shall denote the *gradient* $\nabla\varphi=(D_1\varphi,\cdots,D_n\varphi)$ and *Laplacian* $\Delta\varphi=\sum_{j=1}^{n}D_j^2\varphi$ as indicated. Also, ν will denote the unit outward normal on ∂G, and $D_\nu\varphi=\nabla\varphi\cdot\nu$ is the directional normal derivative. See [12] for details.

Example 1. Define $A: V\to V'$ by

$$\langle A\varphi, \psi\rangle=\int_G \nabla\varphi\cdot\overline{\nabla\psi}, \qquad \varphi, \psi\in V.$$

For each $\varphi\in V$, the restriction of $A\varphi$ to the space $C_0^\infty(G)$ is the distribution $-\Delta\varphi$. Regularity theory for elliptic equations shows that Green's formula

$$\langle A\varphi, \psi\rangle=\int_G(-\Delta\varphi)\bar{\psi}+\int_{\partial G}D_\nu\varphi\bar{\psi}$$

is meaningful whenever $\Delta\varphi\in L^2(G)$. Take $W=L^2(G)$ and $\langle B\varphi, \psi\rangle=(\varphi, \psi)_{L^2(G)}$. Let $R\geqq 0$ and $r\geqq 0$ be given and define

$$\langle C\varphi, \psi\rangle=R\int_G\varphi\bar{\psi}+r\int_{\partial G}\varphi\bar{\psi}, \qquad \varphi, \psi\in V.$$

Finally, let $F(x, t)$ be a real-valued function in $C^1(\bar{G}\times[0,\infty))$ and set $f(t)=F(\cdot, t)$, $t\geqq 0$. Propositions 2 and 3 show that for each pair u_1, $u_2\in V$ with $\Delta u_1\in L^2(G)$ and $D_\nu u_1+ru_2=0$ on Γ_1 there is a unique generalized solution $u=u(x, t)$ of

$$\begin{aligned}
&D_t^2u+RD_tu-\Delta u=F(x, t), && x\in G, \quad t\geqq 0,\\
&u(x, 0)=u_1(x), \qquad D_tu(x, 0)=u_1(x), &&\\
&u(x, t)=0, && x\in\Gamma_0,\\
&D_\nu u(x, t)+rD_tu(x, t)=0, && x\in\Gamma_1.
\end{aligned}$$

This hyperbolic problem is the classical wave equation with weak dissipation distributed through $G(R>0)$ or along $\partial G(r>0)$.

Example 2. Let A and B be as above and set

$$\langle C\varphi, \psi\rangle = \varepsilon \int_G \nabla\varphi \cdot \overline{\nabla\psi}, \qquad \varphi, \psi \in V,$$

where $\varepsilon > 0$. Propositions 2 and 4 show that for each pair $u_1 \in V$ and $u_2 \in L^2(G)$ there is a unique generalized solution $u = u(x, t)$ of

$$\begin{aligned} &D_t^2 u - \varepsilon \Delta D_t u - \Delta u = F(x, t), && x \in G, \quad t > 0,\\ &u(x, 0) = u_1(x), \qquad D_t u(x, 0) = u_2(x), && \\ &u(x, t) = 0, && x \in \Gamma_0,\\ &D_\nu(u(x, t) + \varepsilon D_t u(x, t)) = 0, && x \in \Gamma_1. \end{aligned}$$

This is a parabolic problem arising from certain models in classical hydrodynamics or viscoelasticity. Strong dissipation results from the presence of the positive constant ε which represents *viscosity* in the model [8].

Example 3. Take A as above but set $C = 0$, $W = V$, and define

$$\langle B\varphi, \psi\rangle = \int_G (\varphi\bar{\psi} + \varepsilon \nabla\varphi \cdot \overline{\nabla\psi}), \qquad \varphi, \psi \in V,$$

where $\varepsilon > 0$. Let $G(s, t)$ be a real-valued function in $C^1(\Gamma_1 \times [0, \infty))$ and define $f : [0, \infty) \to V'$ by

$$\langle f(t), \varphi\rangle = \int_G F(\,\cdot\,, t)\varphi + \int_{\partial G} G(\,\cdot\,, t)\varphi, \qquad \varphi \in V.$$

Then either of Propositions 3 or 4 shows that for each pair u_1, $u_2 \in V$ there is a generalized solution $u = u(x, t)$ of

$$\begin{aligned} &D_t^2 u - \varepsilon\, \Delta D_t^2 u - \Delta u = F(x, t), && x \in G, \quad t \geqq 0,\\ &u(x, 0) = u_1(x), \qquad D_t u(x, 0) = u_2(x), && \\ &u(x, t) = 0, && x \in \Gamma_0,\\ &D_\nu(u(x, t) + \varepsilon D_t^2 u(x, t)) = G(x, t), && x \in \Gamma_1. \end{aligned}$$

This problem arises in classical vibration models in which ε represents *inertia* [13, § 278].

Example 4. Here we choose $W = V$ and $C = 0$ as before, but define

$$\langle B\varphi, \psi\rangle = \int_G \nabla\varphi \cdot \overline{\nabla\psi}, \qquad \varphi, \psi \in V,$$

$$\langle A\varphi, \psi\rangle = \int_G \left\{ a \sum_{j=1}^{n-1} D_j\varphi D_j\bar{\psi} + b D_n\varphi D_n\bar{\psi} \right\},$$

where $a \geqq 0$ and $b \geqq 0$. Define f as in the preceding example. From either of Propositions 3 or 4 (and possibly after an exponential shift to obtain an equivalent problem with A replaced by the coercive $A+\lambda^2B$) we obtain for each pair u_1, $u_2 \in V$ the existence of a generalized solution $u=u(x,t)$ of the problem

$$
\begin{aligned}
&-\Delta D_t^2 u - a\sum_{j=1}^{n-1} D_j^2 u - bD_n^2 u = F(x,t), && x\in G, \quad t\geqq 0,\\
&u(x,0)=u_1(x), \qquad D_t u(x,0)=u_2(x), && \\
&u(x,t)=0, && x\in\Gamma_0,\\
&D_\nu(D_t^2 u) + a\sum_{j=1}^{n-1} \nu_j D_j u + b\nu_n D_n u = 0, && x\in\Gamma_1.
\end{aligned}
$$

Such problems arise in models of "fat bodies" of homogeneous incompressible fluid in rotation. These include the internal waves in which the term with $b>0$ results from the rotation while that with $a>0$ is contributed by a vertical temperature gradient [10, § 6]. Similarly, certain models of wave motion in a rotating stratified fluid [17] lead to the equation

$$(D_t + D_1)^2\,\Delta u + dD_1^2 u = 0.$$

An elementary change of variable reduces this to the form above.

Various models of diffusion processes lead to problems similar to Examples 3 and 4 but with D_t^2 replaced by D_t and without the initial condition on $D_t u(x,0)$. These are resolved as Problem 2 with $M=B$ and $L=A$ in the respective examples [14].

Many other similar problems arising from models of waves in fluids or solids could be added. If one considers *transverse vibrations* (instead of longitudinal vibrations) of rods, then we obtain problems like Examples 1 and 3 but with Δu replaced by $\Delta^2 u$. Consideration of *shear* forces could add a term $\Delta^2 u$ to Example 3. Finally, we mention the models of coupled heat-sound systems and plate vibrations which lead to *systems* in the form of (1.1) in which the operators are 2×2 matrix-operators. Our results apply to these as well.

4. Approximation by projection. In order to describe the approximation methods we shall discuss, we denote as indicated the following forms:

$$
\begin{aligned}
a(x,y) &= \langle Ax, y\rangle, \qquad c(x,y)=\langle Cx,y\rangle, && x,y\in V,\\
b(x,y) &= \langle Bx, y\rangle, && x,y\in W,\\
m(x,y) &= \langle Mx, y\rangle, && x,y\in V_m = V\times W,\\
l(x,y) &= \langle Lx, y\rangle, && x,y\in V_l = V\times V.
\end{aligned}
$$

These forms permit a weak formulation of Problem 2.

LEMMA 2. *If $w(\cdot)$ is a solution of Problem 2, then*

$$m(w'(t), v) + l(w(t), v) = \langle (0, f(t)), v\rangle, \qquad v\in V_l, \quad t>0. \tag{4.1}$$

Let S be a closed subspace of V. We shall consider an approximation of $w(\cdot)$ by a function $W:[0,\infty)\to S\times S$ which satisfies

$$m(W'(t), v) + l(W(t), v) = \langle (0, f(t)), v\rangle, \qquad v\in S\times S, \quad t>0, \tag{4.2}$$

and for which $W(0)$ is specified below. We note that if $U:[0,\infty)\to S$ is the corresponding approximation of a solution u of Problem 1, i.e.,

$$b(U''(t), v)+c(U'(t), v)+a(U(t), v)=\langle f(t), v\rangle, \qquad v\in S, \quad t>0, \tag{4.3}$$

then the pair $W=(U, U')$ satisfies (4.2). If $\ker(A)=\{0\}$, then (4.2) and (4.3) are equivalent. When S has finite dimension, (4.3) is the expansion method of S (Faedo [5]).

We obtain error estimates in the energy norm $\|x\|_m\equiv m(x,x)^{1/2}$ by comparing each of $w(\cdot)$ and $W(\cdot)$ with the pointwise *L-projection* $W_l(t)$ of $w(t)$ onto $S\times S$: for each $t>0$, $W_l(t)\in S\times S$ is defined by

$$l(W_l(t), v)=l(w(t), v), \qquad v\in S\times S. \tag{4.4}$$

From (4.1), (4.2) and (4.4) we obtain for each $v\in S\times S$,

$$m(w'(t)-W_l'(t), v)=m(W'(t)-W_l'(t), v)+l(W(t)-W_l(t), v).$$

Setting $v=W(t)-W_l(t)$ and using the monotonicity of L give

$$D_t\|W(t)-W_l(t)\|_m^2\leqq 2\|w'(t)-W_l'(t)\|_m\|W(t)-W_l(t)\|_m.$$

Since the function $t\mapsto\|W(t)-W_l(t)\|_m$ is absolutely continuous, hence, differentiable almost everywhere with

$$D_t\|W(t)-W_l(t)\|_m^2=2\|W(t)-W_l(t)\|_m D_t\|W(t)-W_l(t)\|_m,$$

we obtain the estimate

$$D_t(\|W(t)-W_l(t)\|_m)\leqq\|w'(t)-W_l'(t)\|_m \tag{4.5}$$

off of the set of $t>0$ for which $\|W(t)-W_l(t)\|_m=0$. But (4.5) trivially holds at an accumulation point of this set, and there are at most a countable number of isolated points of this set, so (4.5) holds almost everywhere on $(0,\infty)$. Integrating (4.5) yields the following lemma.

LEMMA 3. *Let A and B be symmetric and monotone and let C be monotone. If u is such that $w\in C([0,\infty), V_l)$ (cf. Proposition 3) and if $w', W_l'\in L^1((0,\varepsilon), V_m)$ for some $\varepsilon>0$, then*

$$\|W(t)-W_l(t)\|_m\leqq\|W(0)-W_l(0)\|_m+\int_0^t\|w'-W_l'\|_m, \qquad t\geqq 0.$$

If the initial value $W(0)\in S\times S$ is chosen by M-projection, i.e.,

$$m(W(0), v)=m(w(0), v), \qquad v\in S\times S,$$

then $\|W(0)-w(0)\|_m\leqq\|W_l(0)-w(0)\|_m$, so the triangle inequality yields

$$\|W(0)-W_l(0)\|_m\leqq 2\|w(0)-W_l(0)\|_m.$$

If $W(0)$ is chosen by L-projection (4.4), then $W(0)=W_l(0)$ and the preceding estimate holds trivially. Either way we obtain the following proposition.

PROPOSITION 5. *In the situation of Lemma* 3, *if* $W(0)$ *is chosen by* M-*projection or by* L-*projection, then*

$$\|w(t)-W(t)\|_m \leqq \|w(t)-W_l(t)\|_m + 2\|w(0)-W_l(0)\|_m + \int_0^t \|w'-W_l'\|_m, \quad t \geqq 0. \tag{4.6}$$

Thus, the error in approximating (4.1) by (4.2) is determined by the error in the corresponding stationary Galerkin approximation (4.4).

Hereafter we restrict our attention to the case of $C=\varepsilon A$, $\varepsilon \geqq 0$, for then (4.4) factors into a pair of A-projections of V onto S. That is, denoting the error by $e(t)=w(t)-W_l(t)\in V_l$, we see that (4.4) is equivalent to ($j=1,2$)

$$a(e_j(t), v)=0, \qquad v\in S, \quad t\geqq 0, \tag{4.7}$$

so $U_l(t)(U_l'(t))$ is the A-projection of $u(t)$ (respectively, $u'(t)$) onto S, where $W_l(t)=(U_l(t), U_l'(t))$, $t\geqq 0$. This gives

$$\|u(t)-U_l(t)\|_a \leqq \inf\{\|u(t)-v\|_a : v\in S\},$$

and similar estimates hold for the various derivatives of the error.

We shall combine the preceding remarks with approximation-theoretic results. Denote by $H^k(G)$ the space of functions φ which with all derivatives $D^\alpha\varphi$ of order $|\alpha|$ at most k belong to $L^2(G)$. Such a space is complete with the norm

$$\|\varphi\|_{H^k}^2 = \sum\{\|D^\alpha\varphi\|_{L^2(G)}^2 : |\alpha|\leqq k\}.$$

For appropriate functions v from an interval $[0, T]$ into a normed space N with norm $\|\cdot\|_N$, we recall the norms

$$\|v\|_{L^p(N)} = \left(\int_0^T \|v(t)\|_N^p\,dt\right)^{1/p}, \qquad 1\leqq p<\infty,$$

$$\|v\|_{L^\infty(N)} = \operatorname{ess\,sup}\{\|v(t)\|_N : 0\leqq t\leqq T\}.$$

Our approximation result is based on an approximation assumption that is typical of multivariate spline and finite element spaces [16].

PROPOSITION 6. *Let V be a closed subspace of $H^1(G)$ (as in § 3), and $\{S_h : 0<h<1\}$ a collection of finite-dimensional subspaces of V which satisfy the following approximation assumption: There are a constant M and an integer $k\geqq 1$ such that*

$$\inf\{\|\varphi-\psi\|_{H^1} : \psi\in S_h\} \leqq Mh^{k-1}\|\varphi\|_{H^k}, \qquad \varphi\in V\cap H^k(G), \quad 0<h<1. \tag{4.8}$$

Let A and B be symmetric and monotone, A be V-coercive, and set

$$K_a \equiv \sup\{a(\varphi,\varphi)^{1/2} : \|\varphi\|_{H^1}\leqq 1\},$$

$$K_b \equiv \sup\{b(\varphi,\varphi)^{1/2} : \|\varphi\|_{H^1}\leqq 1\}.$$

Let $u\in C^1([0,T], V)$ be a solution of Problem 1 *with $C=\varepsilon A$ for some $\varepsilon\geqq 0$ and assume that*

$$u, u'\in L^\infty([0,T], H^k(G)), \qquad u''\in L^1([0,T], H^k(G)).$$

Then the approximate solution U defined by (4.3) *with $S=S_h$ and initial data chosen by M-projection* (*or L-projection*) *satisfies the estimate*

$$(\|u(t)-U(t)\|_a^2+\|u'(t)-U'(t)\|_b^2)^{1/2}\leqq Ch^{k-1}, \qquad 0\leqq t\leqq T, \tag{4.9}$$

where $C=M\{3K_a\|u\|_{L^\infty(H^k)}+(3K_b+TK_a)\|u'\|_{L^\infty(H^k)}+K_b\|u''\|_{L^1(H^k)}\}$.

Additional Remarks. The coercivity of A implies that (4.9) bounds the H^1-norm of $u(t)-U(t)$. Similar remarks apply to $u'(t)-U'(t)$ when B is coercive.

The preceding proofs give estimates for problems of first order in time in the form of Problem 2.

Proposition 6 applies directly to Examples 2 and 3 of § 3. After an elementary change of variable, Example 1 with $r=0$ is included. In the following section we indicate how Example 4 can be perturbed into a "nearby" problem to which Proposition 6 applies.

Since B is not required to be coercive, Proposition 6 gives error estimates for problems like the following:

$$\begin{aligned}
&-\Delta u(x,t)=F(x,t), && x\in G,\quad t\geqq 0,\\
&D_t^2u(x,t)+D_\nu u(x,t)=0, && x\in\Gamma_1,\\
&u(x,0)=u_1(x),\qquad D_tu(x,0)=u_2(x), &&\\
&u(x,t)=0, && x\in\Gamma_0,\quad t\geqq 0.
\end{aligned}$$

Such problems arise as linear approximations of gravity waves [11], [15].

The preceding techniques lead directly to energy estimates of error in the approximation of equations with higher order elliptic coefficients. Such examples were mentioned at the end of § 3. For related results, see [1], [2], [3], [16], [18].

5. Perturbations. Three methods will be given for perturbing (1.1) into "nearby" equations with desirable properties. We shall assume that A, B and C are all monotone and that A and B are symmetric. None are necessarily coercive, so the functions $\|\cdot\|_a$ and $\|\cdot\|_b$ are continuous seminorms on V and W, respectively; denote the corresponding seminorm spaces by V_a and W_b. The first two methods are appropriate for the most common situation (e.g., Example 1) in which A is strictly stronger than B and C. The first method corresponds to an introduction of artificial viscosity for strong dissipation in the model (cf., Example 2) while the second method is suggestive of an introduction of artificial inertia. The third method is a means of perturbing (1.1) into an equation to which we can apply our approximation results of § 4. It is appropriate for situations (e.g., Example 4 with $a=0$ or $b=0$) in which B is an elliptic operator and A is not coercive.

Parabolic regularization. We modify (1.1) by replacing C with $C+\varepsilon A$, $\varepsilon>0$. If u_ε is the corresponding solution of Problem 1 on $[0, T]$ and $w_\varepsilon=(u_\varepsilon, u_\varepsilon')$, then we have

$$Mw_\varepsilon'(t)+L_\varepsilon w_\varepsilon(t)=(0,f(t)), \qquad 0\leqq t\leqq T, \tag{5.1}$$

where $L_\varepsilon=L+\varepsilon\begin{pmatrix}0&0\\0&A\end{pmatrix}$. If u is a solution of Problem 1 and $w=(u, u')$ the corresponding solution of Problem 2, then

$$M(w'(t)-w'_\varepsilon(t))+L_\varepsilon(w(t)-w_\varepsilon(t))=(0,\varepsilon Au'(t)),\qquad 0\leqq t\leqq T, \tag{5.2}$$

and from (2.2) we obtain

$$\|w(t)-w_\varepsilon(t)\|_m^2+2\operatorname{Re}\int_0^t\langle(C+\varepsilon A)(u'-u'_\varepsilon),u'-u'_\varepsilon\rangle$$
$$=2\operatorname{Re}\int_0^t\langle\varepsilon Au',u'-u'_\varepsilon\rangle\leqq\varepsilon\int_0^t(\|u'\|_a^2+\|u'-u'_\varepsilon\|_a^2).$$

Since C is monotone, it follows that

$$\|w(t)-w_\varepsilon(t)\|_m^2+\varepsilon\int_0^t\|u'-u'_\varepsilon\|_a^2\leqq\varepsilon\int_0^t\|u'\|_a^2,\qquad 0\leqq t\leqq T.$$

PROPOSITION 7. *If $u'\in L^2([0,T],V_a)$, then*

$$\|u(t)-u_\varepsilon(t)\|_a^2+\|u'(t)-u'_\varepsilon(t)\|_b^2+\varepsilon\int_0^t\|u'-u'_\varepsilon\|_a^2\leqq\varepsilon\int_0^t\|u'\|_a^2.$$

In particular, $u_\varepsilon\to u$ ($u'_\varepsilon\to u'$) in $L^\infty([0,T],V_a)$ (respectively, $L^\infty([0,T],W_b)$) and u'_ε is bounded in $L^2([0,T],V_a)$.

From (5.1) and (2.3) one shows easily that $\|u_\varepsilon\|_{L^\infty(V_a)}$, $\|u'_\varepsilon\|_{L^\infty(W_b)}$ and $\sqrt{\varepsilon}\|u'_\varepsilon\|_{L^2(V_a)}$ are bounded. The existence of a solution of (1.1) can be deduced from existence for (5.1) and weak*-compactness of closed balls in L^∞ [12, Chap. 3.8].

When $C=0$ and A is coercive, Proposition 6 applies both to (1.1) and (5.1). However, the strongly dissipative parabolic equation (5.1) may be more desirable for numerical work [16, Chap. 7.3].

Sobolev regularization. In this perturbation of (1.1) we replace B with $B+\varepsilon A$, $\varepsilon>0$. Denoting by u_ε a solution of the perturbed problem and letting $w_\varepsilon=(u_\varepsilon,u'_\varepsilon)$ as before, we have

$$M_\varepsilon w'_\varepsilon(t)+Lw_\varepsilon(t)=(0,f(t)),\qquad 0\leqq t\leqq T, \tag{5.3}$$

where $M_\varepsilon=M+\varepsilon\begin{pmatrix}0&0\\0&A\end{pmatrix}$. With u and w as before, we have

$$M_\varepsilon(w'(t)-w'_\varepsilon(t))+L(w(t)-w_\varepsilon(t))=(0,\varepsilon Au''(t)),\qquad 0\leqq t\leqq T, \tag{5.4}$$

so (2.2) gives the estimate

$$\|w(t)-w_\varepsilon(t)\|_m^2+\varepsilon\|u'(t)-u'_\varepsilon(t)\|_a^2\leqq2\varepsilon\int_0^t\operatorname{Re}\langle Au'',u'-u'_\varepsilon\rangle$$
$$\leqq\varepsilon\int_0^t(\|u''\|_a^2+\|u'-u'_\varepsilon\|_a^2),\quad 0\leqq t\leqq T.$$

Setting $H(t)=\int_0^t\|u''\|_a^2+\|u'-u'_\varepsilon\|_a^2$, we have

$$H'(t)\leqq\|u''(t)\|_a^2+H(t),\qquad 0\leqq t\leqq T,$$

and hence,

$$H(t) \leqq \int_0^t e^{t-\tau} \|u''(\tau)\|_a^2 \, d\tau \leqq e^t \int_0^t \|u''\|_a^2.$$

Our original estimate now gives the following proposition.

PROPOSITION 8. *If $u'' \in L^2([0, T], V_a)$, then*

$$\|u(t) - u_\varepsilon(t)\|_a^2 + \|u'(t) - u_\varepsilon'(t)\|_b^2 + \varepsilon \|u'(t) - u_\varepsilon'(t)\|_a^2 \leqq \varepsilon \, e^t \int_0^t \|u''\|_a^2, \qquad 0 \leqq t \leqq T.$$

In particular, $u_\varepsilon \to u$ ($u_\varepsilon' \to u'$) in $L^\infty([0, T], V_a)$ (respectively, $L^\infty([0, T], W_b)$) and u_ε' is bounded in $L^\infty([0, T], V_a)$.

From (5.3) and (2.3) it follows that $\|u_\varepsilon\|_{L^\infty(V_a)}$, $\|u_\varepsilon'\|_{L^\infty(W_b)}$ and $\sqrt{\varepsilon}\|u_\varepsilon'\|_{L^\infty(V_a)}$ are bounded. We can obtain existence proofs from such a priori inequalities.

Discrete analogues of this method appear as Laplace-modified Galerkin techniques [1], [2] for equation (1.1) with $B = I$ and first order equations, (1.1) with $B = 0$. In these numerical schemes, ε is chosen as a first or second power of the time increment.

A nonsingular perturbation. For our final method we modify Problem 1 by replacing A with $A + \varepsilon B$. Letting u_ε denote the corresponding solution and $w_\varepsilon = (u_\varepsilon, u_\varepsilon')$ as before, we have

$$M_\varepsilon w_\varepsilon'(t) + L_\varepsilon w_\varepsilon(t) = (0, f(t)), \qquad 0 \leqq t \leqq T, \tag{5.5}$$

where $M_\varepsilon = M + \varepsilon \begin{pmatrix} B & 0 \\ 0 & 0 \end{pmatrix}$, $L_\varepsilon = L + \varepsilon \begin{pmatrix} 0 & -B \\ B & 0 \end{pmatrix}$. If u and w are respective solutions of Problems 1 and 2, then we have

$$M_\varepsilon(w'(t) - w_\varepsilon'(t)) + L_\varepsilon(w(t) - w_\varepsilon(t)) = (0, \varepsilon Bu(t)), \qquad 0 \leqq t \leqq T, \tag{5.6}$$

so Proposition 1 gives us the following.

PROPOSITION 9. *If u and u_ε are respective solutions of Problem 1 and the indicated perturbed problem, then*

$$\|u(t) - u_\varepsilon(t)\|_a^2 + \|u'(t) - u_\varepsilon'(t)\|_b^2 + \varepsilon \|u(t) - u_\varepsilon(t)\|_b^2 \leqq \varepsilon T e \int_0^t \|u\|_b^2, \qquad 0 \leqq t \leqq T.$$

The point of Proposition 9 is to perturb Example 4 into a form to which Proposition 6 can be applied. An attempt to do so by introducing the unknown $v(t) = e^{-\lambda t} u(t)$ leads to Problem 1 for v with A replaced by the coercive $A + \lambda^2 B$ but at the expense of introducing a term $2\lambda A u'(t)$, thus making Proposition 6 nonapplicable.

Similar techniques work for corresponding problems with a first order time derivative. Such a problem arises with the equation

$$D_t(\Delta u(x, t)) + \beta D_1 u(x, t) = 0$$

for divergence-free Rossby waves [10, § 7].

REFERENCES

[1] J. DENDY, *An analysis of some Galerkin schemes for the solution of nonlinear time-dependent problems*, to appear.

[2] J. DOUGLAS, JR. AND T. DUPONT, *Alternating direction Galerkin methods on rectangles*, Proc. Symp. Numerical Solution of Partial Differential Equation, II, B. Hubbard, ed., Academic Press, New York, 1971, pp. 133–214.

[3] T. DUPONT, *L^2-estimates for Galerkin methods for second order hyperbolic equations*, SIAM J. Numer. Anal., 10 (1973), pp. 880–889.

[4] R. EWING, *Numerical solution of Sobolev partial differential equations*, Ibid., 12 (1975), pp. 345–363.

[5] S. FAEDO, *Un nuovo metodo per l'analisi esistenziale e quantitativa dei problemi di propagazione*, Ann. Scuola Norm. Sup. Pisa, 1 (1949), pp. 1–40.

[6] W. FORD AND T. W. TING, *Uniform error estimates for difference approximations to nonlinear pseudo-parabolic partial differential equations*, SIAM J. Numer. Anal., 11 (1974), pp. 155–169.

[7] J. GOLDSTEIN, *Some remarks on infinitesimal generators of analytic semigroups*, Proc. Amer. Math. Soc., 22 (1969), pp. 91–93.

[8] J. GREENBERG, R. MACCAMY AND V. MIZEL, *On the existence, uniqueness and stability of solutions of the equation $\sigma'(u_x)u_{xx} + \lambda u_{xtx} = \rho u_{tt}$*, J. Math. Mech., 17 (1968), pp. 707–728.

[9] T. KATO, *Perturbation Theory for Linear Operators*, Springer-Verlag, Berlin–Heidelberg–New York, 1966.

[10] M. LIGHTHILL, *Dynamics of rotating fluids: a survey*, J. Fluid Mech., 26 (1966), pp. 411–436.

[11] J. LIONS, *On the numerical approximation of some equations arising in hydrodynamics*, Numerical Solutions of Field Problems in Continuum Physics, SIAM–AMS Proceedings, 2 (1969), pp. 11–23.

[12] J. LIONS AND E. MAGENES, *Probleme aux Limites non Homogenes et Applicationes*, vol. 1, Dunod, Paris, 1968.

[13] A. E. H. LOVE, *A Treatise on the Mathematical Theory of Elasticity*, Dover, New York, 1944.

[14] R. SHOWALTER, *Existence and representation theorems for a semilinear Sobolev equation in Banach space*, this Journal, 3 (1972), pp. 527–543.

[15] ———, *Nonlinear degenerate evolution equations and partial differential equations of mixed type*, this Journal, 6 (1975), pp. 25–42.

[16] G. STRANG AND G. FIX, *An Analysis of the Finite Element Method*, Prentice-Hall, Englewood Cliffs, N.J., 1973.

[17] K. TRUSTRUM, *An Oseen model of the two-dimensional flow of a stratified fluid over an obstacle*, J. Fluid Mech., 50 (1971), pp. 177–178.

[18] M. F. WHEELER, *L^∞ estimates of optimal orders for Galerkin methods for one-dimensional second order parabolic and hyperbolic equations*, SIAM J. Numer. Anal., 10 (1973), pp. 908–193.