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Dynamic plasticity models

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Abstract

The evolution of an elastic-plastic material is modeled as an initial boundary value problem consisting of the dynamic momentum equation coupled with a constitutive law for which the hysteretic dependence between stress and strain is described by a system of variational inequalities. This system is posed as an evolution equation in Hilbert space for which is proved the existence and uniqueness of three classes of solutions which are distinguished by their regularity.

1. Introduction

Classical models of elastic–plastic material lead to an initial-boundary-value problem consisting of the dynamic momentum equation

$$u_{tt} + D^*\sigma = f(x, t) \tag{1.1a}$$

coupled with a constitutive law

$$\sigma = F(\varepsilon) \tag{1.1b}$$

which contains a system of variational inequalities. Here, u is the displacement vector, σ is the tensor of internal stress, f is the volume density of body force, and ε is the strain tensor

$$\varepsilon = Du\,. \tag{1.1c}$$

The (small) strain is given by the symmetric gradient

$$(Du)_{ij} \equiv \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right),$$

and the corresponding dual operator is the divergence

$$(D^*\sigma)_i = -\sum_{j=1}^{3} \sigma_{ij,j}$$

in (1.1a). The constitutive law (1.1b) permits a variety of classical models of elastic–plastic materials with multi-yield surfaces.

We give a new formulation of this system as an evolution equation in Hilbert space for which we prove the

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existence and uniqueness of three classes of solutions which are distinguished by their regularity. *Weak* solutions are obtained in a very general situation, *strong* solutions arise in the presence of kinematic work-hardening or viscosity, and the solution is even more *regular* under a stability assumption connecting the constraint set with the divergence operator. This approach yields simpler proofs of many classical results, it displays clearly how the regularity arises from work hardening, and it leads in special cases to an even more regular solution that was anticipated from numerical experiments.

A variety of well-known results will be recovered in this setting. The existence and uniqueness of the *weak* solution for the fundamental Prandtl–Reuss model with a single yield surface was given in [6]. The fundamental idea was to express the constitutive relation (1.1b) as a variational equation or inequality

$$\sigma_t + \partial\varphi(\sigma) \ni Dv \tag{1.2}$$

which is coupled to the dynamic equation (1.1a). Here, $\varphi(\cdot)$ denotes either the indicator function $I_K(\cdot)$ of a given closed convex set K characterizing the particular plasticity model or a smooth convex function for the viscosity models, and $\partial\varphi$ is the corresponding subgradient or derivative, respectively. For a weak solution, the strain-rate is *not* in L^2, so it must be understood in a weak form by means of the dual operator, D^*. The extension to more general Prandtl–Ishlinski models with multi-yield surfaces was obtained by Visintin [18]. In these models, the total stress is given as the sum of a collection of stress components, i.e. $\sigma = \Sigma_j \sigma_j$, where the collection of these components $\vec{\sigma} \equiv \{\sigma_j\}$ satisfies a system of the form (1.2). An alternative approach is taken in the work of Krejci [13], where a large class of such general multiple component models is considered. There, the dissipation properties of the hysteresis functional are developed and exploited.

The quasi-static case, in which the dynamic equation (1.1a) is replaced by the corresponding static equation, was developed in [9,10]. There a regularizing effect due to work-hardening of the material appeared, and both weak and strong forms of solutions were obtained. Also, see Babuska and Li [14], Suquet [17], and Han and Reddy [8].

Here, we shall write the system (1.1) in the form

$$v_t + D^*\sigma = f(x,t)\,, \qquad \sigma = \sum_j \sigma_j \tag{1.3a}$$

$$\vec{\sigma}_t + \partial\varphi(\vec{\sigma}) - Dv \ni g(x,t)\,, \tag{1.3b}$$

and show that the dynamics is governed by an *m-accretive* operator in L^2-type spaces. In this framework, we obtain three classes of solutions which we call *weak*, *strong* and *regular*, respectively. The smoother strong solution with strain rate Dv in L^2 results from a boundedness assumption on a non-trivial measurable subset of the subgradients $\partial\varphi_j$ in the system (1.3b). This assumption arises from a *work hardening* component in the stress or in the presence of *viscosity*. This shows that each of these characteristics has a regularizing effect. From an additional stability condition relating the convex sets of the plasticity model to the divergence operator, D^*, we obtain the new regular solution for which each component of $\vec{\sigma}$ is smooth. Details are given here for the one-dimensional case, and it is straightforward to extend most of them to the realistic three-dimensional case. We have not been able to apply our results on the regular solutions to a three-dimensional model of plasticity.

Our plan is as follows. We recall below some topics from convex analysis and evolution equations in Hilbert space. Section 2 consists of some elementary examples of models of plasticity. These motivate the general construction to follow. We introduce in Section 3 an abstract setting for these examples and recover the above mentioned well-known theorems as *weak* solutions, and additionally we give sufficient general conditions under which these solutions are *strong*. The new *regular* solution is obtained for the one-dimensional case.

A (possibly multi-valued) *operator* or *relation* $\mathbb{C}$ in a real Hilbert space H is a collection of related pairs $[x, y] \in H \times H$ denoted by $y \in \mathbb{C}(x)$; the *domain* Dom($\mathbb{C}$) is the set of all such x and the *range* Rg($\mathbb{C}$) consists of all such y. The operator $\mathbb{C}$ is called *accretive* if for all $y_1 \in \mathbb{C}(x_1)$, $y_2 \in \mathbb{C}(x_2)$, and $\varepsilon > 0$, we have

$$\|x_1 - x_2\| \leq \|x_1 - x_2 + \varepsilon(y_1 - y_2)\|\,.$$

This is equivalent to requiring that $(I + \varepsilon\mathbb{C})^{-1}$ be a contraction on Rg$(I + \varepsilon\mathbb{C})$ for every $\varepsilon > 0$. This is also equivalent to requiring

$$(y_1 - y_2, x_1 - x_2)_H \geq 0 \quad \text{for all } [x_1, y_1], [x_2, y_2] \in \mathbb{C}\,.$$

Additionally, if $\mathrm{Rg}(I + \varepsilon\mathbb{C}) = H$ for some (equivalently, for all) $\varepsilon > 0$, then we say $\mathbb{C}$ is *m-accretive*. For such an operator, the *Cauchy problem* is known to be well-posed, and we shall realize each of our initial-boundary-value problems as such a problem in an appropriate function space.

THEOREM A. Let $\mathbb{C}$ be m-accretive in the Hilbert space H. If $T > 0$, $x_0 \in \mathrm{Dom}(\mathbb{C})$ and $f \in W^{1,1}(0, T; H)$, then there exists a unique solution $x \in W^{1,\infty}(0, T; H)$ of the Cauchy problem

$$x'(t) + \mathbb{C}(x(t)) \ni f(t)\,, \quad t > 0$$
$$x(0) = x_0$$

with $x(t) \in \mathrm{Dom}(\mathbb{C})$ for all $0 \leqslant t \leqslant T$.

We will use some techniques of convex analysis to construct the operators below. For details, see [7,2,3]. Let W be a Hilbert space, and let $\varphi : W \to (-\infty, +\infty]$ be convex, proper, and lower-semi-continuous. Then, the functional $f \in W'$, the dual space, is a *subgradient* of φ at $u \in W$ if

$$f(v - u) \leqslant \varphi(v) - \varphi(u) \quad \text{for all } v \in W\,.$$

The set of all subgradients of φ at u is denoted by $\partial\varphi(u)$. The subgradient is a generalized notion of the derivative, comparable to a directional derivative. We regard $\partial\varphi$ as a multivalued operator from W to W'; it is easily shown to be *monotone*. That is, if $f_1 \in \partial\varphi(u_1)$, $f_2 \in \partial\varphi(u_2)$, then $(f_1 - f_2)(u_1 - u_2) \geqslant 0$.

If K is a closed, convex, nonempty subset of W, then the *indicator function* $I_K(\cdot)$ of K, given by $I_K(w) = 0$ if $x \in K$ and $I_K(w) = +\infty$ otherwise, is convex, proper, and lower-semi-continuous. Its subgradient is characterized by a *variational inequality*: $f \in \partial I_K(w)$ means

$$f \in W'\,, \quad w \in K\colon \quad f(y - w) \leqslant 0 \quad \text{for all } y \in K\,.$$

As an example, we consider first the *indicator function* $I_1(\cdot)$ of the interval $[-1, 1]$. Thus, $I_1 : \mathbb{R} \to +\mathbb{R}_\infty$ is convex, proper, and lower-semi-continuous, and its subgradient is characterized as follows: $f \in \partial I_1(x)$ means

$$|x| \leqslant 1 \quad \text{and} \quad \begin{cases} f \geqslant 0\,, & \text{for } x = 1\,, \\ f = 0\,, & \text{for } -1 < x < 1\,, \\ f \leqslant 0\,, & \text{for } x = -1\,. \end{cases}$$

Thus, ∂I_1 is just the inverse of the *sign graph*,

$$\mathrm{sgn}(x) = \begin{cases} \{1\}\,, & \text{if } x > 0\,, \\ [-1, 1]\,, & \text{if } x = 0\,, \\ \{-1\}\,, & \text{if } x < 0\,. \end{cases}$$

A second example is the corresponding realization on the Hilbert space $W = L^2(0, 1)$ given by

$$\varphi_1(\sigma) = \int_0^1 I_1(\sigma(x))\,\mathrm{d}x\,, \quad \sigma \in W\,, \tag{1.4}$$

and here we have $f \in \partial\varphi_1(\sigma)$ if f, $\sigma \in W = W'$ and $f(x) \in \partial I_1(\sigma(x))$ at a.e. $x \in (0, 1)$. For a third example, let φ_1 be given by (1.4) on the Sobolev space $W = H^1(0, 1)$. Then, the inclusion $f \in \partial\varphi_1(\sigma)$ implies that σ is smoother, but it permits f to be a distribution, so the pointwise characterization above does not necessarily hold.

2. Examples

We shall describe a variety of models of plasticity in one spatial dimension for the ease of exposition. These are intended only to illustrate the theorems which will follow. The full 3-dimensional models can be developed similarly.

2.1. Elastic–perfectly plastic

The momentum and constitutive equations are, respectively,

$$v_t - \sigma_x = f, \quad \sigma_t + \mathrm{sgn}^{-1}(\sigma) \ni \varepsilon_t .$$

The phase diagram showing the relationship between stress σ and strain ε is the *hysteresis* functional shown in Fig. 1. This model results from the *series* addition of an elastic element, $\sigma_t = \varepsilon_t$, and a perfectly-plastic element, $\mathrm{sgn}^{-1}(\sigma) \ni \varepsilon_t$. By equality of mixed derivatives, $u_{xt} = u_{tx}$, the resulting dynamical system is (formally) given by

$$v_t - \sigma_x = f, \quad 0 < x < 1, \quad 0 < t, \quad v(0,t) = 0 \tag{2.1a}$$

$$\sigma_t - v_x + \mathrm{sgn}^{-1}(\sigma) \ni 0, \quad \sigma(1,t) = 0 \tag{2.1b}$$

with appropriate initial conditions on v and σ.

We shall write this as an evolution equation

$$\frac{d}{dt}[v, \sigma] + \mathbb{C}([v, \sigma]) \ni [f, 0] \tag{2.2}$$

in the appropriate product space. Define the Hilbert space

$$W = (\sigma \in H^1(0,1) : \sigma(1) = 0).$$

Let the function φ_1 be defined on this space W by (1.4). For $\varepsilon > 0$, the corresponding resolvent equation, $(I + \varepsilon\mathbb{C})[v, \sigma] \ni [f, g]$, is given by

$$v \in L^2: \quad v - \varepsilon\sigma_x = f, \quad 0 < x < 1, \quad \sigma(1) = 0,$$

$$\sigma \in W: \quad \sigma - \varepsilon v_x + \varepsilon\, \partial\varphi_1(\sigma) \ni g.$$

Note that εv_x and $\varepsilon\, \partial\varphi_1(\sigma)$ are in W', so this is a *weak* solution in our notation below, and there is no boundary value assigned to $v(0)$.

It will follow easily that $\mathbb{C}$ is m-accretive, and then Theorem A shows directly that there is a unique *weak* solution of (2.1) with

$$v, \frac{\partial v}{\partial t}, \frac{\partial \sigma}{\partial t} \in L^\infty(0, T; L^2(0,1)), \quad \sigma \in L^\infty(0, T; W).$$

This is the content of Theorem W in the next section. This *weak* solution was already obtained as Theorem 4.2 of [6] and Theorem 1 of [18]. See [1] for regularity of the solution and the interpretation of (2.1b).

2.2. Kinematic hardening

Here, we assume that the material work-hardens each time the yield stress is reached. Momentum and constitutive equations are, respectively,

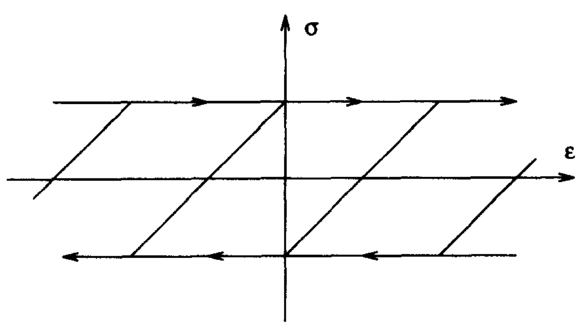

Fig. 1.

$$
\begin{aligned}
&\frac{\partial}{\partial t} v - \sigma_x = f\,, \qquad \sigma = \beta_1 \sigma_1 + \beta_2 \sigma_2\,, \\
&\frac{\partial}{\partial t} \sigma_1 + \partial\varphi_1(\sigma_1) \ni \beta_1 \frac{\partial}{\partial x} v\,, \qquad \frac{\partial}{\partial t} \sigma_2 = \beta_2 \frac{\partial}{\partial x} v\,.
\end{aligned}
\tag{2.3}
$$

This model results from the *parallel* addition of the elastic–plastic stress from Section 1 (corresponding to σ_1) with a purely elastic stress (corresponding to σ_2) which records the position of the center of the yield stress interval. The lines in Fig. 2 representing the upper and lower yield surfaces have slope β_2^2.

We shall write the system (2.3) as an evolution equation (2.2) in the appropriate product space. The corresponding operator $\mathbb{C}$ is m-accretive in the space $H \equiv L^2(0, 1)^3$ and, since $v_x \in L^2(0, 1)$, it leads to a *strong* solution. This solution agrees with that of Theorem 1.2 of Chapter III in [13] where much more general situations are obtained. To this end, as well as to motivate our notation in the next section, we introduce the following:

$$V = \{v \in H^1(0, 1) : v(0) = 0\} \qquad D = \frac{\mathrm{d}}{\mathrm{d}x} : V \to L^2(0, 1)$$

$D^* = -\dfrac{\mathrm{d}}{\mathrm{d}x} : L^2(0, 1) \to V'$ is the *continuous* dual operator

$\beta = [\beta_1 I, \beta_2 I] : L^2(0, 1) \to L^2(0, 1)^2$ where $\beta_1, \beta_2 \in \mathbb{R}$ are given,

$$\beta^*[\sigma] = \beta_1 \sigma_1 + \beta_2 \sigma_2\,, \qquad \beta^* : L^2(0, 1)^2 \to L^2(0, 1)$$

$$W_0 = \{\sigma = [\sigma_1, \sigma_2] \in L^2(0, 1)^2 : \beta^*\sigma \in H^1(0, 1), \beta^*\sigma(1) = 0\}$$

Denote by D_* the $L^2(0, 1)$-adjoint of the closed operator, D. That is,

$D_* w = f \Leftrightarrow w, f \in L^2(0, 1)$ and $(Dv, w) = (v, f)$ for all $v \in \mathrm{Dom}(D) \equiv V$.

Then, $D_* : \mathrm{Dom}(D_*) \to L^2(0, 1)$ is also closed and dense, and it can be characterized as follows.

LEMMA. $D_* w = f \in L^2(0, 1) \Leftrightarrow w \in L^2(0, 1)$, $f = -\mathrm{d}w/\mathrm{d}x$ *and* $w(\cdot)v(\cdot)|_0^1 = 0$ *for all* $v \in \mathrm{dom}(D)$.

This shows how the boundary conditions imposed on D determine those associated with D_*. Then, we set $W = \mathrm{Dom}(D_*)$ so that $D_* : W \to L^2(0, 1)$. Note that for any solution of (2.3), either weak or strong, we have $\beta^*\sigma \in W$ and D_* can be replaced by D_* in the momentum equation. In particular, $\beta^*\sigma$ satisfies the appropriate boundary condition.

The resolvent equation $(I + \mathbb{C})[v, \sigma] \ni [f, g]$ corresponding to (2.3) is equivalent to solving the system

$$
\begin{aligned}
&v \in V\colon \quad v + D^*\beta^*\sigma = f\,, \\
&\sigma \in W_0\colon \quad \sigma - \beta D v + [\partial\varphi_1(\sigma_1), 0] = g \in L^2(0, 1)^2\,.
\end{aligned}
$$

This is equivalent to solving for v the equation

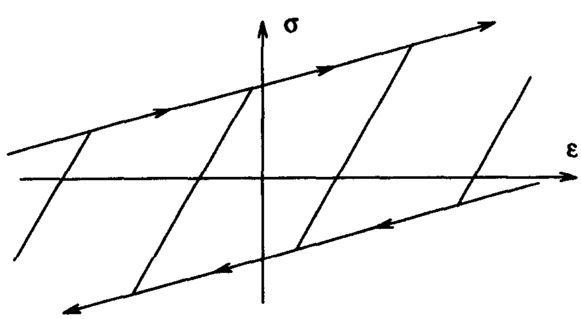

Fig. 2.

$$v \in V: \quad v + D^*(\beta_1(I + \partial\varphi_1)^{-1}(\beta_1 Dv + g_1) + \beta_2^2 Dv + \beta_2 g_2) = f \quad \text{in } V' .$$

Since $\beta_2^2 > 0$, the form is coercive, and existence of a solution follows. The components of $[\sigma_1, \sigma_2] \in W_0$ are obtained directly from the second and third terms in this equation, respectively, and then we check that $\sigma \in W_0$. In particular, the boundary condition at $x = 1$ is satisfied. These remarks show that Theorem A applies directly to give existence and uniqueness of a *strong* solution of (2.3) with

$$v \in L^\infty(0, T; V), \quad \sigma \in L^\infty(0, T; W_0), \quad \frac{\partial v}{\partial t}, \frac{\partial \sigma}{\partial t} \in L^\infty(0, T; L^2(0, 1)) .$$

This is the content of Theorem S in Section 3.

REMARK 1. Since v belongs to V instead of merely to $L^2(I)$, the solution here is *smoother* than that of Section 1. This is made possible here by the coercivity resulting from the β_2 term.

REMARK 2. We can include a viscous element in parallel to the above by adding a third equation of the form

$$\frac{1}{k}\frac{\partial}{\partial t}\sigma_3 + \frac{1}{\mu}\sigma_3 = \beta_3 \frac{\partial}{\partial x} v .$$

More generally, we can include *visco-elastic* elements in the form

$$\frac{1}{k}\frac{\partial}{\partial t}\sigma_3 + J'(\sigma_3) = c\frac{\partial}{\partial x} v .$$

where J has a *bounded* derivative. This represents a series combination of elastic element and a purely viscous element, and one obtains *strong* solutions as above. See Theorem 3.1 of [6] for the case of a single stress component.

Finally, we give a simple but important extension of the preceding example to a plasticity model built on four stress components. This will motivate the consideration of generalized sums or *integrals* of a collection or even a *continuum* of such components. The system is given by

$$\begin{aligned}
&\frac{\partial}{\partial t} v - \sigma_x = f , \\
&\sigma = \sigma_1 + \frac{1}{2}\sigma_2 + \frac{1}{4}\sigma_3 + \frac{1}{4}\sigma_4 , \\
&\frac{\partial}{\partial t}\sigma_1 + \partial\varphi_1(\sigma_1) \ni \frac{\partial}{\partial x} v , \\
&\frac{\partial}{\partial t}\sigma_2 + \partial\varphi_2(\sigma_2) \ni \frac{1}{2}\frac{\partial}{\partial x} v , \\
&\frac{\partial}{\partial t}\sigma_3 + \partial\varphi_3(\sigma_3) \ni \frac{1}{4}\frac{\partial}{\partial x} v , \\
&\frac{\partial}{\partial t}\sigma_4 = \frac{1}{4}\frac{\partial}{\partial x} v .
\end{aligned} \tag{2.4}$$

For each $j = 1, 2, 3$, φ_j is the indicator function of the interval $[-j, j]$, so the corresponding stress component σ_j is constrained to lie within that interval. The relation between total stress σ and strain ε is indicated by Fig. 3. (Recall that $\partial\varepsilon/\partial t = \partial v/\partial x$ is the *strain rate*.) Here, we begin with all components at 0. We increase the strain, ε, from 0 to 5, decrease it to -5, then increase it to 2, and we follow the resulting stress, σ.

The limiting positive slope $1/4$ is the *work-hardening* component, and it is this component of the stress that leads to a *strong* solution of (2.4) as before. Since the boundary lines in this hysteresis functional are straight lines, such models are called *multilinear*. By using a collection of such components, one can approximate a large class of convex bounding curves; with a continuum of such components, the corresponding class of convex functions can be matched. Most models of plasticity involve such multiple yield surfaces, and these provide an approximation of the observed smooth transitions between elastic and plastic regimes. Such smooth transitions are best modeled by a continuum of elastic–plastic elements with varying yield surfaces.

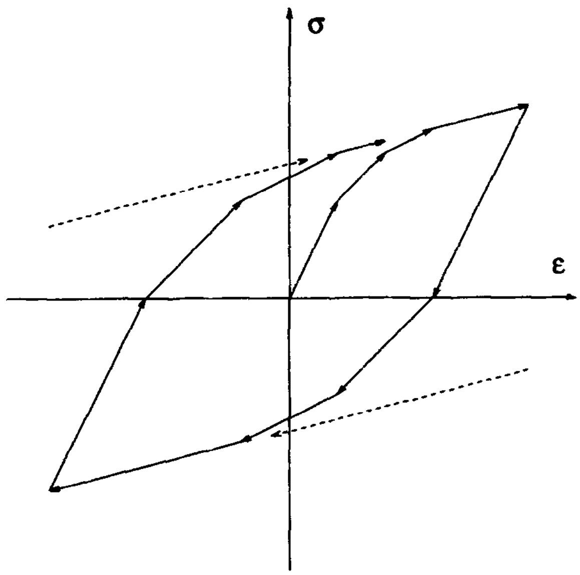

Fig. 3.

3. A general plasticity model

Let $D : \mathrm{Dom}(D) \to L^2(0, 1)$ be a closed operator with dense domain $\mathrm{Dom}(D)$ in $L^2(0, 1)$. Let D_* be the $L^2(0, 1)$-adjoint of this closed operator. That is,

$$D_* w = f \Leftrightarrow w, f \in L^2(0, 1) \quad \text{and} \quad (Dv, w) = (v, f) \quad \text{for all } v \in \mathrm{Dom}(D)\,.$$

Therefore, $D_* : \mathrm{Dom}(D_*) \to L^2(0, 1)$ is also closed and dense. We set $W \equiv \mathrm{Dom}(D_*)$ and give it the graph norm. Then, $D_* : W \to L^2(0, 1)$, is a *bounded* operator between Banach spaces. The *continuous* dual operator will be denoted by $D_*^* = (D_*)^* : L^2(0, 1) \to \mathrm{Dom}(D_*)' = W'$. Note that

$$\begin{aligned} D_*^* v(w) &= (v, D_* w)_{L^2} \quad \text{for all } w \in \mathrm{Dom}(D_*)\,, \quad v \in L^2\,, \\ &= (Dv, w)_{L^2} \quad \text{for all } w \in \mathrm{Dom}(D_*)\,, \quad v \in \mathrm{dom}(D)\,, \end{aligned}$$

so we have $D_*^* \supset D$ in the sense of graphs. Similarly, we put the graph norm on $\mathrm{Dom}(D)$ and denote the resulting space by V. We define the continuous dual $D^* : L^2(0, 1) \to V'$ and note that $D^* \supset D_*$.

Let S, μ be a measure space. Define $\beta : L^2(0, 1) \to L^2(S, d\mu; L^2(0, 1)) = L^2(S \times (0, 1))$ by $(\beta g)(s, x) \equiv \beta(x, s) g(x)$ where $\beta(\cdot, \cdot) \in L^\infty((0, 1),\ L^2(S))$. Then the continuous dual is an operator β^*: $L^2(S, \mathrm{d}\mu; L^2(0, 1)) \to L^2(0, 1)$, and we have

$$\beta^* \rho(x) = \int_S \beta(x, s) \rho(s, x)\, \mathrm{d}\mu_s\,, \quad \text{a.e. } x \in (0, 1)\,, \quad \rho \in L^2(S, \mathrm{d}\mu; L^2(0, 1))\,.$$

Define $W_S \equiv \{\sigma \in L^2(S, \mathrm{d}\mu; L^2(0, 1)) : \beta^* \sigma \in \mathrm{Dom}(D_*)\}$. Let $\beta_* : W_S \to W$ be the indicated restriction, which is bounded on W_S with the graph norm, and denote its *continuous* dual by $\beta_*^* : W' \to W'_S$. We shall denote the space $L^2(S, \mathrm{d}\mu; L^2(0, 1))$ by $L^2(S \times (0, 1))$. The various operators are summarized in the following diagram.

$$\begin{array}{ccccc} L^2(I) & \xrightarrow{D_*^*} & W' & \xrightarrow{\beta_*^*} & W'_S \\ \cup & & \cup & & \cup \\ V & \xrightarrow{D} & L^2(I) & \xrightarrow{\beta} & L^2(S \times I) \end{array} \qquad \begin{array}{ccccc} L^2(S \times I) & \xrightarrow{\beta^*} & L^2(I) & \xrightarrow{D^*} & V' \\ \cup & & \cup & & \cup \\ W_S & \xrightarrow{\beta_*} & W & \xrightarrow{D_*} & L^2(I) \end{array}$$

Let $\varphi : W_S \to \mathbb{R}_\infty$ be proper, convex and lower-semicontinuous, and denote its subgradient by $\partial\varphi : W_S \to W'_S$.

DEFINITION. The *weak Cauchy Problem* is to find $v(t)$, $\sigma(t)$ for $0<t<T$ such that

$$v,\ \frac{\partial v}{\partial t} \in L^{\infty}(0,T;L^2(0,1))\,, \quad \sigma \in L^{\infty}(0,T;W_S)\,, \quad \frac{\partial \sigma}{\partial t} \in L^{\infty}(0,T;L^2(S\times(0,1)))\,,$$

and they satisfy the system

$$\frac{\mathrm{d}v(t)}{\mathrm{d}t} + D_*\beta_*\sigma(t) = f(t) \quad \text{in } L^2(0,1)\,, \tag{3.1a}$$

$$\frac{\mathrm{d}\sigma(t)}{\mathrm{d}t} + \partial\varphi(\sigma(t)) - \beta_*^{\ddagger}D_*^{\ddagger}v(t) \ni g(t) \quad \text{in } W_S'\,, \tag{3.1b}$$

$$v(0) = v_0 \quad \text{in } L^2(0,1)\,, \qquad \sigma(0) = \sigma_0 \quad \text{in } L^2(S,\mathrm{d}\mu;L^2(0,1))\,, \tag{3.1c}$$

where the four functions $v_0 \in L^2(0,1)$, $\sigma_0 \in W_S$, $f \in L^{\infty}(0,T;L^2(0,1))$ and $g \in L^{\infty}(0,T;L^2(S,\mathrm{d}\mu;L^2(0,1)))$ are given.

Note that the variational form of (3.1b) is

$$\begin{aligned}\sigma(t)\in W_S:\quad & -\left(\frac{\mathrm{d}\sigma(t)}{\mathrm{d}t},\ \rho-\sigma(t)\right)_{L^2(S\times(0,1))} + (v(t), D_*\beta_*(\rho-\sigma(t)))_{L^2(0,1)} \\ & + (g(t),\rho-\sigma(t))_{L^2(S\times(0,1))} \leqslant \varphi(\rho)-\varphi(\sigma(t)) \quad \text{for all } \rho\in W_S\,.\end{aligned}$$

THEOREM W. Assume that the linear operator $D: V \to L^2(0,1)$, the function $\beta(\cdot,\cdot) \in L^{\infty}((0,1),L^2(S))$, and the convex functional $\varphi: W_S \to \mathbb{R}_{\infty}$ are given as above, and define the corresponding operators

$$D_*: W \to L^2(0,1),\ D_*^{\ddagger}: L^2(0,1) \to W'\,,$$

$$\beta: L^2(0,1) \to L^2(S,\mathrm{d}\mu;L^2(0,1)),\ \beta_*: W_S \to W\,.$$

Let $v_0 \in L^2(0,1)$ and $\sigma_0 \in W_S$ be given with $(\partial\varphi(\sigma_0) - \beta_*^{\ddagger}D_*^{\ddagger}v_0) \cap L^2(S,\mathrm{d}\mu;L^2(0,1))$ non-empty. Let $f \in W^{1,1}(0,T;L^2(0,1))$ and $g \in W^{1,1}(0,T;L^2(S,\mathrm{d}\mu;L^2(0,1)))$ be given. Then there is a unique weak solution of (3.1) with $v(0)=v_0$, $\sigma(0)=\sigma_0$.

Under additional assumptions we can obtain $Dv \in L^2(0,1)$ and thus $v \in V$. Then, the pair v, σ is a *strong* solution of the resolvent equation corresponding to the *strong Cauchy Problem*. By this we mean the weak Cauchy Problem (3.1) in which we additionally require that $v \in L^{\infty}(0,T;V)$. Hence, one can then replace $D_*^{\ddagger}$ with D and $\beta_*^{\ddagger}$ with β. This takes the form of a system

$$\frac{\partial}{\partial t}v(x,t) + D_*\int_S \beta(x,s)\sigma(s,x,t)\,\mathrm{d}\mu_s = f(x,t)\,, \tag{3.2a}$$

$$\frac{\partial}{\partial t}\sigma(s,x,t) + \partial\varphi(\sigma(s,x,t)) - \beta(x,s)Dv(x,t) \ni g(x,s,t)\,, \quad \text{a.e. } s\in S \tag{3.2b}$$

for a.e. $x \in (0,1)$, $t>0$,

$$v(0)=v_0 \quad \text{in} \quad L^2(0,1)\,, \qquad \sigma(0)=\sigma_0 \quad \text{in } L^2(S,\mathrm{d}\mu;L^2(0,1))\,. \tag{3.2c}$$

Assume that φ is given in the form

$$\varphi(\sigma) = \int_S\int_0^1 \varphi_s(\sigma(s,x))\,\mathrm{d}x\,\mathrm{d}\mu_s\,, \quad \sigma \in L^2(S\times(0,1))\,, \tag{3.3}$$

with a *normal integrand* [15] for which each $\varphi_s: L^2(0,1) \to \mathbb{R}_{\infty}$ is convex, lower-semicontinuous, and takes its minimum at $\varphi_s(\mathbf{0})=0$. We shall require that *some* of the $\partial\varphi_s$'s be *regular*, i.e. that they are linearly bounded. In order to quantify this condition, we set $\alpha_s \equiv (I+\partial\varphi_s)^{-1}$. Note that each α_s is (uniformly) Lipschitz and that we have

$$\alpha_s(\xi)(\xi) \geqslant 0\,, \quad s\in S\,, \quad \xi \in \mathbb{R}\,.$$

We shall assume additionally that there is an $\varepsilon>0$ and a measurable set $S_0 \subset S$ such that

$$\alpha_s(\xi)(\xi) \geqslant \varepsilon|\xi|^2\,, \quad s \in S_0\,, \quad \xi \in \mathbb{R}\,,$$

and (3.4)

$$\int_{S_0} (\beta(x,s))^2 \,\mathrm{d}\mu_s \geqslant \varepsilon\,.$$

THEOREM S. In the situation of Theorem W, assume in addition that the function φ is given on $L^2(S \times (0,1))$ by the formula (3.3) and the normal family φ_s of convex and lower-semicontinuous non-negative functionals for which $\varphi_s(\mathbf{0}) = 0$, $s \in S$, and the estimates (3.4) hold. Also, let $v_0 \in V$. Then the weak solution is a strong solution, i.e. $v \in L^\infty(0, T; V)$ and the strong Cauchy Problem has a unique solution.

The momentum equation (3.1a) requires only that the generalized sum, $\beta_* \sigma(t)$, belong to W at each $t > 0$. We show that when $\beta(\cdot,\cdot)$ is independent of x one may obtain a solution for which *each component*, $\sigma(s,t)$, belongs to W at each $t > 0$. Define the *distributed operator* $\mathbb{D} : L^2(S; V) \to L^2(S \times I)$ by

$$\mathbb{D}(v)(s) = Dv(s)\,, \quad s \in S\,, \quad v \in L^2(S; V)\,,$$

and denote its L^2-adjoint by $\mathbb{D}_* : L^2(S; W) \to L^2(S \times I)$. The corresponding continuous duals are $\mathbb{D}^*$ and $\mathbb{D}_*^*$ as before.

$$\begin{array}{ccccccccc}
L^2(I) & \xrightarrow{D_*^*} & W' & \xrightarrow{\beta_*^*} & W'_S & \qquad & L^2(S \times I) & \xrightarrow{\mathbb{D}_*^*} & L^2(S; W') \\
\cup & & \cup & & \cup & & \cup & & \cup \\
V & \xrightarrow{D} & L^2(I) & \xrightarrow{\beta} & L^2(S \times I) & & L^2(S; V) & \xrightarrow{\mathbb{D}} & L^2(S \times I)
\end{array}$$

Assume that the function $\beta(\cdot,\cdot)$ is independent of x, so we have $\beta(\cdot) \in L^2(S)$. We summarize the resulting structure as follows:

$$\begin{array}{ccccccccc}
L^2(S \times I) & \xrightarrow{\beta^*} & L^2(I) & \xrightarrow{D^*} & V' & \qquad & L^2(S \times I) & \xrightarrow{\mathbb{D}^*} & L^2(S; V') \\
\cup & & \cup & & \cup & & \cup & & \cup \\
W_S & \xrightarrow{\beta_*} & W & \xrightarrow{D_*} & L^2(I) & & L^2(S; W) & \xrightarrow{\mathbb{D}_*} & L^2(S \times I) \\
\cup & & & & \uparrow \beta^* & & & & \\
L^2(S; W) & & \xrightarrow{\mathbb{D}_*} & & L^2(S \times I) & & & &
\end{array}$$

Moreover, β^* commutes with both $\mathbb{D}_*$ and $\mathbb{D}$.

A *regular* solution of the resolvent equation is a strong solution (with $v \in V$) for which, in addition, $\sigma \in L^2(S; W)$. For this, we assume that

$$(A_1\sigma, A_2\sigma) \geqslant 0,\ \sigma \in \mathrm{Dom}(A_1)\,, \tag{3.5a}$$

$$(A_1\sigma, \partial\varphi_\varepsilon(\sigma)) > 0,\ \sigma \in \mathrm{Dom}(A_1)\,, \tag{3.5b}$$

$$A_2 + \partial\varphi \text{ is } m\text{-accretive}\,. \tag{3.5c}$$

The effort is to show that the resolvent of the operator $\mathbb{C}$ is *stable* under the norm of $V \times L^2(S; W)$. That is, the lower semicontinuous norm

$$\Phi([v, \sigma]) \equiv (\|Dv\|^2 + \|\mathbb{D}_*\sigma\|^2)^{1/2}$$

is a *Liapunov functional* for the Cauchy problem (3.1). When additionally $f(t) = 0$ and $g(t) = 0$, each closed ball in $H = L^2(0,1) \times L^2(S \times (0,1))$ of the form

$$B_R \equiv \{[v, \sigma] \in V \times L^2(S; W): \quad \Phi([v, \sigma]) \leqslant R\}$$

is *invariant* under the evolution equation in (3.1). For the nonhomogeneous case we show that each solution remains in such a ball, B_R, and thereby is a regular solution.

THEOREM R. In the situation of Theorem S, assume in addition that the function $\beta(\cdot, \cdot)$ *is independent of* x, *that is,* $\beta(\cdot) \in L^2(S)$, *and assume* $[f(\cdot), g(\cdot)] \in L^1(0, T; V \times L^2(S; W))$ *and* $[v_0, \sigma_0] \in V \times L^2(S; W)$. *Then the strong solution* $[v(t), \sigma(t)]$ *from Theorem S satisfies*

$$(\|Dv(t)\|^2 + \|\mathbb{D}_*\sigma(t)\|^2)^{1/2} \le (\|Dv_0\|^2 + \|\mathbb{D}_*\sigma_0\|^2)^{1/2} + \int_0^t (\|Df(s)\|^2 + \|\mathbb{D}_*g(s)\|^2)^{1/2}\, ds\,, \quad 0 \le t \le T\,, \tag{3.6}$$

hence, $\sigma \in L^\infty(0, T; L^2(S; W))$.

For the plasticity problems, the estimate (3.6) is a substantial regularity result for solutions. In particular, whereas a strong solution is one for which the *average* stress $\beta_*\sigma$ is regular in the sense that $\beta_*\sigma \in W$, that is, it is differentiable, the regular solution is one for which *each component* of the stress is differentiable, i.e. $\sigma(s, \cdot) \in W$ for a.e. $s \in S$. The proof given above for Theorem R depends on the assumptions (3.5). We note that (3.5a) follows from the lack of dependence of $\beta(s)$ on x, and (3.5c) also follows rather generally. However, although the verification of (3.5b) appears to be easy in one dimension, it is difficult to find examples in $\mathbb{R}^n$ which satisfy this condition.

If we have $\partial\varphi_s = 0$ or, more generally, $\partial\varphi_s : W \to W$ is bounded uniformly in s, for $s \in S_0$, then from the restriction to S_0 of the identity $\sigma + \partial\varphi(\sigma) = \beta Dv + g$ we obtain a regularity result for the velocity in the stationary resolvent equation. That is, we get $Dv \in W$ and consequently $D_*Dv \in L^2(I)$. This occurs, for example, when $\partial\varphi_s$ arises from kinematic hardening or from a viscosity regularization, respectively. A corresponding regularity result for the displacement of a regular solution of the Cauchy problem is the following.

COROLLARY. Assume additionally that $\partial\varphi_s = 0$ *for* $s \in S_0$ *and that* $u_0 \in V$ *with* $Du_0 \in W$. *Let* $[v(t), \sigma(t)]$ *be the regular solution from Theorem R, and denote the displacement by* $u(t) = u_o + \int_0^t v(\tau)\, d\tau$. *Then,* $u \in W^{1,\infty}(0, T; V)$ *and* $Du \in L^\infty(0, T; W)$, *i.e.* $D_*Du \in L^\infty(0, T; L^2(I))$.

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