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DIFFUSION WITH PRESCRIBED CONVECTION IN FISSURED MEDIA

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Abstract. A system consisting of an ordinary differential equation coupled to a non-symmetric degenerate elliptic equation provides a model for heat transfer with phase change in a fissured medium, a collection of material blocks separated by a system of fissures, when a fluid of prescribed velocity transports heat within the fissures. Additional transport in the fissures occurs by diffusion, and phase changes can occur independently in the two components along respective free-boundaries. With the physically appropriate initial and boundary conditions, the system is shown to be a well-posed problem; special regularity properties of the solution will be established.

1. Introduction. We shall formulate the problem as a model for the melting of paraffin sediments in oil-saturated media by means of hot liquid injection. Thus, heated fluid is flowing through a fissured medium at a prescribed velocity in the fissure system; it has been injected in order to remove paraffin sediments [5]. Some volume fraction, $0 < a < 1$, is fixed in the blocks while another fraction $0 < b < 1$, is carried along by the fluid in the fissures. The temperature varies both in the blocks and in the fissures within a range which includes the melting temperature of paraffin; we normalize this to zero. The temperature u_1 in the blocks is given as a function of the enthalpy or heat energy content w_1 in the blocks, $u_1 = \alpha(w_1)$. A typical model is $\alpha(w) = \alpha_1 w^- + \alpha_2(w-a)^+$ where $x^- = \min(x,0)$ and $x^+ = \max(x,0)$ denote negative and positive parts, respectively, α_1 and α_2 are reciprocals of specific heat above and below the melting temperature, and "a" is a measure of the latent heat released with this phase change of the paraffin in the blocks [7]. Similarly, the temperature u_2 in the fissures is given as a function, $u_2 = \beta(w_2)$, of the enthalpy in the fissures. An example is $\beta(w) = b_1 w_1^- + b_2(w-b)^+$ where b_1, $b_2 > 0$ are reciprocals of specific heat and "b" is a measure of latent heat of the fluid-paraffin mixture.

Let $\vec{v}$ be the prescribed velocity field of the water in the fissures and $k(u_2)$ denote the conductivity of the fluid-paraffin mixture in the fissures. Typically, $k(u) > 0$ is essentially

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constant on each of $u < 0$ and $u > 0$. Thus, the *heat flux* in the fissures is given by Fourier's law as

$$\vec{v}w_2 - k(u_2)\vec{\nabla}u_2,$$

where $\vec{\nabla}$ is the gradient operator. The rate of heat transfer between the blocks and the fissure system per unit surface area of contact is proportional to the temperature difference times the average conductivity of the fluid, averaged over that temperature range, between blocks and fissures; *i.e.*,

$$\tilde{k}(u)(u_2 - u_1) = \int_{u_1}^{u_2} k(s)\, ds\ .$$

The rate of heat transfer per unit volume of the medium is then of the form

$$\frac{1}{\varepsilon}(K(u_2) - (K(u_1)),$$

where $K'(s) = k(s)$, $K(0) = 0$, and the coefficient $\frac{1}{\varepsilon}$ is proportional to the specific surface of the fissures; *i.e.*, the surface area of fissures per unit volume. This is proportional to $1/\ell$ where $\ell > 0$ is the average linear dimension of the blocks, so $\varepsilon \sim \ell$ and $\frac{1}{\varepsilon}$ is a measure of the degree of fissuring.

Conservation of heat energy computed respectively in the blocks and in the fissures leads to the pair of equations

$$\frac{\partial w_1}{\partial t} + \frac{1}{\varepsilon}\big(K(u_1) - K(u_2)\big) = f_1\ , \qquad u_1 = \alpha(w_1)\,, \tag{1.a}$$

$$\vec{\nabla}\cdot\{\vec{v}w_2 - \vec{\nabla}K(u_2)\} + \frac{1}{\varepsilon}\big(K(u_2) - K(u_1)\big) = f_2\ , \qquad u_2 = \beta(w_2)\,, \tag{1.b}$$

where $\vec{\nabla}$ is divergent. The derivation of (1) follows [2]. There is no block-to-block diffusion, since the system of fissures essentially isolates the blocks, so (1.a) balances the storage of heat with the volume exchange rate. In (1.b) occurs the dual situation in which the storage of heat in the fissure system is negligible, due to the smallness of its volume, but the relatively high flow rates in the fissures balance the exchange of heat with the blocks. To determine the temperatures u_1, u_2 and enthalpies w_1, w_2 in a region Ω in $\mathbf{R}^n$ for all time $t > 0$, we require, in addition to (1), that they satisfy a prescribed initial condition,

$$w(x,0) = w_0(x)\ , \qquad x \in \Omega\,, \tag{2}$$

and boundary conditions of the form

$$u_2(s,t) = g_1(s)\ , \qquad s \in S_1\,,\ \ t > 0 \tag{3.a}$$

$$\big(\vec{v}w_2 - \vec{\nabla}K(u_2)\big)\cdot\vec{\nu} = g_2(s)\ , \qquad s \in S_2\,,\ \ t > 0, \tag{3.b}$$

where $S_1 \cup S_2$ is an appropriate decomposition of the boundary, $\partial\Omega$, and $\vec{\nu}$ denotes the unit outward normal. Specifically, the fluid temperature is prescribed by (3.a) on the *outflow* region S_1, where $\vec{v}\cdot\vec{\nu} \geq 0$, and the flux is prescribed by (3.b) on the *inflow* region S_2, where $\vec{v}\cdot\vec{\nu} \leq 0$. By making a change-of-variable for $K(u_1)$ and $K(u_2)$ and relabeling α and β, we shall hereafter take K equal to the identity in this system without any loss of generality.

2. The Elliptic Operator. We shall write the system above as an O.D.E. and an elliptic operator equation in the Banach space L^1. We begin by defining the notation we shall use to construct the elliptic operator in (1.b) and (3).

Let Ω be an open, bounded subset of $\mathbf{R}^n$ with smooth boundary $\partial\Omega$. There is given a C^2 vector field $\vec{v} : \overline{\Omega} \to \mathbf{R}^n$ which describes the velocity of the flow in the fissures. This vector field determines, nonuniquely, a splitting of the boundary $\partial\Omega$ as follows:

$$\begin{aligned}
&S_1 \text{ and } S_2 \text{ are measurable with } \operatorname{meas}(S_1) > 0\,,\\
&\partial\Omega = S_1 \cup S_2\,, \qquad S_1 \cap S_2 = \phi\,,\\
&S_1 \subseteq \{s \in \partial\Omega \mid \vec{v}(s)\cdot\vec{\nu}(s) \geq 0\}\,, \text{ and}\\
&S_2 \subseteq \{s \in \partial\Omega \mid \vec{v}(s)\cdot\vec{\nu}(s) \leq 0\}\,,
\end{aligned}$$

where $\vec{\nu}$ denotes the outward unit normal on $\partial\Omega$.

We shall use the standard notation L^p and $W^{m,p}$ for the Lebesgue and Sobolev spaces of functions, and we denote $W^{m,2}$ by H^m. The trace operator $\gamma_0 : H^1(\Omega) \to H^{1/2}(\partial\Omega)$ corresponds to restricting function values to the boundary. We denote by V the subspace of functions g in $H^1(\Omega)$ for which $\gamma_0(g) = 0$ a.e. on S_1, endowed with the norm $\|\nabla u\|_{L^2} = \|u\|_V$. As in [8], we denote by $\mathcal{E}$ those vector valued functions $\vec{g} : \Omega \to \mathbf{R}^n$ in $\{L^2(\Omega)\}^n$ for which the divergence $\operatorname{div}(\vec{g}) \in L^2(\Omega)$. The trace operator $\gamma_\nu : \mathcal{E} \to H^{-1/2}(\partial\Omega)$ corresponds to restricting the normal component of the vector field, $\vec{\nu}\cdot\vec{g}$, to $\partial\Omega$.

The functions α and $\beta : \mathbf{R} \to \mathbf{R}$ are assumed to be Lipschitz and nondecreasing. We further require that $\beta(0) = 0$ and that the graph of β^{-1} be bounded above and below by affine functions; *i.e.*,

$$|s| \leq K\big(\beta(s) + 1\big)\,, \qquad s \in \mathbf{R}\,, \tag{2.0}$$

for some $K \geq 0$.

The elliptic operator equation will correspond to a well-posed problem provided appropriate boundary conditions are specified. Let A be the closure in $L^1 \times L^1$ of the operator A_2 defined on the domain

$$\begin{aligned}
D(A_2) = \big\{w \in L^2 \mid \beta(w) - g_1 \in V\,,\ \vec{v}w - \nabla\beta(w) \in \mathcal{E}\\
\text{and } \ \langle\gamma_\nu(\vec{v}w - \nabla\beta(w) - \vec{g_2})\,, \gamma_0(v)\rangle = 0 \quad \text{for all } v \in V\big\}\,,
\end{aligned}$$

by $A_2(w) = \operatorname{div}(\vec{v}w - \vec{\nabla}\beta(w))$. In the above definition, $g_1 \in H^1$ and $\vec{g_2} \in \mathcal{E}$ are fixed and the angular brackets denote the duality betwen $H^{-1/2}(\partial\Omega)$ and $H^{1/2}(\partial\Omega)$. Note that the "temperature" $\beta(w)$ is specified as g_1 on S while the normal component $\vec{\nu}\cdot(\vec{v}w - \vec{\nabla}\beta(w))$ of the flux is specified as $\vec{\nu}\cdot\vec{g_2}$ on S_2. The technical conditions which the data should satisfy are

$$g_1 \in H^1(\Omega) \cap L^\infty(\Omega), \gamma_0(g_1) \in L^\infty(S_1), \gamma_0(g_1) \in \text{ Range of } \beta \quad \text{a.e. on } S_1\,,$$

and

$$\beta_0^{-1}\big(\gamma_0(g_1)\big) \text{ is bounded on } S_1\,.$$

The function β_0^{-1} is the minimal section of the monotone graph β^{-1}.

There is a constant M so that

$$\langle\gamma_\nu(\vec{g}_2), \gamma_0(v)\rangle \leq M|\langle\gamma_\nu(\vec{\nu}), \gamma_0(v)\rangle| \qquad \text{for all } v \in V\,,$$

and

$$\operatorname{div}(\vec{v}) \geq 0 \quad \text{and} \quad \vec{v} = \vec{o} \text{ on } \overline{S}_1 \cap \overline{S}_2\,.$$

These conditions are discussed in [6], where it is shown that the operator A is m-accretive. The following lemma is a modification of those results.

Lemma 2.1. *For any $f \in L^\infty$ and for each $\varepsilon > 0$, there is a solution $w \in D(A_2)$ to the problem*

$$\beta(w) + \varepsilon A_2(w) = f\,. \tag{2.1}$$

If $\hat{w}$ is a solution to (2.1) with right side equal to $\hat{f}$, then we have the estimate

$$\|\big(\beta(w) - \beta(\hat{w})\big)^+\|_{L^1} \le \|(f - \hat{f})^+\|_{L^1}\,. \tag{2.2}$$

Furthermore, there is a constant K depending only on the data g_1, $\vec{g}_2$, $\vec{v}$, and the domain Ω such that

$$\|\beta(w)\|_{L^\infty} \le \max\{K, \|f\|_{L^\infty}\}\,, \tag{2.3}$$

and

$$\|\beta(w) - g_1\|_V \le \frac{1}{\varepsilon} K\big(1 + \|f\|_{L^2}\big)\,. \tag{2.4}$$

Remarks: The solution w to (2.1) may not be unique, but we shall only need to use the properties of $\beta(w)$ in the sequel.

Proof: If w and $\hat{w}$ satisfy (2.1) with right side, respectively, equal to f and $\hat{f}$, then for N, $\varepsilon > 0$ we have

$$(w - \hat{w}) + \frac{\varepsilon}{N}\big(A(w) - A(\hat{w})\big) = \big(w - \frac{1}{N}\beta(w)\big) - \big(\hat{w} - \frac{1}{N}\beta(\hat{w})\big) + \frac{1}{N}(f - \hat{f})\,.$$

Since A is m-accretive it follows that

$$\begin{aligned}\|w - \hat{w}\|_{L^1} &\le \|(w - \hat{w}) - \frac{1}{N}\big(\beta(w) - \beta(\hat{w})\big)\|_{L^1} + \frac{1}{N}\|f - \hat{f}\|_{L^1} \\ &= \|w - \hat{w}\|_{L^1} - \frac{1}{N}\|\beta(w) - \beta(\hat{w})\|_{L^1} + \frac{1}{N}\|f - \hat{f}\|_{L^1}\,,\end{aligned}$$

provided N is larger than the Lipschitz constant for the monotone function β. We conclude that, although the solution w to (2.1) may not be unique, the function $\beta(w)$ *is* unique. This means that w may be nonunique only on the set where $\beta'(w) = 0$.

To obtain a solution to (2.1), we shall solve a sequence of approximating problems. Since A is m-accretive, the resolvent

$$J_\lambda \equiv (I + \lambda A)^{-1}$$

is a contraction on L^1. Define $T_N : L^1 \to D(A)$ by

$$T_N(w) = J_{\varepsilon/N}\Big(\big(1 - \frac{1}{N^2}\big)w - \frac{1}{N}\beta(w) + \frac{1}{N}f\Big)\,.$$

Then, for $N - 1$ larger than the Lipschitz constant for β, T_N is a strict contraction on L^1:

$$\|T_N(w) - T_N(\hat{w})\|_{L^1} \le \big(1 - \frac{1}{N^2}\big)\|w - \hat{w}\|_{L^1} - \frac{1}{N}\|\beta(w) - \beta(\hat{w})\|_{L^1}\,.$$

The contraction mapping principle guarantees that a unique fixed point w_N of T_N exists. This function satisfies the equation

$$\frac{1}{N}w_N + \beta(w_N) + \varepsilon A(w_n) = f\,. \tag{2.5}$$

If $f \in L^\infty(\Omega)$, then so is $\frac{1}{N}w_N + \beta(w_N)$; in fact, the maximum principle

$$\|\frac{1}{N}w_N + \beta(w_N)\|_{L^\infty} \leq \max\{K, \|f\|_{L^\infty}\} \tag{2.6}$$

holds and may be proven exactly as in [6]. Similarly,

$$\|\beta(w_N) - g_1\|_V \leq \frac{K}{\varepsilon}(1 + \|f\|_{L^2}), \tag{2.7}$$

as may be verified by multiplying (2.5) by the test function $\beta(w_N) - g_1 \in V$, integrating by parts and applying Hölder's inequality and the estimate (2.0) on β. Finally, we have the estimate

$$\Big\| \Big[\frac{1}{N}w_N + \beta(w_N) - \big(\frac{1}{N}\hat{w}_N + \beta(\hat{w}_N)\big)\Big]^+ \Big\|_{L^1} \leq \|[f - \hat{f}]^+\|_{L^1}, \tag{2.8}$$

where $\hat{w}_N$ is the solution to (2.5) with right side equal to $\hat{f}$. This estimate follows from the accretiveness and order estimates for A.

For $f \in L^\infty(\Omega)$, we obtain a solution to (2.1) by letting $N \to \infty$. From (2.6) and (2.0), $\{w_N, \beta(w_N)\}$ is a bounded subset of $L^\infty(\Omega)$, hence of $L^2(\Omega)$. From (2.7), $\{\beta(w_N)\}$ is bounded in $H^1(\Omega)$, so we may choose subsequences (still denoted by subscript N) for which

$$w_N \rightharpoonup w \qquad \text{in } L^2$$

$$\beta(w_N) \to \beta \qquad \text{in } L^2, \text{ weakly in } H^1$$

and pointwise a.e. In fact, $\beta = \beta(w)$ since

$$\begin{aligned}\|\beta - \beta(w)\|_{L^2}^2 &= \lim_{N\to\infty} \int \big(\beta(w_N) - \beta(w)\big)^2 \\ &\leq \lim_{N\to\infty} c_\beta \int \big(\beta(w_N) - \beta(w)\big)(w_N - w) = 0\,,\end{aligned}$$

where we have used the facts that $w_N \rightharpoonup w$, $\beta(w_N) \to \beta$ in L^2 and β is nondecreasing and Lipschitz. The limit w is a solution to (2.1). ∎

Corollary 2.2. *The solution operator $\beta \circ (\beta + \varepsilon A_2)^{-1} : L^\infty \to L^1$ extends uniquely to an L^1 contraction defined on all of L^1.*

Proof: Since $\beta \circ (\beta + \varepsilon A_2)^{-1}$ satisfies (2.2) on the dense subspace L^∞ of L^1, the result follows easily. Moreover, equation (2.4) continues to hold whenever $w = (\beta + \varepsilon A_2)^{-1} f$ for $f \in L^2$. ∎

We shall denote the extended operator described in the corollary by $\beta \circ \tilde{J}_\varepsilon$, where $\tilde{J}_\varepsilon$ is meant to be the operator $(\beta + \varepsilon A)^{-1}$. Note that we have not actually proven that $\beta + \varepsilon A$ is invertible, hence the notation is only figurative.

Finally, note that if $f \in L^{p/2}(\Omega)$ for $p > n$ and if $u = \beta \circ \tilde{J}_\varepsilon(f) \in L^p \cap H^1(\Omega)$, then for some $\gamma \in (0,1)$, $u \in C^\gamma(\overline{\Omega'})$ for any $\Omega' \subset\subset \Omega$, and

$$\|u\|_{C^\gamma} \leq C\big(\|f\|_{L^{p/2}} + \|u\|_{L^p} + 1\big)\,, \tag{2.9}$$

where C is independent of f and u. This fact follows from the Hölder continuity of solutions to the weakly formulated problem

$$\int_\Omega u\varphi + \varepsilon\vec{\nabla}u\cdot\nabla\varphi = \int_\Omega \varepsilon\vec{v}w\cdot\nabla\varphi + f\varphi\,,$$

where φ is any testing function from the class $C_0^1(\Omega)$. Here, the function w satisfies $\beta(w) = u$; hence, if $u \in L^p$, $w \in \beta^{-1}(u) \in L^p$ by (2.0). We conclude that $\varepsilon\vec{v}w \in (L^p(\Omega))^n$, and may apply the interior regularity theory [4] to conclude that $u \in C^\gamma(\Omega')$.

3. The Evolution Problem. If we rewrite (1.b) and (3) in terms of the operator A, we have

$$\beta(w_2) + \varepsilon A(w_2) = u_1 + \varepsilon f_2\,.$$

We solve this problem to obtain (uniquely) $\beta(w_2)$ and substitute into (1.a) to obtain the O.D.E.

$$\frac{d}{dt}w_1(t) + \frac{1}{\varepsilon}\left(\alpha(w_1) - \beta\Big(\tilde{J}_\varepsilon\big(\alpha(w_1) + \varepsilon f_2\big)\Big)\right) = f_1 \tag{3.1}$$

subject to the initial condition $w_1(0) = w_0$. This equation is known as the *fissured medium equation* and is formally equivalent to the equation studied in [2], but is considerably complicated by the convective term $\mathrm{div}(\vec{v}w)$ in the operator A. Because the "resolvent" $\tilde{J}_\varepsilon$ of A retains smoothing properties similar to those of the resolvent of the nonconvective operator in [2], properties of the solution to (3.1) will be similar to those obtained there. The following theorem follows from the ordinary differential equation structure of (3.1).

Theorem 1. *If A, α, β, f_1, f_2, ε and w_0 satisfy the conditions specified above, then there is a unique solution $w_1 \in W^{1,\infty}(0,T;L^1)$ to (3.1) subject to the initial condition*

$$w_1(0) = w_0\,.$$

Proof: Define $f : [0,T]\times L^1 \to L^1$ by

$$f(t,w) = \frac{1}{\varepsilon}(\beta\circ\tilde{J}_\varepsilon - I)\big(\alpha(w) + \varepsilon f_2(t)\big) + f_1(t) + f_2(t)\,.$$

Since α and β are Lipschitz,

$$\|f(t,w) - f(t,\hat{w})\|_{L^1} \le \frac{2k}{\varepsilon}\|w - \hat{w}\|_{L^1}\,.$$

The mapping $t \mapsto f(t,w)$ is measurable for each fixed $w \in L^1$; thus, $f(t,w)$ is integrable for all $w \in L^1$.

Define $F : L^\infty(0,T;L^1) \to L^\infty(0,T;L^1)$ by

$$F(w)(t) = w_0 + \int_0^t f\big(s,w(s)\big)\,ds\,.$$

We seek a fixed point of F. Let

$$b(t) = \int_0^t \exp(2k(t-s)/\varepsilon)\|f(s,w_0)\|_{L^1}\,ds\,,$$

and define $\mathcal{M} \subseteq L^\infty(0,T;L^1)$ by

$$\mathcal{M} = \{w \in L^\infty(0,T;L^1) \big| \|w(t) - w_0)\|_{L^1} \le b(t)\}\,.$$

Then F maps $\mathcal{M}$ into $\mathcal{M}$:

$$\begin{aligned}\|F(w)(t) - w_0\|_{L^1} &\le \|F(w)(t) - F(w_0)(t)\|_{L^1} + \|F(w_0)(t) - w_0\|_{L^1} \\ &\le \int_0^t \|f(s,w(s)) - f(s,w_0)\|_{L^1}\,ds + \int_0^t \|f(s,w_0)\|_{L^1}\,ds \\ &\le \int_0^t \frac{2k}{\varepsilon}\|w(s) - w_0\|_{L^1} + \|f(s,w_0)\|_{L^1}\,ds\,.\end{aligned}$$

If $w \in \mathcal{M}$, then we have

$$\begin{aligned}&\le \int_0^t \frac{2k}{\varepsilon}b(s) + \|f(s,w_0)\|_{L^1}\,ds = \int_0^t \frac{d}{dS}b(s)\,ds \\ &= b(t)\,.\end{aligned}$$

We conclude that $F(w) \in \mathcal{M}$ whenever $w \in \mathcal{M}$.

We shall prove that $F^N : \mathcal{M} \to \mathcal{M}$ is a strict contraction provided N is large enough. If w, $\hat{w} \in \mathcal{M}$, then

$$\|F(w)(t) - F(\hat{w})(t)\|_{L^1} \le \frac{2k}{\varepsilon}\int_0^t \|w(s) - \hat{w}(s)\|_{L^1}\,ds\,,$$

and from here it follows successively that

$$\begin{aligned}\|F(w) - F(\hat{w})\|_{L^\infty(0,t;L^1)} &\le \frac{2k}{\varepsilon}t\|w - \hat{w}\|_{L^\infty(0,T;L^1)} \\ \|F^2(w)(t) - F^2(\hat{w})(t)\|_{L^1} &\le \left(\frac{2k}{\varepsilon}\right)\int_0^t \left(\frac{2k}{\varepsilon}\right)s\,ds\|w - \hat{w}\|_{L^\infty(0,T;L^1)} \\ &= \left(\frac{2k}{\varepsilon}\right)^2\frac{t^2}{2}\|w - \hat{w}\|_{L^\infty(0,T;L^1)}\end{aligned}$$

and by an easy induction that

$$\|F^N(w) - F^N(\hat{w})\|_{L^\infty(0,T;L^1)} \le \left[\left(\frac{2kT}{\varepsilon}\right)^N\Big/N!\right]\|w - \hat{w}\|_{L^\infty(0,T;L^1)}\,.$$

Thus, F^N is a strict contraction for N sufficiently large, so F has a unique fixed point $w_1 \in \mathcal{M}$.

The operator $\beta \circ \tilde{J}_\varepsilon$ satisfies (2.2) and (2.3), and we may interpolate this operator to the spaces L^p, $1 \le p \le \infty$. In fact, the order estimate included in (2.2) is stronger than we need; all that is necessary is that $\beta \circ \tilde{J}_\varepsilon$ be accretive (*i.e.*, satisfy (2.2) without the "+" *s*.) The following lemma is an extension of Lemma 3 from [3].

Lemma 3.1. *Suppose $T : L^1(\Omega) \to L^1(\Omega)$ satisfies*

$$\|(T(f) - T(\hat{f}))^+\|_{L^1} \le \|(f - \hat{f})^+\|_{L^1} \tag{3.2}$$

and

$$\|T(f)\|_{L^\infty} \le \max\{K\,, \|f\|_{L^\infty}\} \tag{3.3}$$

and let $j : \mathbf{R} \to \mathbf{R}^+$ be a convex, lower semicontinuous function satisfying $j(0) = \min_x j(x) = 0$. Then,

$$\int_\Omega j(T(f)) \le \int_\Omega [j(f) + j(k)] = j(k)\ meas(\Omega) + \int_\Omega j(f)\,.$$

Proof: It suffices to prove the inequality for the functions $j_1(r) = (r - t)^+$ and $j_2(r) = (-r - t)^+$, where $t \ge 0$ (cf., [3]). Let $g = f - (f - t)^+ = \min\{f, t\}$. Then, (3.3) implies that

$$Tg \le \max\{\sup(g), k\} \le \max\{t, k\}\,.$$

It follows that

$$(Tf - \max\{t, k\})^+ \le (Tf - Tg)^+ \le |Tf - Tg|\,,$$

so (3.2) (without the "+") implies

$$\int_\Omega (Tf - \max\{t, k\})^+ \le \int_\Omega |f - g| = \int_\Omega (f - t)^+\,.$$

Since $\max\{t, k\} = t + (k - t)^+$, and since

$$(a - b)^+ \le (a - c)^+ + (c - b)^+$$

for all a, b, $c \in \mathbf{R}$, we have the estimate we desire, namely

$$\int_\Omega (Tf - t)^+ \le \int_\Omega [(f - t)^+ + (k - t)^+]\,.$$

The estimate for j_2 follows by replacing $T(g)$ with $-T(-g)$. The conclusion of the lemma now follows exactly as in [3]. ∎

Lemma 3.1 allows us to interpolate the operator $\beta \circ J_\varepsilon$ as an operator on L^p by taking $j(r) = r^p$:

$$\|\beta(J_\varepsilon(f))\|_{L^p} \le \left(\|f\|_{L^p}^p + k^p|\Omega|\right)^{1/p} \le \|f\|_{L^p} + k|\Omega|^{1/p}\,.$$

Let the set $\mathcal{M}$ used in the proof of Theorem 1 be redefined as

$$\mathcal{M} = \{w \in L^\infty(0, T; L^1) : \|w(t) - w_0\|_{L^1} \le b(t) \text{ and } \|w(t)\|_{L^p} \le c(t)\}\,,$$

where $b(t)$ is the same function defined in the proof of Theorem 1 and $c(t)$ is defined by the integral equation

$$c(t) \equiv \|w_0\|_{L^p} + \frac{Kt}{\varepsilon}|\Omega|^{1/p} + \int_0^t [(2 + c_\beta)\|f_2(s)\|_{L^p} + \|f_1(s)\|_{L^p}]\,ds + \frac{c_\alpha(1 + c_\beta)}{\varepsilon}\int_0^t c(s)\,ds\,.$$

In this expression, $c_\alpha \geq \|\alpha\|_{\mathrm{Lip}}$ and $c_\beta \geq \|\beta\|_{\mathrm{Lip}}$. The operator F maps $\mathcal{M}$ into $\mathcal{M}$, since

$$\begin{aligned}
\|F(w)(t)\|_{L^p} &\leq \|w_0\|_{L^p} + \int_0^t \|f(s,w)\|_{L^p}\,ds \\
&\leq \|w_0\|_{L^p} + \int_0^t \frac{1}{\varepsilon}\Big[\|\alpha(w)+\varepsilon f_2(s)\|_{L^p} + K|\Omega|^{1/p} + \|\beta\big(\alpha(w)+\varepsilon f_2(s)\big)\|_{L^p}\Big] \\
&\qquad + \|f_1(s)\|_{L^p} + \|f_2(s)\|_{L^p}\,ds \\
&\leq \|w_0\|_{L^p} + \frac{Kt}{\varepsilon}|\Omega|^{1/p} \\
&\qquad + \int_0^t \Big[\frac{c}{\varepsilon}\alpha(1+c_\beta)\|w(s)\|_{L^p} + (2+c_\beta)\|f_2(s)\|_{L^p} + \|f_1(s)\|_{L^p}\Big]\,ds \\
&\leq c(t).
\end{aligned}$$

If we modify the proof of Theorem 1 by using the modified set $\mathcal{M}$, we obtain the first part of the following.

Corollary. *If, in addition to the assumptions in Theorem 1, f_1, $f_2 \in L^q(0,T;L^p)$ for $q \geq 1$ and $w_0 \in L^p$, then $w_1 \in L^\infty(0,T;L^p)$. Consequently, $w_1 \in W^{1,q}(0,T;L^p)$. Furthermore, if f_1 and $f_2 \in L^2(0,T;H^1)$ and $w_0 \in H^1$, then $w_1 \in L^\infty(0,T;H^1)$ and, consequently, $w_1 \in H^1(\Omega \times (0,T))$.*

Proof: (continued). We shall make use of the splitting of (3.1) described in [2], namely

$$\begin{cases} \frac{dw_1}{dt} + \frac{1}{\varepsilon}\alpha(w_1(t)) = \frac{1}{\varepsilon}\beta\Big(\tilde{J}_\varepsilon\big(\alpha(w_1)(t)\big) + \varepsilon f_2(t)\big)\Big) + f_1(t) + f_2(t) \\ w_1(0) = w_0\,. \end{cases} \tag{3.4}$$

Suppose $w_1 \in H^1(0,T;L^2)$ is the solution to (3.1) and let $g(t) = \frac{1}{\varepsilon}\beta\big(\tilde{J}_\varepsilon(\alpha(w_1(t))+\varepsilon f_2(t))\big) \in L^\infty(0,T;H^1)$ by the regularizing property (2.4) of $\beta \circ \tilde{J}_\varepsilon$. The solution to (3.4) may be found as the fixed point of the operator

$$G(s)(t) \equiv w_0 + \int_0^t -\frac{1}{\varepsilon}\alpha\big(w(s)\big) + g(s) + f_1(s) + f_2(s)\,ds\,.$$

It is easy to see that G is a strict contraction on $L^2(0,T;L^2)$ for small enough T. In fact, if $\tilde{\mathcal{M}}$ is the set

$$\tilde{\mathcal{M}} = \{w \in L^2(0,T;L^2) |\ \|\nabla w(t)\|_{L^2} \leq a(t)\}$$

where $a(t)$ is the function which is defined by the integral equation

$$a(t) = \|\nabla w_0\|_{L^2} + \int_0^t \frac{c_\alpha}{\varepsilon}a(s) + \|\nabla\big(g(s)+f_1(s)+f_2(s)\big)\|_{L^2}\,ds$$

then it is easily verified that $G : \tilde{\mathcal{M}} \to \tilde{\mathcal{M}}$. Since $a(t)$ is bounded on $(0,T)$, we conclude that $\tilde{\mathcal{M}}$ is a closed subset of $L^2(0,T;L^2)$; hence, $w(t) \in L^\infty(0,T;H^1)$.

4. Smoothness of solutions. It was shown in [2] that solutions to the fissured medium equation behave differently from solutions to the porous medium equation. Theorem 1 in [2] describes the persistence of discontinuities in space as a function of time. Analogous results hold for solutions to the equation with convection.

Theorem 2. *Let $\vec{\nu}$ be a unit vector in R^n and define $\sigma(f(x)) = \lim_{h\downarrow 0}(f(x+h\vec{\nu})-f(x-h\vec{\nu}))$, the jump in the function f in the direction ν at the point x. Suppose $f_2 \equiv 0$, $f_1 \in L^\infty(0,T;L^p)$, where $p > n$. Furthermore, assume that for some point $x \in \Omega$, there is a number $h_0 > 0$ and a function $g \in L^1(0,T)$ for which*

$$|f_1(x + h\vec{\nu}, t)| \leq g(t)\,, \qquad |h| < h_0\,,$$

and that each of $\sigma(w_0(x))$ and $\sigma(f_1(x,t))$ exists for almost all $t \in [0,T]$. Then, $\sigma(w_1(x,t))$ exists for each $t \in [0,T]$ and

$$\sigma^+\big(w_1(x,t)\big) \leq \sigma^+\big(w_0(x)\big) + \int_0^t \sigma^+\big(f(x,s)\big)\,ds\,, \tag{4.1}$$

$$\sigma^+\big(w_1(x,t)\big) \geq e^{-Kt/\varepsilon}\left\{\sigma^+\big(w_0(x)\big) + \int_0^t \sigma^-\big(f(x,s)\big)\,ds\right\}, \tag{4.2}$$

and similar estimates hold for $\sigma^-(w_1(x,t))$ and $\sigma(w_1(x,t))$.

Remarks: If the initial data or the external source f_1 has a jump discontinuity at the point x in the direction $\vec{\nu}$, then so does the solution w_1; furthermore, the discontinuity remains located at the point x for all time. Solutions to the Stefan problem do not enjoy this property, even though the Stefan equation is formally the limit as $\varepsilon \downarrow 0$ of (3.1). Discontinuities of w in the Stefan problem propagate with the free boundary and are not stationary in space.

Proof: As in [2], we begin with the representation (3.4) of the solution. From the regularity assumptions on the data and the remarks about the regularity of $\beta \circ \tilde{J}_\varepsilon(\cdot)$ at the end of section 2, $\beta \circ \tilde{J}_\varepsilon(\alpha(w_1(t)))$ belongs to $C^\gamma(\Omega)$ for each t; thus, $\sigma(\beta \circ \tilde{J}_\varepsilon(\alpha(w_1(t)))) = 0$. We compute σ of both sides of (3.4) to obtain the differential equation

$$\frac{d\sigma(w_1(x,t))}{dt} + \sigma\Big(\frac{1}{\varepsilon}\,\alpha\big(w_1(x,t)\big)\Big) = \sigma\big(f_1(x,t)\big)\,.$$

After multiplying both sides by $\mathrm{sgn}_0^+(\sigma(w_1(t))) \in \mathrm{sgn}^+(\sigma(\alpha(w_1(t))))$ and integrating over $[0,t]$, we have

$$\sigma^+\big(w_1(x,t)\big) \leq \sigma^+\big(w_1(x,0)\big) + \int_0^t \sigma^+\big(f_1(x,s)\big) - \sigma^+\Big(\frac{1}{\varepsilon}\,\alpha\big(w_1(x,s)\big)\Big)\,ds\,.$$

The upper bound (4.1) now follows immediately. To obtain the lower bound (4.2), we note that if C_α is the Lipschitz constant for α, then

$$\sigma^+\Big(\alpha\big(w_1(t)\big)\Big) \leq C_\alpha \sigma^+\big(w_1(t)\big)\,.$$

We use this estimate after multiplying both sides of the differential equation by $\mathrm{sgn}_0^+(\sigma(w_1(t)))$ to obtain

$$\sigma^+\big(w_1(x,t)\big) \geq \sigma^+\big(w_1(x,0)\big) + \int_0^t \sigma^-\big(f_1(x,s)\big) - \frac{1}{\varepsilon}\,C_\alpha\sigma^+\big(w_1(x,s)\big)\,ds\,.$$

Estimate (4.2) now follows from Gronwall's inequality.

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