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Homogenization of a pseudoparabolic system

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Pseudoparabolic equations in periodic media are homogenized to obtain upscaled limits by asymptotic expansions and two-scale convergence. The limit is characterized and convergence is established in various linear cases for both the classical binary medium model and the highly heterogeneous case. The limit of vanishing time-delay parameter in either medium is included. The double-porosity limit of Richards' equation with dynamic capillary pressure is obtained.

Keywords: homogenization; pseudoparabolic equations; fractured porous media; dynamic capillary pressure

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1. Introduction

Pseudoparabolic equations arise in a range of applications from radiation with time-delay [1], degenerate double-diffusion and heat-conduction models [2,3] and resolution of ill-posed problems [4] through recently developed applications in level set methods [5] and models of lightning propagation [6]. They were first analysed in [7–9]; see [10] for an extensive review and bibliography. Here we are interested in a degenerate pseudoparabolic equation arising from modelling dynamic capillary pressure in unsaturated flow; specifically, we study the case of flow in heterogeneous media in which the coefficients are periodic on a fine scale.

The classical Richards equation for flow through a partially saturated porous medium with porosity $\phi(x)$ and permeability $K(x)$ takes the form

$$\phi(x)\frac{\partial u(t,x)}{\partial t} + \nabla\cdot K(x)\frac{k_w(u(t,x))}{\mu_w}\nabla(P_c(u(t,x)) - \rho GD(x)) = 0, \tag{1}$$

where u denotes saturation, and gravitational effects depend on depth $D(x)$ and (constant) density ρ. Here $k_w(u)$, $P_c(u)$ denote relative permeability and capillary pressure relationships, respectively. This standard model follows from Darcy's law extended to multiphase flow and conservation of mass [11,12] with the assumption

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that atmospheric pressure of air is constant. The model has been analysed in [13–15] and elsewhere.

The experimental determination of the pressure–saturation relationship $p = -P_c(u)$ is based on the assumption that this is an instantaneous process, although in reality it requires substantial time to approach an equilibrium before measurements can be taken. This led to the introduction of *dynamic capillary pressure* [16] in which $P_c(u)$ is replaced by $P_{c,dyn}(u) \equiv P_c(u) - \tau \frac{\partial u}{\partial t}$ with $\tau > 0$. Other dynamic models had been introduced earlier [17,18]; see [19–22] for supporting experimental evidence. A similar model was derived in [23] by homogenization from standard two-phase models with special interface conditions.

The dynamic capillary pressure model of [16] leads to the nonlinear pseudoparabolic equation

$$\phi(x)\frac{\partial u(t,x)}{\partial t} + \nabla \cdot K(x)\frac{k_w(u(t,x))}{\mu_w}\nabla(P_c(u(t,x)) - \rho G D(x)) \\ - \nabla \cdot K(x)\frac{k_w(u(t,x))}{\mu_w}\nabla\tau(x)\frac{\partial u(t,x)}{\partial t} = 0. \tag{2}$$

When written in terms of pressure $u \mapsto -P_c(u)$ (see Section 4) and linearized about a known solution u_0, with $\kappa(x) \equiv K(x)\frac{k_w(u_0)}{\mu_w}$, ϕ replaced by $\phi\frac{\partial u}{\partial p}|_{u_0}$ and τ by $\frac{\tau}{\phi}$, Equation (2) takes the form

$$\phi(x)\frac{\partial u(t,x)}{\partial t} - \nabla \cdot \kappa(x)\nabla\left(u(t,x) + \tau(x)\phi(x)\frac{\partial u(t,x)}{\partial t}\right) = \nabla \cdot \kappa(x)\rho G D(x). \tag{3}$$

If the convective term is dropped, i.e. set $D(x) = 0$, we obtain

$$\phi(x)\frac{\partial u(t,x)}{\partial t} - \nabla \cdot \kappa(x)\nabla\left(u(t,x) + \tau(x)\phi(x)\frac{\partial u(t,x)}{\partial t}\right) = 0. \tag{4}$$

In realistic porous media there is substantial variation of $\phi(x)$ and $K(x)$, as well as the nonlinear relationships $k_w(\cdot)$, $P_c(\cdot)$, $\tau(\cdot)$ in (2). Consequently the coefficients in linearized models (3) and (4) vary similarly. In this article we derive homogenized models for (2) and (4), and in particular for the special case of *binary media* in which $\phi(x)$, $K(x)$, $\tau(x)$ and consequently $\kappa(x)$ oscillate between two respective constant values. See [24,25] for further discussion of heterogeneous dynamic capillary pressure models, references and numerical results.

The multiscale analysis is aided by the structure of the *pseudoparabolic system*

$$\phi(x)\frac{\partial u(t,x)}{\partial t} + \frac{1}{\tau(x)}\big(u(t,x) - v(t,x)\big) = 0, \tag{5a}$$

$$-\nabla \cdot \big(\kappa(x)\nabla v(t,x)\big) + \frac{1}{\tau(x)}\big(v(t,x) - u(t,x)\big) = 0, \quad x \in \Omega. \tag{5b}$$

This system is equivalent to a single equation: if we eliminate v we obtain the *pseudoparabolic equation* (4) for the variable $u(t,x)$; v satisfies a similar equation. It is supplemented with corresponding boundary and initial conditions. Here we take homogeneous Dirichlet boundary conditions

$$v(t,s) = 0, \quad \text{a.e. } s \in \partial\Omega, \tag{5c}$$

and the initial condition

$$\phi(x)u(0,x) = \phi(x)u_*(x), \quad \text{a.e. } x \in \Omega. \tag{5d}$$

The well-posedness of the system (5) follows from very general assumptions on the coefficients and initial function. The following suffices for our purposes here.

Theorem 1.1 *Assume that functions $\phi(\cdot)$, $\kappa(\cdot)$, $\tau(\cdot) \in L^\infty(\Omega)$ are given, each with a strictly positive lower bound, and let $u_*(\cdot) \in L^2(\Omega)$. Then there is a unique pair $u(\cdot) \in H^1((0,T); L^2(\Omega))$ and $v(\cdot) \in L^2((0,T); H_0^1(\Omega))$ such that $u(0,\cdot) = u_*(\cdot)$ and*

$$\begin{aligned}\int_\Omega \bigg(\phi(x)\frac{\partial u(t,x)}{\partial t}\varphi(x) + \frac{1}{\tau(x)}\big(u(t,x) - v(t,x)\big)\big(\varphi(x) - \psi(x)\big) \\ + \kappa(x)\nabla v(t,x) \cdot \nabla\psi(x)\bigg)\mathrm{d}x = 0\end{aligned} \tag{6}$$

for all $\varphi(\cdot) \in L^2(\Omega)$ and $\psi(\cdot) \in H_0^1(\Omega)$.

Corresponding results hold under much more general conditions of non-negativity of the coefficients. See [10,26–29]. The initial value u_* need be chosen only with $\phi(\cdot)^{1/2}u_*(\cdot) \in L^2(\Omega)$. Also, the *a priori* estimates show explicitly that $u - v \to 0$ as $\tau \to 0$.

Our objective is to homogenize the system (5) and thereby the corresponding pseudoparabolic equation (4) when the coefficients depend (periodically) on a small parameter ε. The precise description of coefficients will follow below. Bensoussan et al. [30] briefly investigated the homogenization of pseudoparabolic equations as an example for which the limiting problem is of a different type, and perhaps *non-local*, not even a partial differential equation. (See [30] Chapter II, Section 3.9, pp. 318, 338.) We shall see below that this occurs when certain variables are eliminated or *hidden*. The limited regularity and estimates for solutions of the corresponding pseudoparabolic equation (4) makes the homogenization more delicate. Only in special cases there is a purely upscaled limit.

In Section 2, we obtain the formal asymptotic expansion of the solution for the linear equation (4) in the classical case and find the dependence of the limit on ϕ and τ. The analysis and homogenization of the linear system (5) by two-scale convergence is developed in Section 3 for ε-periodic binary coefficients and includes cases of $\tau \to 0$ with parabolic or first-order kinetic systems as limits. Finally, Section 4 contains the asymptotic expansion for a nonlinear highly heterogeneous case arising from Richards' equation with dynamic capillary pressure.

2. Asymptotic expansion

First we introduce periodic coefficients into the pseudoparabolic system (5) and use formal asymptotic expansions to obtain the limiting problem as the period scale $\varepsilon > 0$ tends to zero. Let Y denote the unit cube in $\mathbb{R}^N$, let there be given the Y-periodic functions $\phi(y)$, $\tau(y)$, $\kappa(y)$ and then define $\phi^\varepsilon(x) = \phi(\frac{x}{\varepsilon})$, $\tau^\varepsilon(x) = \tau(\frac{x}{\varepsilon})$, $\kappa^\varepsilon(x) = \kappa(\frac{x}{\varepsilon})$. The three functions ϕ^ε, τ^ε, κ^ε are the respective ε-periodic coefficients in (5), so the corresponding solution u^ε, v^ε to (5) depends on ε. We write these as formal asymptotic expansions

$$u^\varepsilon(t,x) = \sum_{p=0}^{\infty} \varepsilon^p u_p(t,x,y), \quad v^\varepsilon(t,x) = \sum_{p=0}^{\infty} \varepsilon^p v_p(t,x,y), \quad y = \frac{x}{\varepsilon}, \tag{7}$$

with each $u_p(t,x,\cdot)$, $v_p(t,x,\cdot)$ being Y-periodic.

Substitute (7) into (5) and collect terms by powers ε^p for $p \geq -2$. Note that the gradient $\nabla = \nabla_x + \frac{1}{\varepsilon}\nabla_y$ is used in calculations where $y = x/\varepsilon$. The ordinary differential equation (5a) gives (at $p=0$)

$$\phi(y)\frac{\partial u_0(t,x,y)}{\partial t} + \frac{1}{\tau(y)}(u_0(t,x,y) - v_0(t,x,y)) = 0.$$

The initial condition will always be assumed to be independent of the local variable, $y \in Y$.

The procedure for the elliptic equation (5b) is standard [30–32]. Equating to zero the coefficient of ε^{-2} in the expansion of (5b) gives

$$-\nabla_y \cdot \kappa(y)\nabla_y v_0(t,x,y) = 0, \quad y \in Y.$$

With the Y-periodic boundary conditions on v_0, we conclude that $\nabla_y v_0(t,x,y) = 0$, and so $v_0 = v_0(t,x)$ is independent of $y \in Y$. From the combined coefficients of ε^{-1} in the expansion of (5b) we obtain

$$-\nabla_y \cdot \kappa(y)(\nabla_y v_1(t,x,y) + \nabla_x v_0(t,x)) - \nabla_x \cdot \kappa(y)\nabla_y v_0(t,x) = 0.$$

The last term is null, so the function $v_1(t,x,y)$ is the solution of an elliptic periodic boundary-value problem on Y, and we can represent it in terms of Y-periodic solutions $\omega_j(y)$ of the *cell problem* (see (17))

$$-\nabla_y \cdot \kappa(y)\big(\nabla_y \omega_j + \boldsymbol{e}_j\big) = 0, \quad j = 1 \ldots N.$$

This representation $v_1(t,x,y) = \sum_{j=1}^{N} \omega_j(y)\frac{\partial}{\partial x_j} v_0(t,x)$ (up to a function of x) will be used to compute the effective tensor κ^* below. Finally, collecting terms with ε^0 in the expansion of (5b) gives

$$\begin{aligned} &- \nabla_y \cdot \kappa(y)(\nabla_y v_2 + \nabla_x v_1) - \nabla_x \cdot \kappa(y)(\nabla_x v_0(t,x) + \nabla_y v_1(t,x,y)) \\ &\quad + \frac{1}{\tau(y)}(v_0(t,x) - u_0(t,x,y)) = 0. \end{aligned}$$

Integrate this equation over Y. The first term vanishes due to Y-periodicity of each v_r, and the second becomes the effective elliptic contribution with the tensor κ^*. The third term gets averaged to yield the second equation of the system

$$\phi(y)\frac{\partial u_0(t,x,y)}{\partial t} + \frac{1}{\tau(y)}(u_0(t,x,y) - v_0(t,x)) = 0, \tag{8a}$$

$$-\nabla \cdot \kappa^* \nabla v_0(t,x) + \int_Y \frac{1}{\tau(y)}(v_0(t,x) - u_0(t,x,y))\mathrm{d}y = 0, \tag{8b}$$

the first being copied from above. The effective tensor κ^* is obtained in this calculation as $\kappa^*_{ij} = \int_Y \kappa(y)(\nabla_y \omega_i(y) + \boldsymbol{e}_i) \cdot (\nabla_y \omega_j(y) + \boldsymbol{e}_j)\mathrm{d}y$.

Only if the product $\phi(\cdot)\tau(\cdot)$ is constant we get $u_0(t,x,y)=u_0(t,x)$ independent of $y\in Y$, and in that case we can eliminate v_0 from the system to obtain the upscaled pseudoparabolic equation

$$\phi^*\frac{\partial u_0(t,x)}{\partial t}-\nabla\cdot\kappa^*\nabla u_0(t,x)-\nabla\cdot\kappa^*\nabla\phi^*\tau^*\frac{\partial u_0(t,x)}{\partial t}=0. \tag{9}$$

The homogenized porosity is the average $\phi^*=\int_Y\phi(y)\mathrm{d}y$ and the homogenized time-delay is the *harmonic average* $\tau^*=(\int_Y\frac{1}{\tau(y)}\mathrm{d}y)^{-1}$. In the general situation, u_0 depends also on the local variable $y\in Y$, and then the limit system (8) is *partially upscaled*, a combination of the local equation (8a) and the upscaled (8b). We will make similar but much more interesting calculations below when $\phi(\cdot)$ and $\tau(\cdot)$ are *piecewise constant*.

3. The pseudoparabolic system

Next we extend the models to include binary media of classical or highly heterogeneous type, and then we obtain the homogenized limit problems by two-scale convergence.

3.1. *The heterogeneous micro-models*

We use a binary medium to emphasize the dependence of singularities on geometry. Let the unit cube Y be given in open disjoint complementary parts, Y_1 and Y_2, so $Y_1\cap Y_2=\emptyset$ and Y is the interior of $\overline{Y_1}\cup\overline{Y_2}$. We denote by $\chi_j(y)$ the characteristic function of Y_j for $j=1,2$, extended Y-periodically to all of $\mathbb{R}^N$. Thus, $\chi_1(y)+\chi_2(y)=1$ for a.e. y in $\mathbb{R}^N$. It is assumed that the sets $\{y\in\mathbb{R}^N:\chi_j(y)=1\}$ for $j=1,2$, have smooth boundary, but we do not require these sets to be connected. The corresponding ε-periodic characteristic functions are defined by

$$\chi_j^\varepsilon(x)\equiv\chi_j\left(\frac{x}{\varepsilon}\right),\quad x\in\mathbb{R}^N,\quad j=1,2,$$

and these naturally partition the global domain Ω into two sub-domains, Ω_1^ε and Ω_2^ε by $\Omega_j^\varepsilon\equiv\{x\in\Omega:\chi_j^\varepsilon(x)=1\}$, $j=1,2$. We use the characteristic functions as multipliers to denote the *zero-extension* of various functions. Let $\Gamma\equiv\partial Y_1\cap\partial Y_2\cap Y$ be the part of the interface between Y_1 and Y_2 that is interior to the local cell Y. Then $\Gamma^\varepsilon\equiv\partial\Omega_1^\varepsilon\cap\partial\Omega_2^\varepsilon\cap\Omega$ represents the corresponding interface between Ω_1^ε and Ω_2^ε that is interior to Ω. We denote by γ_j the *boundary trace* of functions on Y_j to Γ and by γ_j^ε the *boundary trace* of functions on Ω_j^ε to Γ^ε. (See [29,33].)

3.1.1. *The classical case*

Let the strictly positive lower-bounded functions $\phi_j(\cdot,\cdot)$, $\kappa_j(\cdot,\cdot)$, $\tau_j(\cdot,\cdot)\in L^\infty(\Omega;C(\overline{Y_j}))$ be given, and define Y-periodic functions in $L^\infty(\Omega;L^2_\#(Y))$ by

$$\phi(x,y)\equiv\phi_j(x,y),\quad\kappa(x,y)\equiv\kappa_j(x,y),\quad\tau(x,y)\equiv\tau_j(x,y),\quad y\in Y_j,\quad j=1,2,\quad x\in\Omega.$$

The subscript # denotes the subspace of Y-periodic functions in any function space. Corresponding functions on Ω_j^ε are defined by

$$\phi_j^\varepsilon(x)\equiv\phi_j\left(x,\frac{x}{\varepsilon}\right),\quad\kappa_j^\varepsilon(x)\equiv\kappa_j\left(x,\frac{x}{\varepsilon}\right),\quad\tau_j^\varepsilon(x)\equiv\tau_j\left(x,\frac{x}{\varepsilon}\right),\quad x\in\Omega_j^\varepsilon,\quad j=1,2,$$

and the coefficients for the pseudoparabolic system (5) are given by

$$\phi^\varepsilon(x) \equiv \chi_1^\varepsilon(x)\phi_1^\varepsilon(x) + \chi_2^\varepsilon(x)\phi_2^\varepsilon(x), \tag{10a}$$

$$\kappa^\varepsilon(x) \equiv \chi_1^\varepsilon(x)\kappa_1^\varepsilon(x) + \chi_2^\varepsilon(x)\kappa_2^\varepsilon(x), \tag{10b}$$

$$\tau^\varepsilon(x) \equiv \chi_1^\varepsilon(x)\tau_1^\varepsilon(x) + \chi_2^\varepsilon(x)\tau_2^\varepsilon(x). \tag{10c}$$

These are ε-periodic on the *fine scale*. Theorem 1.1 gives a unique solution of the ε-problem: $u^\varepsilon(\cdot) \in H^1((0,T);\, L^2(\Omega))$ and $v^\varepsilon(\cdot) \in L^2((0,T);\, H_0^1(\Omega))$ satisfy

$$\int_\Omega \left(\phi^\varepsilon(x)\frac{\partial u^\varepsilon(t,x)}{\partial t}\varphi(x) + \frac{1}{\tau^\varepsilon(x)}\big(u^\varepsilon(t,x) - v^\varepsilon(t,x)\big)\big(\varphi(x) - \psi(x)\big) \right.$$
$$\left. + \kappa^\varepsilon(x)\nabla v^\varepsilon(t,x)\cdot\nabla\psi(x)\right)\mathrm{d}x = 0 \tag{11}$$

for all $\varphi(\cdot) \in L^2(\Omega)$ and $\psi(\cdot) \in H_0^1(\Omega)$, together with the initial condition $u^\varepsilon(0,\cdot) = u_*(\cdot)$. The initial value u_* is independent of ε.

If the coefficients κ_j^ε are continuous on Ω_j^ε, the strong form of (11) is the *transmission problem*

$$\phi^\varepsilon(x)\frac{\partial u^\varepsilon(t,x)}{\partial t} + \frac{1}{\tau^\varepsilon(x)}\big(u^\varepsilon(t,x) - v^\varepsilon(t,x)\big) = 0, \quad x \in \Omega, \tag{12a}$$

$$-\nabla\cdot\big(\kappa_1^\varepsilon(x)\nabla v^\varepsilon(t,x)\big) + \frac{1}{\tau_1^\varepsilon(x)}\big(v^\varepsilon(t,x) - u^\varepsilon(t,x)\big) = 0, \quad x \in \Omega_1^\varepsilon, \tag{12b}$$

$$-\nabla\cdot\big(\kappa_2^\varepsilon(x)\nabla v^\varepsilon(t,x)\big) + \frac{1}{\tau_2^\varepsilon(x)}\big(v^\varepsilon(t,x) - u^\varepsilon(t,x)\big) = 0, \quad x \in \Omega_2^\varepsilon, \tag{12c}$$

$$\gamma_1^\varepsilon v^\varepsilon(t,s) = \gamma_2^\varepsilon v^\varepsilon(t,s), \tag{12d}$$

$$\kappa_1^\varepsilon(s)\nabla v^\varepsilon(t,s)\cdot\nu = \kappa_2^\varepsilon(s)\nabla v^\varepsilon(t,s)\cdot\nu, \quad s \in \Gamma^\varepsilon, \tag{12e}$$

where ν denotes the unit outward normal on $\partial\Omega_1^\varepsilon$. We have homogeneous Dirichlet boundary conditions

$$v^\varepsilon(t,s) = 0 \quad \text{a.e. } s \in \partial\Omega, \tag{12f}$$

and the initial condition $u^\varepsilon(0,x) = u_*(x)$, a.e. $x \in \Omega$. This is the *exact micro-model*. If κ^ε is continuous on Γ^ε, there are no interface conditions and (12) reduces to the single system (5) over Ω. Even then, the fine-scale dependence on the coefficients and geometry make it numerically intractable for realistically small values of $\varepsilon > 0$.

3.1.2. *The highly heterogeneous case*

In the highly heterogeneous case, the permeability is scaled by ε^2 in the second region Ω_2^ε, so the flux is given by $-\varepsilon^2\kappa_2^\varepsilon(x)\nabla v^\varepsilon$ in Ω_2^ε:

$$\kappa^\varepsilon(x) \equiv \chi_1^\varepsilon(x)\kappa_1^\varepsilon(x) + \varepsilon^2\chi_2^\varepsilon(x)\kappa_2^\varepsilon(x). \tag{13}$$

Then the system (11) becomes

$$\phi^\varepsilon(x)\frac{\partial u^\varepsilon(t,x)}{\partial t}+\frac{1}{\tau^\varepsilon(x)}\big(u^\varepsilon(t,x)-v^\varepsilon(t,x)\big)=0,\quad x\in\Omega, \tag{14a}$$

$$-\nabla\cdot\big(\kappa_1^\varepsilon(x)\nabla v^\varepsilon(t,x)\big)+\frac{1}{\tau_1^\varepsilon(x)}\big(v^\varepsilon(t,x)-u^\varepsilon(t,x)\big)=0,\quad x\in\Omega_1^\varepsilon, \tag{14b}$$

$$-\nabla\cdot\big(\varepsilon^2\kappa_2^\varepsilon(x)\nabla v^\varepsilon(t,x)\big)+\frac{1}{\tau_2^\varepsilon(x)}\big(v^\varepsilon(t,x)-u^\varepsilon(t,x)\big)=0,\quad x\in\Omega_2^\varepsilon, \tag{14c}$$

$$\gamma_1^\varepsilon v^\varepsilon(t,s)=\gamma_2^\varepsilon v^\varepsilon(t,s), \tag{14d}$$

$$\kappa_1^\varepsilon(s)\nabla v^\varepsilon(t,s)\cdot\nu=\varepsilon^2\kappa_2^\varepsilon(s)\nabla v^\varepsilon(t,s)\cdot\nu,\quad s\in\Gamma^\varepsilon. \tag{14e}$$

The ε-problem for the model developed by Arbogast et al. [34] is recovered by letting $\tau^\varepsilon\to 0$.

3.2. *Homogenization of the classical case*

3.2.1. *The two-scale limit*

Let the coefficients in (5) be given by (10). Denote the gradient in the y-variable by ∇_y, and use the symbol '$\stackrel{2}{\to}$' to denote *two-scale convergence* [35].

LEMMA 3.1 *For each $\varepsilon>0$, let $u^\varepsilon(\cdot)$, $v^\varepsilon(\cdot)$ denote the unique solution to the pseudoparabolic ε-problem* (11). *These satisfy the estimates*

$$\|u^\varepsilon\|_{L^2((0,T)\times\Omega)}+\|v^\varepsilon\|_{L^2((0,T);H_0^1(\Omega))}\le C,$$

so there exist

(i) *a function U in $L^2((0,T)\times\Omega;\,L^2_\#(Y))$,*
(ii) *a function v in $L^2((0,T);\,H_0^1(\Omega))$,*
(iii) *a function V in $L^2((0,T)\times\Omega;\,H^1_\#(Y)/\mathbb{R})$,*

and a subsequence, hereafter denoted by u^ε, v^ε, which two-scale converges as follows:

$$u^\varepsilon\stackrel{2}{\to}U(t,x,y), \tag{15a}$$

$$v^\varepsilon\stackrel{2}{\to}v(t,x), \tag{15b}$$

$$\nabla v^\varepsilon\stackrel{2}{\to}\nabla v(t,x)+\nabla_y V(t,x,y). \tag{15c}$$

This suggests use of the corresponding test functions

$$\tilde\varphi(x)=\Phi(x,x/\varepsilon),\quad \tilde\psi(x)=\psi(x)+\varepsilon\Psi(x,x/\varepsilon),$$

where $\psi\in H_0^1(\Omega)$, $\Phi,\Psi\in C_0^\infty\big(\Omega;\,C_\#^\infty(Y)\big)$. Setting these in (11), we obtain

$$\int_\Omega\Big(\phi^\varepsilon(x)\frac{\partial u^\varepsilon(t,x)}{\partial t}\Phi(x,x/\varepsilon)+\frac{1}{\tau^\varepsilon(x)}\big(u^\varepsilon(t,x)-v^\varepsilon(t,x)\big)\big(\Phi(x,x/\varepsilon)-(\psi(x)+\varepsilon\Psi(x,x/\varepsilon))\big)$$
$$+\kappa^\varepsilon(x)\nabla v^\varepsilon(t,x)\cdot\nabla(\psi(x)+\varepsilon\Psi(x,x/\varepsilon))\Big)\mathrm{d}x=0.$$

Take the limit as $\varepsilon \to 0$ to obtain the *two-scale limit system*

$$\int_\Omega \int_Y \Big(\phi(x,y) \frac{\partial U(t,x,y)}{\partial t} \Phi(x,y) + \frac{1}{\tau(x,y)} \big(U(t,x,y) - v(t,x)\big)\big(\Phi(x,y) - \psi(x)\big) \\ + \kappa(x,y)\big(\nabla v(t,x) + \nabla_y V(t,x,y)\big) \cdot \big(\nabla \psi(x) + \nabla_y \Psi(x,y)\big)\Big) \mathrm{d}y\, \mathrm{d}x = 0, \tag{16}$$

for all Φ, ψ, Ψ as above, and $U(0,x,y) = u_*(x)$. From the uniqueness of the solution of the initial-value problem for (16), it follows that the original sequence u^ε, v^ε two-scale converges as above.

In order to eliminate the function $V(t,x,y)$ from this system, we use the *periodic cell problem*: for each $k = 1, 2, \ldots, N$, define ω_k by

$$\omega_k \in L^2(\Omega; H^1_\#(Y)) : \\ \int_Y \kappa(x,y)\big(\nabla_y \omega_k(x,y) + \boldsymbol{e}_k\big) \cdot \nabla_y \Psi(x,y) \mathrm{d}y = 0 \quad \text{for all } \Psi \in L^2(\Omega; H^1_\#(Y)). \tag{17}$$

(Let us ask that $\int_Y \omega_k(x,y)\mathrm{d}y = 0$ to fix the constant.) Then we have the representation $V(t,x,y) = \sum_{i=1}^N \frac{\partial v(t,x)}{\partial x_i} \omega_i(x,y)$. Specify similar test functions $\Psi(x,y) = \sum_{j=1}^N \frac{\partial \psi(x)}{\partial x_j} \omega_j(x,y)$ above to obtain the following theorem.

Theorem 3.2 *The limits U, v in Lemma* 3.1 *are the solution of the* partially homogenized *pseudoparabolic system*

$$U \in H^1\big((0,T); L^2\big(\Omega; L^2_\#(Y)\big)\big), \quad v \in L^2\big((0,T); H^1_0(\Omega)\big) : \\ \int_\Omega \int_Y \Big(\phi(x,y) \frac{\partial U(t,x,y)}{\partial t} \Phi(x,y) + \frac{1}{\tau(x,y)} \big(U(t,x,y) - v(t,x)\big)\big(\Phi(x,y) - \psi(x)\big)\Big) \mathrm{d}y\, \mathrm{d}x \\ + \int_\Omega \Bigg(\sum_{i,j=1}^N \kappa^*_{ij}(x) \frac{\partial v(x)}{\partial x_i} \frac{\partial \psi(x)}{\partial x_j} \Bigg) \mathrm{d}x = 0, \quad \textit{for all } \Phi \in L^2\big(\Omega; L^2_\#(Y)\big),\ \psi \in H^1_0(\Omega), \tag{18}$$

and $U(0,x,y) = u_*(x)$, *where the* effective coefficients *are given by*

$$\kappa^*_{ij}(x) = \int_Y \kappa(x,y)(\nabla_y \omega_i(x,y) + \boldsymbol{e}_i) \cdot (\nabla_y \omega_j(x,y) + \boldsymbol{e}_j)\mathrm{d}y.$$

3.2.2. *Summary*

The strong formulation of the system (18) is

$$\phi(x,y) \frac{\partial U(t,x,y)}{\partial t} + \frac{1}{\tau(x,y)} \big(U(t,x,y) - v(t,x)\big) = 0, \quad y \in Y, \tag{19a}$$

$$\int_Y \frac{1}{\tau(x,y)} \big(v(t,x) - U(t,x,y)\big) \mathrm{d}y - \nabla \cdot \kappa^* \nabla v(t,x) = 0. \tag{19b}$$

This extends (8) from ε-periodic coefficients to those which depend also on the slow variable, $x \in \Omega$.

Consider the case of a binary medium in which each of ϕ_j, $\tau_j \in L^\infty(\Omega)$ is independent of $y \in Y_j$. Then the same is true of

$$U(t,x,y) \equiv \begin{cases} U_1(t,x), & y \in Y_1\,, \\ U_2(t,x), & y \in Y_2\,, \end{cases}$$

and we have the homogenized *binary system*

$$|Y_1|\phi_1(x)\frac{\partial U_1(t,x)}{\partial t}+\frac{|Y_1|}{\tau_1(x)}\big(U_1(t,x)-v(t,x)\big)=0, \tag{20a}$$

$$|Y_2|\phi_2(x)\frac{\partial U_2(t,x)}{\partial t}+\frac{|Y_2|}{\tau_2(x)}\big(U_2(t,x)-v(t,x)\big)=0, \tag{20b}$$

$$\frac{|Y_1|}{\tau_1(x)}\big(v(t,x)-U_1(t,x)\big)+\frac{|Y_2|}{\tau_2(x)}\big(v(t,x)-U_2(t,x)\big)-\nabla\cdot\kappa^*\nabla v(t,x)=0. \tag{20c}$$

This is the binary medium analogue of (9).

3.3. *Homogenization of the highly heterogeneous case*

3.3.1. *The two-scale limit*

Here the permeability is given by (13), so we obtain weaker *a priori* estimates and correspondingly weaker convergence results.

LEMMA 3.3 *For each* $\varepsilon>0$, *let* $u^\varepsilon(\cdot)$, $v^\varepsilon(\cdot)$ *denote the unique solution to the pseudoparabolic* ε*-problem* (11). *These satisfy the estimates*

$$\|u^\varepsilon\|_{L^2((0,T)\times\Omega)}+\|v^\varepsilon\|_{L^2((0,T)\times\Omega)}+\|v^\varepsilon\|_{L^2((0,T);H^1(\Omega_1^\varepsilon))}+\|\varepsilon v^\varepsilon\|_{L^2((0,T);H^1(\Omega_2^\varepsilon))}\le C,$$

so there exist

(i) *a function* U *in* $L^2((0,T)\times\Omega;L^2_\#(Y))$,
(ii) *a function* v_1 *in* $L^2((0,T);H^1_0(\Omega))$,
(iii) *a pair of functions* V_j *in* $L^2((0,T)\times\Omega;H^1_\#(Y_j)/\mathbb{R})$, $j=1,2$,

and a subsequence, hereafter denoted by u^ε, v^ε, *which two-scale converges as follows*:

$$u^\varepsilon(t,x)\overset{2}{\to}U(t,x,y), \tag{21a}$$

$$\chi_1^\varepsilon v^\varepsilon\overset{2}{\to}\chi_1(y)v_1(t,x), \tag{21b}$$

$$\chi_1^\varepsilon\nabla v^\varepsilon\overset{2}{\to}\chi_1(y)[\nabla v_1(t,x)+\nabla_y V_1(t,x,y)], \tag{21c}$$

$$\chi_2^\varepsilon v^\varepsilon\overset{2}{\to}\chi_2(y)V_2(t,x,y), \tag{21d}$$

$$\varepsilon\chi_2^\varepsilon\nabla v^\varepsilon\overset{2}{\to}\chi_2(y)\nabla_y V_2(t,x,y). \tag{21e}$$

The function V_2 satisfies $\gamma_2(V_2(t,x,y)=v_1(x)$, $y\in\Gamma$. (See [36].) These suggest use of the corresponding test functions

$$\varphi(x)=\Phi(x,x/\varepsilon),\quad \psi(x)=\begin{cases}\psi_1(x)+\varepsilon\Psi_1(x,x/\varepsilon) & : x\in\Omega_1^\varepsilon,\\ \Psi_2(x,x/\varepsilon)+\varepsilon\Psi_1(x,x/\varepsilon) & : x\in\Omega_2^\varepsilon,\end{cases}$$

where $\psi_1\in H^1_0(\Omega)$, $\Phi,\Psi_1\in C_0^\infty(\Omega;C_\#^\infty(Y))$ and $\Psi_2\in C_0^\infty(\Omega;C_\#^\infty(Y_2))$ with $\gamma_2\Psi_2(x,\cdot)=\psi_1(x)$ on Γ. Setting these in (11) yields

$$\int_\Omega\Bigg(\phi^\varepsilon\frac{\partial u^\varepsilon(t,x)}{\partial t}\Phi(x,x/\varepsilon)+\frac{\chi_1^\varepsilon(x)}{\tau_1^\varepsilon(x)}\big(u^\varepsilon(t,x)-v^\varepsilon(t,x)\big)\big(\Phi(x,x/\varepsilon)-(\psi_1(x)+\varepsilon\Psi_1(x,x/\varepsilon))\Bigg)$$
$$+\frac{\chi_2^\varepsilon(x)}{\tau_2^\varepsilon(x)}\big(u^\varepsilon(t,x)-v^\varepsilon(t,x)\big)\big(\Phi(x,x/\varepsilon)-(\Psi_2(x,x/\varepsilon)+\varepsilon\Psi_1(x,x/\varepsilon))\big)$$

$$+ \chi_1^\varepsilon(x)\kappa_1^\varepsilon(x)\nabla v^\varepsilon(t,x)\cdot\nabla(\psi_1(x)+\varepsilon\Psi_1(x,x/\varepsilon))$$
$$+ \chi_2^\varepsilon(x)\kappa_2^\varepsilon(x)\varepsilon\nabla v^\varepsilon(t,x)\cdot\varepsilon\nabla(\Psi_2(x,x/\varepsilon)+\varepsilon\Psi_1(x,x/\varepsilon))\big)\mathrm{d}x = 0.$$

Take the limit as $\varepsilon \to 0$ to obtain the *two-scale limit system*

$$\begin{aligned}
&\int_\Omega\int_Y \bigg(\phi(x,y)\frac{\partial U(t,x,y)}{\partial t}\Phi(x,y) + \frac{\chi_1(y)}{\tau_1(x,y)}\big(U(t,x,y)-v_1(t,x)\big)\big(\Phi(x,y)-\psi_1(x)\big)\\
&\quad + \frac{\chi_2(y)}{\tau_2(x,y)}\big(U(t,x,y)-V_2(t,x)\big)\big(\Phi(x,y)-\Psi_2(x,y)\big)\\
&\quad + \chi_1(y)\kappa_1(x,y)\big(\nabla v_1(t,x)+\nabla_y V_1(t,x,y)\big)\cdot\big(\nabla\psi_1(x)+\nabla_y\Psi_1(x,y)\big)\\
&\quad + \chi_2(y)\kappa_2(x,y)\nabla_y V_2(t,x,y)\cdot\nabla_y\Psi_2(x,y)\bigg)\mathrm{d}y\,\mathrm{d}x = 0, \qquad (22)
\end{aligned}$$

for all Φ, ψ_1, Ψ_1, Ψ_2 as above, and $U(0,x,y)=u_*(x)$. The uniqueness of the solution to the corresponding initial-value problem shows that the original sequence converges to it.

As before, we can represent each $V_1(t,x,\cdot)$ by a *cell problem*: define $\omega_k(x,y)$ by

$$\begin{aligned}
&\omega_k\in L^2(\Omega;H^1_\#(Y_1)) : \int_{Y_1}\kappa_1(x,y)\big(\nabla_y\omega_k(x,y)+\boldsymbol{e}_k\big)\cdot\nabla_y\Psi_1(x,y)\mathrm{d}y = 0\\
&\quad\text{for all } \Psi_1\in L^2(\Omega;H^1_\#(Y_1)),\ \int_{Y_1}\omega_k(x,y)\mathrm{d}y = 0. \qquad (23)
\end{aligned}$$

Then we have $V_1(t,x,y)=\sum_{i=1}^N \frac{\partial v_1(t,x)}{\partial x_i}\omega_i(x,y)$, and we specify the test functions $\Psi_1(x,y)=\sum_{j=1}^N\frac{\partial\psi_1(x)}{\partial x_j}\omega_j(x,y)$ above to obtain

THEOREM 3.4 *The limits U, v_1, V_2 in Lemma* 3.3 *are the solution of the* partially homogenized *pseudoparabolic system*

$$\begin{aligned}
&U\in H^1\big((0,T);L^2\big(\Omega;L^2_\#(Y)\big)\big),\quad v_1\in L^2\big((0,T);H^1_0(\Omega)\big),\\
&V_2\in L^2\big((0,T)\times\Omega;H^1_\#(Y_2)\big)\quad \textit{with } \gamma V_2|_\Gamma = v_1 :\\
&\int_\Omega\int_Y\big(\phi(x,y)\frac{\partial U(t,x,y)}{\partial t}\Phi(x,y)+\frac{\chi_1(y)}{\tau_1(x,y)}\big(U(t,x,y)-v_1(t,x)\big)\big(\Phi(x,y)-\psi_1(x)\big)\\
&\quad+\frac{\chi_2(y)}{\tau_2(x,y)}\big(U(t,x,y)-V_2(t,x,y)\big)\big(\Phi(x,y)-\Psi_2(x,y)\big)\big)\mathrm{d}y\,\mathrm{d}x\\
&\quad+\int_\Omega\bigg(\sum_{i,j=1}^N\kappa^*_{ij}(x)\frac{\partial v_1(t,x)}{\partial x_i}\frac{\partial\psi_1(x)}{\partial x_j}\bigg)\mathrm{d}x\\
&\quad+\int_\Omega\int_{Y_2}\kappa_2(x,y)\nabla_y V_2(t,x,y)\cdot\nabla_y\Psi_2(x,y)\mathrm{d}y\,\mathrm{d}x = 0,\\
&\textit{for all } \Phi\in L^2\big(\Omega;L^2_\#(Y)\big),\ \psi_1\in H^1_0(\Omega),\quad \Psi_2\in L^2\big(\Omega;H^1_\#(Y_2)\big)\ \textit{with } \gamma\Psi_2|_\Gamma=\psi_1, \qquad (24)
\end{aligned}$$

and $U(0,x,y)=u_*(x)$, *where the* effective coefficients *are given by*

$$\kappa^*_{ij}(x)=\int_{Y_1}\kappa_1(x,y)(\nabla_y\omega_i(x,y)+\boldsymbol{e}_i)\cdot(\nabla_y\omega_j(x,y)+\boldsymbol{e}_j)\mathrm{d}y.$$

Next we separate the components of the system. First write the part over Y_2

$$\phi_2(x,y)\frac{\partial U(t,x,y)}{\partial t}+\frac{1}{\tau_2(x,y)}\big(U(t,x,y)-V_2(t,x,y)\big)=0 \quad \text{and}$$
$$\frac{1}{\tau_2(x,y)}\big(V_2(t,x,y)-U(t,x,y)\big)-\nabla_y\cdot\kappa_2(x,y)\nabla_y V_2(t,x,y)=0, \quad y\in Y_2,$$
$$\gamma V_2(t,x,y)=v_1(t,x),\ \ y\in\Gamma,$$

and then substitute these back into (24) and use Stokes' theorem on Y_2 to get

$$\int_\Omega\int_{Y_1}\left(\phi_1(x,y)\frac{\partial U(t,x,y)}{\partial t}\Phi(x,y)+\frac{1}{\tau_1(x,y)}\big(U(t,x,y)-v_1(t,x)\big)\big(\Phi(x,y)-\psi_1(x)\big)\right)\mathrm{d}y\,\mathrm{d}x$$
$$+\int_\Omega\left(\sum_{i,j=1}^N\kappa^*_{ij}(x)\frac{\partial v_1(t,x)}{\partial x_i}\frac{\partial\psi_1(x)}{\partial x_j}\right)\mathrm{d}x+\int_\Omega\int_\Gamma\kappa_2(x,y)\nabla_y V_2(t,x,y)\cdot\nu\,\mathrm{d}S\psi_1(x)\mathrm{d}x=0.$$

3.3.2. *Summary*

The strong form of the partially homogenized system (24) is

$$\phi_1(x,y)\frac{\partial U(t,x,y)}{\partial t}+\frac{1}{\tau_1(x,y)}\big(U(t,x,y)-v_1(t,x)\big)=0, \quad y\in Y_1,$$
$$\int_{Y_1}\frac{1}{\tau_1(x,y)}\big(v_1(t,x)-U(t,x,y)\big)\mathrm{d}y-\nabla\cdot\kappa^*\nabla v_1(t,x)$$
$$+\int_\Gamma\kappa_2(x,y)\nabla_y V_2(t,x,y)\cdot\nu\,\mathrm{d}S=0, \tag{25a}$$

and for each $x\in\Omega$,

$$\phi_2(x,y)\frac{\partial U(t,x,y)}{\partial t}+\frac{1}{\tau_2(x,y)}\big(U(t,x,y)-V_2(t,x,y)\big)=0,$$
$$\frac{1}{\tau_2(x,y)}\big(V_2(t,x,y)-U(t,x,y)\big)-\nabla_y\cdot\kappa_2(x,y)\nabla_y V_2(t,x,y)=0, \quad y\in Y_2,$$
$$\gamma V_2(t,x,y)=v_1(t,x), \quad y\in\Gamma. \tag{25b}$$

Note the coupling in the system: the function v_1 from (25a) is input to (25b), and the total flux from (25b) is the distributed source in (25a).

Suppose now that ϕ_1 and τ_1 are independent of $y\in Y_1$, and therefore so also is $u(t,x)\equiv U(t,x,y)$, $y\in Y_1$. Then (25a) is homogenized:

$$\phi_1(x)\frac{\partial u(t,x)}{\partial t}+\frac{1}{\tau_1(x)}\big(u(t,x)-v_1(t,x)\big)=0,$$
$$\frac{1}{\tau_1(x)}\big(v_1(t,x)-u(t,x)\big)-\frac{1}{|Y_1|}\nabla\cdot\kappa^*\nabla v_1(t,x)$$
$$+\frac{1}{|Y_1|}\int_\Gamma\kappa_2(x,y)\nabla_y V_2(t,x,y)\cdot\nu\,\mathrm{d}S=0, \tag{26a}$$

and for each $x \in \Omega$,

$$\phi_2(x,y)\frac{\partial U(t,x,y)}{\partial t} + \frac{1}{\tau_2(x,y)}\big(U(t,x,y) - V_2(t,x,y)\big) = 0,$$
$$\frac{1}{\tau_2(x,y)}\big(V_2(t,x,y) - U(t,x,y)\big) - \nabla_y \cdot \kappa_2(x,y)\nabla_y V_2(t,x,y) = 0, \quad y \in Y_2,$$
$$\gamma V_2(t,x,y) = v_1(t,x), \;\; y \in \Gamma. \tag{26b}$$

Note that (26a) is the upscaled fissured medium system, and (26b) is the local fissured medium system at each $x \in \Omega$.

3.4. *Vanishing time-delay*

Suppose that $\tau_1^{\varepsilon} = o(\varepsilon)$ in the classical system (12). Then $\|u^{\varepsilon} - v^{\varepsilon}\|_{L^2(Y1)} = o(\varepsilon^{1/2})$, so in the limit we obtain $U(t,x,y)|_{Y_1} = v(t,x)$. Choose test functions $\Phi(x,y) = \psi(x) + \varepsilon\Psi(x,y)$ in the weak form, with the equations added, and take the limit to get the homogenized mixed parabolic–pseudoparabolic system (compare (20))

$$\phi_1^*(x)\,\frac{\partial v(t,x)}{\partial t} - \nabla \cdot \kappa^* \nabla v(t,x) + \int_{Y_2} \frac{1}{\tau_2(x,y)}\big(v(t,x) - U(t,x,y)\big)\mathrm{d}y = 0, \tag{27a}$$

$$\phi_2(x,y)\frac{\partial U(t,x,y)}{\partial t} + \frac{1}{\tau_2(x,y)}\big(U(t,x,y) - v(t,x)\big) = 0, \quad y \in Y_2, \tag{27b}$$

with effective porosity $\phi_1^*(x) = \int_{Y_1} \phi_1(x,y)\mathrm{d}y$. Then (27a) is a parabolic equation with a memory term determined by (27b). See Peszyńska [37] for results and additional references to memory functionals in parabolic equations; also see [31] for first-order kinetic models.

Suppose that $\tau_1^{\varepsilon} = o(\varepsilon)$ in the highly heterogeneous system (14). Then $U(t,x,y)|_{Y_1} = v_1(t,x)$ and instead of the system (25a) we obtain the homogenized parabolic equation

$$\phi_1^*(x)\frac{\partial v_1(t,x)}{\partial t} - \nabla \cdot \kappa^* \nabla v_1(t,x) + \int_{\Gamma} \kappa_2(x,y)\nabla_y V_2(x,y) \cdot \nu \, \mathrm{d}S = 0. \tag{28a}$$

Suppose that $\tau_2^{\varepsilon} = o(\varepsilon)$ in (14). Then $U(t,x,y)|_{Y_2} = V_2(t,x,y)$ and instead of the system (25b) we obtain the local parabolic equations

$$\phi_2(x,y)\frac{\partial V_2(t,x,y)}{\partial t} - \nabla_y \cdot \kappa_2(x,y)\nabla_y V_2(t,x,y) = 0, \quad y \in Y_2, \tag{28b}$$

$$\gamma V_2(x,y) = v_1(x), \quad y \in \Gamma. \tag{28c}$$

If both vanish in the limit, then we recover the Arbogast–Douglas–Hornung [34] double-porosity model (28) of a fractured porous medium.

4. Partially saturated flow with dynamic capillary pressure

4.1. *Microscopic equations*

Let us consider the unsaturated flow in a highly heterogeneous medium Ω with the ε-periodic structure of Section 3. Here Y_2 is the matrix block and Y_1 is the surrounding fracture domain. Each of the subdomains Ω_i^ε is characterized by a rock permeability tensor K^i, a porosity ϕ^i, the relative permeability $k_w^i(u^i)$ and the capillary pressure function $P_c^i(u^i)$. Here u^i denotes the saturation in Ω_i^ε. The fluid has constant viscosity μ and density ρ. It has been observed that the dynamic effects in capillary pressure equilibrium are much more significant in media with low conductivity than those with high conductivity, so we assume that the unsaturated flow can be locally described by the original Richards equation (1) in the fracture domain Ω_1^ε and by the pseudoparabolic Richards equation (2) in the porous matrix Ω_2^ε:

$$\phi^1 \frac{\partial u^1}{\partial t} + \nabla \cdot \frac{1}{\mu} K^1 k_w^1(u^1) \nabla \big(P_c^1(u^1) - \rho G D(x) \big) = 0, \quad x \in \Omega_1^\varepsilon, \tag{29a}$$

$$\phi^2 \frac{\partial u^2}{\partial t} + \varepsilon^2 \nabla \cdot \frac{1}{\mu} K^2 k_w^2(u^2) \nabla \left(P_c^2(u^2) - \tau \frac{\partial u^2}{\partial t} - \rho G D(x) \right) = 0, \quad x \in \Omega_2^\varepsilon. \tag{29b}$$

Hereafter for simplicity we set depth $D(x) = x_3$. Introduce $p^i = -P_c^i(u^i)$, $u^i = \alpha^i(p^i)$, $\kappa^i(p^i) = \frac{1}{\mu} K^i k_w^i(u^i)$, so $\alpha^i(\cdot)$ is inverse to $-P_c^i(\cdot)$, and Equations (29a) and (29b) can be rewritten as

$$\phi^1 \frac{\partial \alpha^1(p^1)}{\partial t} - \nabla \cdot \kappa^1(p^1) \big(\nabla p^1 + \rho G \mathbf{e}_3 \big) = 0, \tag{30a}$$

$$\phi^2 \frac{\partial \alpha^2(p^2)}{\partial t} - \varepsilon^2 \nabla \cdot \kappa^2(p^2) \big(\nabla p^2 + \tau \nabla \frac{\partial \alpha^2(p^2)}{\partial t} + \rho G \mathbf{e}_3 \big) = 0, \tag{30b}$$

and are subject to the interface conditions

$$p^1 = p^2 + \tau \frac{\partial \alpha^2(p^2)}{\partial t}, \quad x \in \Gamma^\varepsilon, \tag{30c}$$

$$\kappa^1(p^1) \big(\nabla p^1 + \rho G \mathbf{e}_3 \big) \cdot \nu = \varepsilon^2 \kappa^2(p^2) \left(\nabla p^2 + \tau \nabla \frac{\partial \alpha^2(p^2)}{\partial t} + \rho G \mathbf{e}_3 \right) \cdot \nu, \quad x \in \Gamma^\varepsilon, \tag{30d}$$

where ν is the unit normal on Γ^ε out of Ω_2^ε, and the initial conditions are

$$p^i(x, 0) = p_*^i(x), \quad x \in \Omega_i^\varepsilon, \quad i = 1, 2. \tag{30e}$$

4.2. *Asymptotic expansions*

We shall expand the solution in powers of ε in the form

$$p^i(t, x) = p_0^i(t, x, y) + \varepsilon p_1^i(t, x, y) + \varepsilon^2 p_2^i(t, x, y) + \cdots, \quad i = 1, 2, \tag{31}$$

where p_k^i are Y-periodic in $y \in Y_i$ for $k = 0, 1, 2, \ldots$. Following methods of [38,39], we develop various nonlinear quantities $\theta(p)$ in powers of ε by

$$\begin{aligned}
\theta(p^i) &= \theta(p_0^i) + \theta'(p_0^i)(p^i - p_0^i) + \theta''(p_0^i)(p^i - p_0^i)^2/2 + \cdots \\
&= \theta(p_0^i) + \theta'(p_0^i)(\varepsilon p_1^i + \varepsilon^2 p_2^i + \cdots) + \theta''(p_0^i)(\varepsilon p_1^i + \varepsilon^2 p_2^i + \cdots)^2/2 + \cdots \\
&= \theta(p_0^i) + \varepsilon \theta'(p_0^i) p_1^i + \varepsilon^2 (\theta'(p_0^i) p_2^i + \theta''(p_0^i)(p_1^i)^2/2) + \cdots \\
&= \theta(p_0^i) + \varepsilon \hat{\theta}_1^i + \varepsilon^2 \hat{\theta}_2^i + \cdots, \text{ for appropriate } \hat{\theta}_1^i, \hat{\theta}_2^i, \ldots, \quad i = 1, 2.
\end{aligned}$$

Now, we substitute (31) into the microscopic model and expand the gradient according to the relation $\nabla = \nabla_x + \frac{1}{\varepsilon}\nabla_y$. Then, we collect terms by powers of ε. From (30a) we obtain three equations for the combined ε^{-2}, ε^{-1} and ε^0 terms when $x \in \Omega$, $y \in Y_1$:

$$\nabla_y \cdot \left(\kappa^1(p_0^1)\nabla_y p_0^1\right) = 0, \tag{32a}$$

$$\nabla_y \cdot \left(\kappa^1(p_0^1)(\nabla_x p_0^1 + \nabla_y p_1^1 + \rho G\mathbf{e}_3) + \hat{\kappa}_1^1 \nabla_y p_0^1\right) + \nabla_x \cdot \left(\kappa^1(p_0^1)\nabla_y p_0^1\right) = 0, \tag{32b}$$

$$\begin{aligned}\phi^1 \frac{\partial \alpha^1(p_0^1)}{\partial t} &- \nabla_x \cdot \left(\kappa^1(p_0^1)(\nabla_x p_0^1 + \nabla_y p_1^1 + \rho G\mathbf{e}_3) + \hat{\kappa}_1^1 \nabla_y p_0^1\right) - \nabla_y \cdot \left(\kappa^1(p_0^1)(\nabla_x p_1^1 + \nabla_y p_2^1)\right. \\ &\left. + \hat{\kappa}_1^1(\nabla_x p_0^1 + \nabla_y p_1^1 + \rho G\mathbf{e}_3) + \hat{\kappa}_2^1 \nabla_y p_0^1\right) = 0.\end{aligned} \tag{32c}$$

First, equations for ε^0 from (30b) and (30c) for $x \in \Omega$ are

$$\phi^2 \frac{\partial \alpha^2(p_0^2)}{\partial t} - \nabla_y \cdot \kappa^2(p_0^2)\nabla_y\left(p_0^2 + \tau \frac{\partial \alpha^2(p_0^2)}{\partial t}\right) = 0, \quad y \in Y_2, \tag{33a}$$

$$p_0^2 + \tau \frac{\partial \alpha^2(p_0^2)}{\partial t} = p_0^1, \quad y \in \Gamma. \tag{33b}$$

The ε^{-1}, ε^0 and ε^1 equations of (30d) for $x \in \Omega$, $y \in \Gamma$ are

$$\kappa^1(p_0^1)\nabla_y p_0^1 \cdot \nu = 0, \tag{34a}$$

$$\left(\kappa^1(p_0^1)(\nabla_x p_0^1 + \nabla_y p_1^1 + \rho G\mathbf{e}_3) + \hat{\kappa}_1^1 \nabla_y p_0^1\right) \cdot \nu = 0, \tag{34b}$$

$$\begin{aligned}&\left(\kappa^1(p_0^1)(\nabla_x p_1^1 + \nabla_y p_2^1) + \hat{\kappa}_1^1(\nabla_x p_0^1 + \nabla_y p_1^1 + \rho G\mathbf{e}_3) + \hat{\kappa}_2^1 \nabla_y p_0^1\right) \cdot \nu \\ &\quad = \kappa^2(p_0^2)\nabla_y\left(p_0^2 + \tau \frac{\partial \alpha^2(p_0^2)}{\partial t}\right) \cdot \nu.\end{aligned} \tag{34c}$$

Equations (32a) and (34a) form an elliptic system for p_0^1 in terms of y. Since its solution is independent of y, it follows that $p_0^1 = p_0^1(t, x)$, so all terms with $\nabla_y p_0^1$ vanish.

Equations (32b) and (34b) form a linear elliptic system in y whose solution p_1^1 can be represented in terms of p_0^1. Define $\omega_j(y)$ for $j = 1, 2, 3$ as the Y-periodic solution of the cell problem (compare (23))

$$\nabla_y^2 \omega_j = 0 \quad \text{for } y \in Y_1, \tag{35a}$$

$$\nabla_y \omega_j \cdot \nu = -\mathbf{e}_j \cdot \nu = -\nu_j \quad \text{for } y \in \Gamma. \tag{35b}$$

Then from Equation (32b) we obtain the representation

$$p_1^1(x, y, t) = \sum_{j=1}^{3} \omega_j(y)\left(\frac{\partial p_0^1}{\partial x_j}(x, t) + \rho G \delta_{3j}\right). \tag{36}$$

Now, we locally average (32c) by integrating it over Y_1 to remove the y-variable and get

$$\begin{aligned}
|Y_1|\phi^1 \frac{\partial \alpha^1(p_0^1)}{\partial t} - \int_{Y_1} \nabla_x \cdot \kappa^1(p_0^1)(\nabla_x p_0^1 + \nabla_y p_1^1 + \rho G \mathbf{e}_3)\mathrm{d}y \\
= \int_{Y_1} \nabla_y \cdot \Big(\kappa^1(p_0^1)(\nabla_x p_1^1 + \nabla_y p_2^1) + \hat{\kappa}_1^1(\nabla_x p_0^1 + \nabla_y p_1^1 + \rho G \mathbf{e}_3)\Big)\mathrm{d}y. \qquad (37)
\end{aligned}$$

Apply the divergence theorem to the second integral above, use (34c), make a second application of the divergence theorem, and use (33a) to obtain

$$\begin{aligned}
&\int_{Y_1} \nabla_y \cdot \Big(\kappa^1(p_0^1)(\nabla_x p_1^1 + \nabla_y p_2^1) + \hat{\kappa}_1^1(\nabla_x p_0^1 + \nabla_y p_1^1 + \rho G \mathbf{e}_3)\Big)\mathrm{d}y \\
&\quad = \int_{\partial Y_1} \Big(\kappa^1(p_0^1)(\nabla_x p_1^1 + \nabla_y p_2^1) + \hat{\kappa}_1^1(\nabla_x p_0^1 + \nabla_y p_1^1 + \rho G \mathbf{e}_3)\Big) \cdot \nu \,\mathrm{d}S \\
&\quad = -\int_{\partial Y_2} \kappa^2(p_0^2)\nabla_y\left(p_0^2 + \tau \frac{\partial \alpha^2(p_0^2)}{\partial t}\right) \cdot \nu \,\mathrm{d}S \\
&\quad = -\int_{Y_2} \nabla_y \cdot \kappa^2(p_0^2)\nabla_y\left(p_0^2 + \tau \frac{\partial \alpha^2(p_0^2)}{\partial t}\right)\mathrm{d}y \\
&\quad = -\int_{Y_2} \phi^2 \frac{\partial \alpha^2(p_0^2)}{\partial t}\,\mathrm{d}y.
\end{aligned}$$

The first integral in (37) is evaluated using (36). Its integrand becomes (with implied summation)

$$\begin{aligned}
&\nabla_x \cdot \kappa^1(p_0^1)(\nabla_x p_0^1 + \nabla_y p_1^1 + \rho G \mathbf{e}_3) \\
&\quad = \frac{\partial}{\partial x_k}\left[\kappa^1(p_0^1)\left(\frac{\partial p_0^1}{\partial x_k} + \frac{\partial \omega_j}{\partial y_k}\left(\frac{\partial p_0^1}{\partial x_j} + \rho G \delta_{3j}\right) + \rho G \delta_{3k}\right)\right] \\
&\quad = \frac{\partial}{\partial x_k}\left[\kappa^1(p_0^1)\left(\frac{\partial \omega_j}{\partial y_k} + \delta_{jk}\right)\left(\frac{\partial p_0^1}{\partial x_j} + \rho G \delta_{3j}\right)\right].
\end{aligned}$$

Define the effective fracture permeability tensor $K^* = \{K_{jk}^*\}$ and the macroscopic fracture porosity ϕ^* by

$$K_{jk}^* = K^1 \int_{Y_1}\left(\frac{\partial \omega_j}{\partial y_k} + \delta_{jk}\right)\mathrm{d}y, \quad \phi^* = |Y_1|\phi^1.$$

We also define

$$\kappa^*(p) = \frac{1}{\mu} K^* k_w^1(\alpha^1(p)).$$

Then, the equation for p_0^1 is

$$\phi^* \frac{\partial \alpha^1(p_0^1)}{\partial t} - \nabla_x \cdot \kappa^*(p_0^1)(\nabla_x p_0^1 + \rho G \mathbf{e}_3) = -\int_{Y_2} \phi^2 \frac{\partial \alpha^2(p_0^2)}{\partial t}\,\mathrm{d}y.$$

4.3. *Summary*

The complete system of flow equations for $p_0^1(x,t)$, $p_0^2(x,y,t)$ is given by

$$\phi^* \frac{\partial \alpha^1(p_0^1)}{\partial t} + \int_{Y_2} \phi^2 \frac{\partial \alpha^2(p_0^2)}{\partial t} \, \mathrm{d}y - \nabla_x \cdot \kappa^*(p_0^1)(\nabla_x p_0^1 + \rho G \mathbf{e}_3) = 0, \quad x \in \Omega, \tag{38a}$$

$$\phi^2 \frac{\partial \alpha^2(p_0^2)}{\partial t} - \nabla_y \cdot \kappa^2(p_0^2) \nabla_y \left(p_0^2 + \tau \frac{\partial \alpha^2(p_0^2)}{\partial t} \right) = 0, \quad y \in Y_2, \tag{38b}$$

$$p_0^2 + \tau \frac{\partial \alpha^2(p_0^2)}{\partial t} = p_0^1, \quad y \in \Gamma, \tag{38c}$$

$$p_0^1(x,0) = p_{\mathrm{init}}^1(x), \quad p_0^2(x,y,0) = p_{\mathrm{init}}^2(x), \quad y \in Y_2. \tag{38d}$$

This is the double-porosity model consisting of the upscaled equation (38a) together with the distributed family of local boundary-value problems (38b), (38c) for $x \in \Omega$. It is a nonlinear analogue of the system (28a), (26b).

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