MICRO-STRUCTURE MODELS OF POROUS MEDIA

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1. Introduction

Every attempt to exactly model laminar flow through highly inhomogeneous media, e.g., fissured or layered media, leads to very singular problems of partial differential equations with rapidly oscillating coefficients. Various methods of averaging will yield corresponding types of *double porosity* models, and we shall describe some of these.

As a first approximation to flow in a region G which consists of such a composite of two finely interspersed materials, one can consider *averaged solutions*, one for each material and both defined at every point $x \in G$. This leads to a pair of partial differential equations, one identified with each of the two components, and a coupling term that describes the flow across the interface between these components. The values at each point x of the two dependent variables in this system (the solutions) have been obtained by averaging in the respective media over a generic neighborhood, which is located at $x \in G$ and is sufficiently large to contain a representative sample of each component. Since the two components are treated symmetrically in the resulting system of two parabolic partial differential equations, such a double porosity model is said to be of *parallel flow* type. Although appropriate for many situations, this symmetric treatment of the two components can be a real limitation. For a fissured medium, for example, such a representation is particularly restrictive, since the porous and permeable cells within the structure have flow properties radically different from those of the surrounding highly developed system of fissures. Moreover the geometry of the individual cells and the corresponding interface are lost in the averaging process leading to such models. For layered media similar remarks apply, and these could be supplemented by non-isotropic considerations. In general, essential limitations of the parallel flow models are the suppression of the geometry of the cells and their corresponding interfaces on which the coupling occurs and the lack of any distinction between the space and time scales of the two components of the medium.

In order to overcome these deficiencies, we consider the class of double porosity models which we call *distributed microstructure* models. These are known in many

cases to be the limit (by homogenization) as the scale of the inhomogeneity tends to zero, and they provide a way to represent a continuous distribution of cells with prescribed geometry. At each point $x \in G$ there is given a representative model cell, G_x, the flow within each such cell is described independently by an initial-boundary-value problem, and the solution of this local problem on G_x is coupled on the boundary Γ_x to the value near x of the solution of the flow problem in the global region G. Thus, we have a continuum of partial differential equations to describe the local flow on the micro-scale, one at each point $x \in G$, and these are each coupled to a single partial differential equation in G for the macro-scale flow. Such *continuous* models represent a good approximation of the real but discrete (and very singular) case of a finite but very large number of cells. This concept was introduced in the early 1930's (see Section 3), and it has arisen repeatedly in a variety of applications which we mention below.

We shall illustrate these two classes of double porosity models with examples of single phase flow in various types of fissured media. The class of parallel models is briefly treated in Section 2 for motivation and orientation. The distributed microstructure models are introduced in Section 3 by a simple and classical example, and we discuss this class in much more detail. We use both types to describe first the example of a *totally fissured medium* in which the individual cells are completely isolated from each other by the fissure system. In this situation the cells act only as storage sites, and there is no direct diffusion from cell to cell within the matrix. Then we introduce corresponding models for the more general example of a *partially fissured medium* in which there is some fluid flow directly through the cell structure. This flow through the cells is driven indirectly by the pressure gradient in the fissure system, and it contributes an additional component to the velocity field in the fissure system which we call the *secondary flux*. We also display the form of the functional differential equations by which these problems can be represented; here the family of local problems is replaced by convolution terms. We shall describe some typical results on the theory of such systems in Section 4. This is illustrated in the simplest case as an application of continuous direct sums of Hilbert spaces which arise rather naturally as the energy or state spaces for the corresponding (stationary) variational or (temporal) dynamic problems. In Section 5 we indicate in some detail how the distributed microstructure models arise by way of homogenization from exact but singular models, partial differential equations with highly periodic coefficients or geometry, and then we close with some additional remarks.

2. Parallel Flow Models

The classical example of a *parallel flow model* for single phase flow in a composite medium is the parabolic system

$$\begin{aligned} \frac{\partial}{\partial t}(au_1) - \vec{\nabla} \cdot (A\vec{\nabla}u_1) + \frac{1}{\delta}(u_1 - u_2) &= f_1 \\ \frac{\partial}{\partial t}(bu_2) - \vec{\nabla} \cdot (B\vec{\nabla}u_2) + \frac{1}{\delta}(u_2 - u_1) &= f_2 \ . \end{aligned} \tag{2.1}$$

discussed in [8] for which u_1 represents the density of fluid in the first material and u_2 the density in the second. The coefficients $a(x)$ and $A(x)$ are porosity and

permeability of the first material, respectively, while $b(x)$ and $B(x)$ are corresponding properties of the second material. The first equation quantifies the rate of flow in the first component of the composite, and the second equation quantifies the corresponding flow rate in the second. Both of these equations are to be understood macroscopically; that is, they were obtained by averaging over a generic neighborhood sufficiently large to contain contributions from each component. The third term in each equation is an attempt to quantify the exchange of fluid between the two components. See [52] for a corresponding system which describes heat conduction in such a composite medium.

Totally Fissured Media. A *fissured medium* consists of a matrix of porous and permeable material cells through which is intertwined a highly developed system of fissures. The bulk of the flow occurs in the highly permeable fissure system, and most of the storage of fluid is in the matrix of cells which accounts for almost all of the total volume. One approach to constructing a model of such a medium is to regard the fissure system as the first component and the cell matrix as the second component of a general composite by adjusting the coefficients in (2.1) appropriately. These fissured media characteristics are modeled by choosing very small values for the coefficients $a(x)$ and $B(x)$ in (2.1). Since one component is essentially responsible for storage and the other for transport, the distributed exchange of fluid between the two components is of fundamental importance. The parameter δ represents the resistance of the medium to this exchange. (When $\delta = \infty$, no exchange flow is possible, and the system is completely decoupled.) An alternative interpretation is that $1/\delta$ represents the degree of fissuring in the medium. (When the degree of fissuring is infinite, the exchange flow encounters no resistance and $u_1 = u_2$.)

In order to specialize the system (2.1) to a totally fissured medium in which the individual cells are isolated from each other, one sets $B = 0$, because there is no direct flow through the matrix of cells; only an indirect exchange occurs by way of the fissures. Thus, the condition $B = 0$ corresponds to a *totally fissured* medium in which each cell of the matrix is isolated from adjacent cells by the fissure system. The resulting system of parabolic-ordinary differential equations

$$\frac{\partial}{\partial t}(au_1) - \vec{\nabla} \cdot (A\vec{\nabla}u_1) + \frac{1}{\delta}(u_1 - u_2) = f \tag{2.2.a}$$

$$\frac{\partial}{\partial t}(bu_2) + \frac{1}{\delta}(u_2 - u_1) = 0 \tag{2.2.b}$$

is called the *first-order kinetic* model, since the cell storage is regarded as an added kinetic storage perturbation of the global fissure system.

The equation (2.2.b) models the delay that is inherent in the flow between the fissures and blocks. It is precisely this delay that led to the introduction of such models by Barenblatt, Zheltov, and Kochina [8] and Warren and Root [62] three decades ago in order to better match observed reservoir behavior. See [2], [18], [15], [39], [26], [13], [22], [38], [57] for additional applications and mathematical developments of such models.

If we further specialize this model by setting $a = 0$ in order to realize that the relative volume of the fissures is zero, we obtain the *pseudoparabolic* partial

differential equation

$$\frac{\partial}{\partial t} b\Big(u_1 - \delta\ \vec{\nabla}\cdot(A\vec{\nabla}u_1)\Big) - \vec{\nabla}\cdot(A\vec{\nabla}u_1) = f + \delta\frac{\partial f}{\partial t}. \tag{2.3}$$

See [12] for the development of such equations. Their solutions are determined by a *group* of operators on appropriate spaces, and their dynamics is regularity-*preserving*.

Partially Fissured Media. Next we present a parallel flow model for a *partially fissured* medium, a fissured medium in which there are substantial flow paths directly joining the cells in addition to the predominate connection with the surrounding fissure system. Thus, the cells are not completely isolated from one another by the fissure system, and the matrix is somewhat connected. This model allows for a *secondary flux* that arises from this bridging between cells. The model above is based on the assumption that the exchange flow between the components has a spatially distributed density proportional to the pressure differences. That is, the fluid stored in the cell system at a point in space is determined solely by the history of the *values* of the pressures of the components at that point. In order to induce a flow within the cell matrix as a response to local effects, however, it is necessary to apply a pressure *gradient* from the fissure system. Thus, we shall model the flux exchange as a response to both the value and the *gradient* of the pressure. Equivalently, we assume that the local cell structure at a point is responsive to the best linear approximation of the pressure in a neighborhood of that point. Furthermore, if the geometry of the cell matrix is symmetric with respect to the coordinate system, then the response of the cell to the value and to the gradient of the pressure is additive, the terms representing even and odd responses, respectively, to even and odd input. Thus, we are led to model the resulting storage and transport responses within the cell matrix as two independent processes whose effects are additive.

In this situation one must account for the effect of the gradient of the fissure flow on the local flow within the cells, for it is this fissure pressure gradient which necessarily provides the driving force for this transport within the matrix. In order to implement this in a model of parallel flow type, we introduce into the first order kinetic model (2.2) a *secondary flux* $\vec{u}_3$ through the cell system. This flux is assumed to respond to the fissure pressure gradient with a delay analogous to that of the cell storage in response to the value of fissure pressure in (2.2). This leads to the *second-order kinetic model*

$$\frac{\partial}{\partial t}(au_1) - \vec{\nabla}\cdot(A\vec{\nabla}u_1) + \frac{1}{\delta}(u_1 - u_2) + \vec{\nabla}\cdot C^*\vec{u}_3 = f \tag{2.4.a}$$

$$\frac{\partial}{\partial t}(bu_2) + \frac{1}{\delta}(u_2 - u_1) = 0 \tag{2.4.b}$$

$$\frac{\partial}{\partial t}(c\vec{u}_3) + \frac{1}{\beta}(\vec{u}_3 + C\vec{\nabla}u_1) = 0\ . \tag{2.4.c}$$

We assume the responses of the cell structure at a point to the value and to the gradient of fissure pressure are additive, an assumption that is valid for cell structures which are symmetric with respect to coordinate directions. The third and fourth terms in (2.4.a) give the distributed mass flow rate into the cell matrix from

the fissure system at a point. According to (2.4.b), the first of these goes toward the storage of fluid in the cells. Fluid from the fissure system enters the cell system at a point of higher pressure, it flows through the cell matrix to a point of lower pressure, and then it exits the cell back into the fissure system according to the second of these exchange terms in (2.4.a). The secondary flux $\vec{u}_3$ follows the fissure pressure gradient according to (2.4.c). The matrix C^*C arises from the bridging of the cells, and it distinguishes the *partially fissured* model (2.4) from the *totally fissured* model (2.2). Such a system may also be appropriate to describe totally fissured media which are composed of *larger* cells for which the more accurate approximation of u is necessary. See [53] for a discussion and development of such models with multiple nonlinearities.

3. Distributed Microstructure Models

We first describe an elementary example in which a distributed microstructure model arises very naturally. For historical reasons, it will be presented in the framework of a heat conduction model, but it clearly has a meaningful analogue for the analogous problem of diffusion and absorbtion of a dissolved chemical in a fluid flowing through a porous medium. The problem concerns the following situation. A system of pipes is used to absorb heat from circulating hot water and then later to return this heat to colder water, thereby serving as an energy storage device. This system has two sources of singularity, a geometric one due to the high ratio between the pipe length and the cross section, and a material one due to the very different conduction properties of the two materials involved. We seek a model in which these two singularities are properly balanced in order to obtain a good description of the exchange process.

We begin by examining the heat exchange in a single pipe in such a system. The heat transport in the water is primarily convective, due to the unit velocity of the water. The transport in the walls of the pipe is purely conductive and relatively very slow. The cross-sections of the pipe are very small, and these must be properly scaled to accurately model the exchange of heat between them and the water. We introduce a small parameter $\varepsilon > 0$ to quantify this. A reference cell $Y = \{y = (y_1, y_2) \in \mathbb{R}^2 : |y_1| + |y_2| < 2^{3/2}\}$ defines the structure of the cross-sections, and we write the parts of this double component domain as $\overline{Y} = \overline{Y_1} \cup \overline{Y}_2$, with $Y_1 = \{y \in \mathbb{R}^2 : \|y\| < 1\}$ representing the internal flow region and $Y_2 = \{y \in \mathbb{R}^2 : 1 < \|y\| \; and \; |y_1| + |y_2| < 2^{3/2}\}$ the walls of the pipe. The boundary of Y_1 is the unit circle, Γ_{11}, and that of Y is the square, Γ_{22}. The boundary of Y_2 is given by $\Gamma_2 = \Gamma_{11} \cup \Gamma_{22}$. By ν we denote the unit vector normal to Γ_2 which points in the direction *out* of Y_2. We shall represent the very small cross-sections by $Y_1^\varepsilon = \varepsilon\, Y_1 = \{z = (z_1, z_2) = \varepsilon(y_1, y_2) \in \mathbb{R}^2 : (y_1, y_2) \in Y_1\}$ and $Y_2^\varepsilon = \varepsilon\, Y_2 = \{z = \varepsilon y \in \mathbb{R}^2 : y \in Y_2\}$. Similarly, we denote the corresponding boundaries by $\Gamma_{11}^\varepsilon, \Gamma_{22}^\varepsilon$, and Γ_2^ε.

Let the interval $G = (0, 1)$ denote the axis of the very narrow pipe in which water is flowing at unit velocity through the cross section Y_1^ε and for which Y_2^ε is the pipe wall, a material of relatively *very low* conductivity. This situation is described by

the initial-boundary-value problem

$$\begin{aligned}
&\frac{\partial U^\varepsilon}{\partial t} + \frac{\partial U^\varepsilon}{\partial x} = \frac{\partial^2 U^\varepsilon}{\partial x^2} + \nabla_z \cdot \nabla_z U^\varepsilon(x,z,t), && z \in Y_1^\varepsilon,\ x \in G,\\
&\frac{\partial U^\varepsilon}{\partial t} = \varepsilon^2\big(\nabla_z \cdot \nabla_z U^\varepsilon(x,z,t) + \frac{\partial^2 U^\varepsilon}{\partial x^2}\big), && z \in Y_2^\varepsilon,\ x \in G,\\
&\nabla_z U^\varepsilon \cdot \nu|_{Y_1^\varepsilon} = \varepsilon^2 \nabla_z U^\varepsilon \cdot \nu|_{Y_2^\varepsilon}, && z \in \Gamma_{11}^\varepsilon,\ x \in G,\\
&\nabla_z U^\varepsilon \cdot \nu = 0, && z \in \Gamma_{22}^\varepsilon,\ x \in G,
\end{aligned}$$

which is the *exact ε-model.* The ε^2-scaled conductivity permits *very high* gradients in $G \times Y_2^\varepsilon$. However, only those components in the cross-section direction are responsible for the exchange flux, and in order to see that it is balanced with the other transport terms we rescale with $z = \varepsilon y$ and $\nabla_z = \frac{1}{\varepsilon}\nabla_y$ to get

$$\begin{aligned}
&\text{(3.1.a)} && \frac{\partial U^\varepsilon}{\partial t} + \frac{\partial U^\varepsilon}{\partial x} = \frac{\partial^2 U^\varepsilon}{\partial x^2} + \frac{1}{\varepsilon^2}\nabla_y \cdot \nabla_y U^\varepsilon(x,y,t), && y \in Y_1,\ \ x \in G,\\
&\text{(3.1.b)} && \frac{\partial U^\varepsilon}{\partial t} = \nabla_y \cdot \nabla_y U^\varepsilon(x,y,t) + \varepsilon^2 \frac{\partial^2 U^\varepsilon}{\partial x^2}, && y \in Y_2,\ \ x \in G,\\
&\text{(3.1.c)} && \nabla_y U^\varepsilon \cdot \nu|_{Y_1} = \varepsilon^2 \nabla_y U^\varepsilon \cdot \nu|_{Y_2}, && y \in \Gamma_{11},\ x \in G,\\
&\text{(3.1.d)} && \nabla_y U^\varepsilon \cdot \nu = 0, && y \in \Gamma_{22},\ x \in G.
\end{aligned}$$

Now it is easy to recognize the limit as $\varepsilon \to 0$. In Y_1 we get $U(x,y,t) = u(x,t)$ in the limit, i.e., it is independent of $y \in Y_1$. Average (3.1.a) over Y_1 and use (3.1.c) with Gauss' theorem to get an integral over Γ_{11}. This gives the following limiting form of the problem:

$$\begin{aligned}
&\text{(3.2.a)} && \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = \frac{\partial^2 u}{\partial x^2} - \frac{1}{|Y_1|}\int_{\Gamma_{11}} \nabla_y U \cdot \nu \, ds, && && x \in G,\\
&\text{(3.2.b)} && \frac{\partial U}{\partial t} = \nabla_y \cdot \nabla_y U(x,y,t), && y \in Y_2, && x \in G,\\
&\text{(3.2.c)} && U(x,s,t) = u(x,t), && s \in \Gamma_{11}, && x \in G,\\
&\text{(3.2.d)} && \nabla_y U^\varepsilon \cdot \nu = 0, && s \in \Gamma_{22}, && x \in G.
\end{aligned}$$

The integral term in (3.2.a) is the total heat lost to the pipe wall at the cross-section $x \in G$, computed from the normal component of the heat gradient in the wall, and it is comparable to the transport terms representing convection and conduction along the pipe. This balances the heat gained in the water with its moderate gradients against that exchanged with the pipe where the gradients are very high.

This problem approximates the heat transport and the exchange between the water and the pipe walls. It is essentially the *heat recuperator* problem which was introduced by Lowan [40], and it is our first example of a *distributed microstructure model.* The *global* domain is the interval G. For each point $x \in G$ there is identified a *local cell*, Y_2, which is located at or identified with that point. In the exchange of heat with the global medium, this family of cells acts as a distributed source, and the global medium is coupled to the cell at that point only on its boundary, Γ_{11}.

It provides a well-posed problem which is a good aproximation to the apparently singular ε-problem. We see the apparently singular term in (3.1.a) arose in the rescaling from the singular geometry in the original problem, but it is balanced by the corresponding term in (3.1.b) without the small coefficient because of the flux condition (3.1.c). In particular, the choice of ε^2 as coefficient in (3.1.b) exactly balanced the competing singularities introduced by the geometry and the materials.

Remark. *In order to describe the complete system of parallel pipes which makes up a heat recuperator, we need a periodic array of such cells εY to cover a fixed region in $\mathbb{R}^2$. Since the domain then has an ε-dependent structure which is not eliminated by a simple scale change as above, we need a more sophisticated technique to describe the limit. This is provided by homogenization. (See Section 5.)*

The introduction of *distributed microstructure models* for diffusion in porous media represents an attempt to recognize the geometry and the multiple scales in these problems in order to better quantify the exchange of fluid across the intricate interface between the components. We are given a domain G which represents the global region of the model. At each point $x \in G$ there is specified a cell G_x, a magnified or scaled representation of the microstructure that is present near x. One partial differential equation is specified to describe the global flow in the region G, and a separate partial differential equation is specified in each cell G_x to describe the flow internal to that cell. Any coupling between these equations will occur on the boundary of G_x, denoted by Γ_x. It is the collection $\{\Gamma_x : x \in G\}$ which provides the interface on which this exchange takes place. Now we use this concept to model some examples of single phase flow in a fissured medium. These include analogues of the parallel flow models which were given above.

Totally Fissured Media. The global flow in the fissure system is described in the macro-scale x by

$$\frac{\partial}{\partial t}\big(a(x)u(x,t)\big) - \vec{\nabla}\cdot A(x)\vec{\nabla}u + q(x,t) = f(x,t)\ , \qquad x \in G\ , \tag{3.3.a}$$

where $q(x,t)$ is the exchange term representing the flow into the cell G_x. The flow within each local cell G_x is described by

$$\frac{\partial}{\partial t}\big(b(x,y)U(x,y,t)\big) - \vec{\nabla}_y\cdot B(x,y)\vec{\nabla}_y U = F(x,y,t)\ , \qquad y \in G_x\ . \tag{3.3.b}$$

The subscript y on the gradient indicates that the gradient is with respect to the *local* variable y. A gradient operator without any subscript will mean that the gradient is taken with respect to the *global* variable x. Because of the smallness of the cells, the fissure pressure is assumed to be well approximated by the "constant" value $u(x,t)$ at every point of the cell boundary, so the effect of the fissures on the cell pressure is given by the interface condition

$$B(x,s)\vec{\nabla}_y U(x,s,t)\cdot\nu + \frac{1}{\delta}(U(x,s,t) - u(x,t)) = 0\ , \qquad s \in \Gamma_x\ , \tag{3.3.c}$$

where ν is the unit outward normal on Γ_x. (When $\delta = 0$, this becomes (and converges to) the *matched* boundary condition, $u(x,t) = U(x,s,t)$ for $s \in \Gamma_x$.) Finally,

the amount of fluid flux across the interface scaled by the cell size determines the remaining term in (3.3.a) by

$$q(x,t) = \frac{1}{|G_x|}\int_{\Gamma_x} B(x,s)\vec{\nabla}_y U\cdot\nu\, ds\ , \tag{3.3.d}$$

where $|G_x|$ denotes the Lebesgue measure of G_x, and this contributes to the *cell storage*. Thus, the system (3.3) comprises a double porosity model of *distributed microstructure* type for a totally fissured medium. It needs only to be supplemented by appropriate boundary conditions for the global pressure $u(x,t)$ and initial conditions for $u(x,0)$ and $U(x,y,0)$ in order to comprise a well-posed problem. See [40], [51], [50], [19], [61], [9], [25], [6], [7], [3], [5], [19], [20], [23], [24], [28], [29], [30], [31], [32], [33], [34], [36], [35], [43], [59], [58], [60], [56], [55], [54] for applications and mathematical theory for (3.3) and various related problems. A typical development of well-posedness results is given below in Section 4.

Finally, we remark that the system (3.3) can be rewritten as a single equation of functional-differential type. By applying Gauss' theorem to (3.3.b) we obtain from (3.3.d)

$$\frac{\partial}{\partial t}\int_{G_x} bU\, dy = \int_{\Gamma_x} B\frac{\partial U}{\partial \nu}\, ds + \int_{G_x} F\, dy\ .$$

Then by using the Green's function for the problem (3.3.b) to represent the solution $U(x,y,t)$ as an integral over Γ_x of $u(x,t)$, we substitute this into (3.3.a) to get the implicit *convolution evolution equation*

$$\frac{\partial}{\partial t}\left\{a(x)u(x,t) + \int_0^t k(x,t-\tau)u(x,\tau)\, d\tau\right\} - \vec{\nabla}\cdot A(x)\vec{\nabla}u = f(x,t)\ . \tag{3.4}$$

The convolution term represents a storage effect with memory. See [37] for a direct treatment of this equation and particularly [45], [48], [49] where this equation forms the basis for an independent theoretical and numerical analysis. Also see [46], [47] for related work.

Partially Fissured Media. We shall present two *distributed microstructure* models for flow in a partially fissured medium, a porous medium composed of *two* interwoven and connected components, the first being the system of fissures and the second being the matrix of porous cells. (Note that such a construction is impossible in $\mathbf{R}^2$.) In partially fissured media, the matrix of cells is connected, so that some of the flow passes directly through the block interconnections. While the primary flow will continue to be the flow from cells into fissures followed by flow within the fissures, the flow within the porous matrix has more than only a local character, as in the case of a totally fissured medium. This effect is less promiment than the bulk flow in the fractures, but it can have a noticeable effect when the interconnections between the cells are sufficiently large.

For our first model of a partially fissured medium, we are motivated by our parallel flow model above, in which the cell system responds additively to the value and the gradient of fissure pressure, that is, to the *best linear approximation* of the

fissure pressure at each cell location. As before the global fluid flow in the fissures is described by

$$(3.5.a) \qquad \frac{\partial}{\partial t}\big(a(x)u(x,t)\big) - \vec{\nabla}\cdot A(x)\vec{\nabla}u + q(x,t) = f(x,t)\ , \qquad x \in G\ ,$$

and the local flow within the matrix in the cell at each point x is given by

$$(3.5.b) \qquad \frac{\partial}{\partial t}\big(b(x,y)U(x,y,t)\big) - \vec{\nabla}_y \cdot B(x,y)\vec{\nabla}_y U = F(x,y,t)\ , \qquad y \in G_x\ .$$

The boundary values for each cell problem are taken from information about the solution to the global equation in the vicinity of that point. The tacit assumption of the microstructure models is that the matrix cells are so small that the global solution u may be effectively approximated over the matrix cell boundary by an appropriate approximation to u. In the usual models, as above, the approximation used for this purpose is merely the constant value of the global solution $u(x,t)$. Our objective here is to refine this model in order to more accurately describe the flow through the matrix. Thus, we shall assume that the pressure on the matrix boundary is driven by the best linear approximation of the fissure pressure, and this leads to the boundary condition

$$(3.5.c) \quad B(x,s)\vec{\nabla}_y U(x,s,t)\cdot\nu + \frac{1}{\delta}(U(x,s,t) - u(x,t) - \vec{\nabla}u(x,t)\cdot s) = 0\ , \qquad s \in \Gamma_x\ ,$$

Here Γ_x is taken to be that part of the fissure-matrix interface that is *interior* to the unit cell, Y; periodic conditions are prescribed on the remaining part of that interface, Γ_{22}, which intersects the boundary of the unit cell. Finally, the exchange term q in (3.5.a) consists of two parts, the average amount flowing into the cell to be stored and the divergence of the secondary flux flowing through the cell structure. The total exchange is given by

$$(3.5.d) \quad q(x,t) = \frac{1}{|G_x|}\int_{\Gamma_x} B(x,s)\vec{\nabla}_y U\cdot\nu\, ds - \frac{1}{|G_x|}\vec{\nabla}\cdot\left(\int_{\Gamma_x} B(x,y)(\vec{\nabla}_y U\cdot\nu)s\, ds\right)\ .$$

The system (3.5) comprises the first of our two distributed microstructure models for a partially fissured medium. This model was introduced in [14] to describe the highly anisotropic situation in layered media and developed in [16] for more general media. See [4] for an earlier discrete version and numerical work.

When the cells G_x are symmetric in coordinate directions, one can separate the effects of storage from those of the secondary flux. Specifically, the storage can then be expressed in terms of the value of the fissure pressure at the point over a time interval through a convolution integral obtained as before from a Green's function representation of the cell problem, and the secondary flux and its corresponding contribution to the global flow are expressed likewise in terms of the global flux. This leads just as before to a functional partial differential equation of the form

$$(3.6) \qquad \begin{aligned} &\frac{\partial}{\partial t}\big(a(x)u(x,t) + k_1(x,\cdot) * u(x,t)\big) \\ &\quad - \vec{\nabla}\cdot\left(A(x)\vec{\nabla}u(x,t) + \frac{\partial}{\partial t}k_{12}(x,\cdot) * \vec{\nabla}u(x,t) + k_2(x,\cdot) * \vec{\nabla}u(x,t)\right) \\ &\qquad = f(x,t)\ , \qquad x \in G\ ,\ t > 0\ . \end{aligned}$$

which is known as *Nunziato's equation*. This equation was presented in [42] without any physical or philosophical justification as an interesting generalization of heat conduction with memory models due to Gurtin and Chen. See [41] for mathematical development of these equations.

Next we describe our second microstructure model for flow in a partially fissured medium. We use *three* variables in this model. The *fast* diffusion through the fissure system is determined as before by a concentration $u_1(x,t)$ there, but we specify the concentration in the matrix in two components. The first of these, $U(x,y,t)$, results from the ε^2-scaled *slow* diffusion in the matrix as before, but we account also for a *moderate* component of flow in the matrix which has concentration $u_2(x,t)$. Thus, u_2 is the (low spatial frequency) component of the density in the matrix that leads to global diffusion, and U is the (high frequency) component of the density in the matrix that gives the local storage. These three variables, $u_1(x,t)$, $u_2(x,t)$, $U(x,t)$, are defined in the fissures, the matrix, and individual cells, respectively. The model system is given by

$$\Phi_1 \frac{\partial u_1}{\partial t} - \vec{\nabla} \cdot (\Lambda_1 \vec{\nabla} u_1) + q(x,t) = 0, \tag{3.7.a}$$

$$\Phi_2 \frac{\partial u_2}{\partial t} - \vec{\nabla} \cdot (\Lambda_2 \vec{\nabla} u_2) - q(x,t) = 0, \quad x \in G, \tag{3.7.b}$$

$$q(x,t) = \int_\Gamma \lambda_2 \vec{\nabla}_y U(x,s,t) \cdot \nu ds, \tag{3.7.c}$$

$$\phi_2 \frac{\partial U}{\partial t} - \vec{\nabla}_y \cdot (\lambda_2 \vec{\nabla} U) = 0, \qquad s \in Y_2(x), \tag{3.7.d}$$

$$U(x,s,t) \text{ and } \lambda_2 \vec{\nabla}_y U(x,s,t) \cdot \nu \text{ are Y-periodic on } \Gamma_{22}, \text{ and} \tag{3.7.e}$$

$$U(x,s,t) + u_2(x,t) = u_1(x,t), \qquad s \in \Gamma. \tag{3.7.f}$$

Note the different spatial domains on which the problems (3.7.b) and (3.7.d) are to be solved. The equation (3.7.d) is to be solved in the individual blocks which are now artificially disconnected along Γ_{22}, while (3.7.b) is to be solved in the whole of G, which includes all of the blocks and their interfaces to form a globally connected set. In view of (3.7.d), the flux (3.7.c) can be expressed by

$$q = \int_{Y_2} \phi_2 \frac{\partial U}{\partial t} dy \,.$$

With appropriate boundary conditions for u_1 and u_2 and initial conditions on each of u_1, u_2 and U, this is a well posed problem.

The condition (3.7.f) expresses a matching of *total* pressures across the matrix-fissure interface. We note that the condition (3.7.f) is, in a mathematical sense, the dual to the right sides of (3.7.a) and (3.7.b). The system is complemented by a pair of conservation equations (3.7.e) on the artificial interface Γ_{22} where pressure and flux are *localized*. This problem is a *hybrid* between parallel and microstructure models: the introduction of two matrix components is a *parallel* construction which was built into the ε-model, and it persisted in the limit. This is one of a family of such models which were derived in [21] by homogenization (see below) from corresponding ε-models.

4. A Variational Formulation

We illustrate with the case of a *totally fissured medium* the mathematical formulation of microstructure models as evolution equations on various Hilbert spaces. This provides a means of establishing that they are well-posed problems and identifies the natural *energy* and *state* spaces for these dynamical problems. Let G be an open, bounded domain in $\mathbb{R}^3$ and for every $x \in G$, let G_x be a bounded region contained in $\mathbb{R}^3$. Identify the product space $\prod_{x\in G} G_x \equiv \mathbb{Q}$ as a subset of $\mathbb{R}^6$; we require that $\mathbb{Q}$ be a *measurable* subset of $\mathbb{R}^6$, hence, each of the cells $G_x \equiv \{y : (x,y) \in \mathbb{Q}\}$ is a measurable subset of $\mathbb{R}^3$. Here we will formulate the Cauchy-Dirichlet problem for the *linear parabolic system*

$$(4.1.a)\quad \frac{\partial}{\partial t}\big(a(x)u(x,t)\big) - \vec{\nabla}\cdot A(x)\vec{\nabla}u(x,t) + \int_{\Gamma_x} B(x,s)\vec{\nabla}_y U(x,s,t)\cdot\vec{\nu}\,ds = f(x,t)\ , \qquad x\in G\ ,$$

$$(4.1.b)\quad \frac{\partial}{\partial t}\big(b(x,y)U(x,y,t)\big) - \vec{\nabla}_y\cdot B(x,y)\vec{\nabla}_y U(x,y,t) = F(x,y,t)\ , \qquad x\in G\ ,\ y\in G_x\ ,$$

$$(4.1.c)\quad B(x,s)\vec{\nabla}_y U(x,s)\cdot\vec{\nu} = \frac{1}{\delta}\big(U(x,s,t) - u(x)\big)\ , \qquad x\in G\ ,\ s =\in \Gamma_x\ ,$$

as an evolution equation in an appropriate Hilbert space. This is just the system (3.3) in which the measure ds on Γ_x is used to absorb the extra factor of $|G_x|$. We shall assume $a \in L^\infty(G)$, $b \in L^\infty(\mathbb{Q})$, A and B are uniformly positive definite and bounded measurable matrix functions, $\vec{\nu}$ is the unit outward normal on Γ_x and $\delta > 0$. We will further assume that each boundary Γ_x is piecewise C^1 and that the measures $|\Gamma_x|$ and $|G_x|$ are uniformly bounded in x. We shall use the Lebesgue space $L^2(\mathbb{Q}) = L^2(G, L^2(G_x))$ with the norm

$$\|U\|_{L^2(\mathbb{Q})} = \left(\int_G\int_{G_x} |U(x,y)|^2\,dy\,dx\right)^{1/2}\ ,$$

and the Sobolev spaces $W_0^{1,2}(G)$ and

$$L^2(G, W^{1,2}(G_x)) = \Big\{U \in L^2\big(G, L^2(G_x)\big) : \vec{\nabla}_y U \in L^2\big(G, L^2(G_x)\big)\Big\}\ .$$

See [1] for information on Sobolev spaces. For the norm on $L^2(G, W^{1,2}(G_x))$ we employ the notation

$$|U|_2 = \left(\int_G\int_{G_x} |\vec{\nabla}_y U(x,y)|^2\,dy\,dx\right)^{1/2}\ ,$$

so $\|U\|_2^2 = |U|_2^2 + \|U\|^2_{L^2(\mathbb{Q})}$. Denote by $V = W_0^{1,2}(G) \times L^2(G, W^{1,2}(G_x))$ the indicated product space with norm

$$\|[u,U]\|_V = \|u\|_{W_0^{1,2}(G)} + \|U\|_2\ .$$

Let γ_x be the usual trace map of $W^{1,2}(G_x)$ into $L^2(\Gamma_x)$, and define $\mathcal{B} = L^2(G, L^2(\Gamma_x))$ and the distributed trace $\gamma : L^2(G, W^{1,2}(G_x)) \to \mathcal{B}$ by $\gamma U(x,s) = (\gamma_x U(x))(s)$. We will require that the trace maps γ_x be uniformly bounded, so γ is continuous from $L^2(G, W^{1,2}(G_x))$ into $\mathcal{B}$. Define $\lambda : W_0^{1,2}(G) \to \mathcal{B}$ by: $\lambda u(x,s) = u(x)1_s$, $x \in G$, $s \in \Gamma_x$, where $u(x)1_s$ is the constant function on Γ_x with value $u(x)$. We will employ the notation $\tilde{u} = [u, U]$.

Define the Hilbert space $H = L^2(G) \times L^2(G, L^2(G_x))$ with the inner product

$$(\tilde{u}, \tilde{\varphi})_H = \int_G a(x)u(x)\varphi(x)\,dx + \int_G \int_{G_x} b(x,y)U(x,y)\Phi(x,y)\,dy\,dx \ ,$$
$$\text{for } \ \tilde{u} = [u, U] \ , \ \tilde{\varphi} = [\varphi, \Phi] \in H \ .$$

Define $V_h = \{\tilde{u} \in V : \gamma U = \lambda u \text{ in } \mathcal{B}\}$. Since γ and λ are continuous, V_h is a closed subspace of V. Also define $V_0 \equiv \{U \in L^2(G, W^{1,2}(G_x)) : \gamma U = 0\}$. It can be shown that $W_0^{1,2}(G) \times V_0$ and V_h are dense in H.

We shall write the system (4.1) as an evolution equation over the spaces described above. To obtain the variational form for the system, choose $[\varphi, \Phi] \in V$, multiply (4.1.a) by φ and integrate over G. Multiply (4.1.b) by Φ and integrate over both G_x and G. Add these equations and apply Green's Theorem to obtain

$$\int_G \left\{ \frac{\partial}{\partial t}\big(a(x)u(x,t)\big)\varphi(x) + \int_{G_x} \frac{\partial}{\partial t}\big(b(x,y)U(x,y,t)\big)\Phi(x,y)\,dy \right\} dx$$
$$+ \int_G \left\{ A(x)\nabla u(x,t)\cdot\vec{\nabla}\varphi(x) + \int_{\Gamma_x} B(x,s)\vec{\nabla}_y U(x,s,t)\cdot\vec{\nu}\varphi(x)\,ds \right.$$
$$+ \int_{G_x} B(x,y)\vec{\nabla}_y U(x,y,t)\cdot\vec{\nabla}_y\Phi(x,y)\,dy$$
$$\left. - \int_{\Gamma_x} B(x,y)\vec{\nabla}_y U(x,y,t)\cdot\vec{\nu}\gamma_x\Phi(x,s)\,ds \right\} dx$$
$$= \int_G f(x)\varphi(x)\,dx + \int_G \int_{G_x} F(x,y)\Phi(x,y)\,dy\,dx \ .$$

Combining the boundary integrals and substituting for $B(x,s)\vec{\nabla}_y U(x,y,t)\cdot\vec{\nu}$ yields

$$
\begin{aligned}
(4.2)\qquad \big[u(t),U(t)\big]\in V:\ &\int_G \frac{\partial}{\partial t}a(x)u(x,t)\varphi(x)\,dx\\
&+\int_G\int_{G_x}\frac{\partial}{\partial t}b(x,y)U(x,y,t)\Phi(x,y)\,dy\,dx\\
&+\int_G A(x)\vec{\nabla}u(x,t)\cdot\vec{\nabla}\varphi(x)\,dx\\
&+\int_G\int_{G_x}B(x,y)\vec{\nabla}_y U(x,y,t)\cdot\vec{\nabla}_y\Phi(x,y)\,dy\,dx\\
&+\int_G\int_{\Gamma_x}\frac{1}{\delta}\big(\gamma U(x,s,t)-\lambda u(x,s,t)\big)\big(\gamma\Phi(x,s,t)-\lambda\varphi(x,s,t)\big)\,ds\,dx\\
&=\int_G f(x)\varphi(x)\,dx+\int_G\int_{G_x}F(x,y)\Phi(x,y)\,dy\,dx\ ,\quad [\varphi,\Phi]\in V\ .
\end{aligned}
$$

A special case of the above is obtained when (4.1.c) is replaced by

$$
(4.1.c)'\qquad \gamma U(x,s,t)=\lambda u(x,s,t)\qquad x\in G\ ,\ s\in\Gamma_x\ ,\ t>0\ .
$$

This is the formal result obtained by allowing $\delta\to 0^+$, so that (4.1.c)$'$ is forced to hold, and it corresponds to

$$
\begin{aligned}
(4.2)'\qquad \big[u(t),U(t)\big]\in V_h:\ &\int_G \frac{\partial}{\partial t}a(x)u(x,t)\varphi(x)\,dx\\
&+\int_G\int_{G_x}\frac{\partial}{\partial t}b(x,y)U(x,y)\Phi(x,y)\,dy\,dx\\
&+\int_G A(x)\vec{\nabla}u(x,t)\cdot\vec{\nabla}\varphi(x)\,dx\\
&+\int_G\int_{G_x}B(x,y)\vec{\nabla}_x U(x,y,t)\cdot\vec{\nabla}_y\Phi(x,y)\,dy\,dx\\
&=\int_G f(x)\varphi(x)\,dx+\int_G\int_{G_x}F(x,y)\Phi(x,y)\,dy\,dx\ ,\quad [\varphi,\Phi]\in V_h\ .
\end{aligned}
$$

The problem (4.1) will be called the *regularized model*, and (4.1)$'$, i.e., (4.1.a), (4.1.b) and (4.1.c)$'$ will be called the *matched model*. Conversely, starting from (4.2) it is not difficult to recover (4.1).

Define the Hilbert spaces

$$
\begin{aligned}
\mathcal{H}&=L^2(0,T;H)\\
\mathcal{W}_2&=L^2(0,T;W_0^{1,2}(G))\\
\mathcal{V}_2&=L^2(0,T;L^2(G,W^{1,2}(G_x)))\\
\mathcal{V}&=\mathcal{W}_2\times\mathcal{V}_2\ ,\ \text{ and}\\
\mathcal{V}_h&=\big\{\tilde{u}\in\mathcal{V}:\gamma U(t)=\lambda u(t)\ \text{ in }\ \mathcal{B}\ \text{ for almost every }\ t>0\big\}\ .
\end{aligned}
$$

Define $L : V \to V'$ by

$$L\tilde{u}(\tilde{\varphi}) = \int_G A(x)\vec{\nabla}u \cdot \vec{\nabla}\varphi \, dx + \int_G \int_{G_x} B(x,y)\vec{\nabla}_y U \cdot \vec{\nabla}_y \Phi \, dy \, dx \quad \tilde{u}, \tilde{\varphi} \in V \ .$$

The conditions above on A and B imply that L is bounded and coercive from V into V'. The exchange term is given by the operator

$$\mathcal{M}(\tilde{u}, \tilde{\varphi}) \equiv \int_G \int_{\Gamma_x} (\gamma U - \lambda u)(\gamma \Phi - \lambda \varphi) \, ds \, dx \ , \qquad \tilde{u}, \tilde{\varphi} \in V \ .$$

This gives a continuous and linear function $\mathcal{M} : V \to V'$. Let $\tilde{f} \in V'$ be given in the form

$$\tilde{f}(\tilde{\varphi}) = \int_G f\varphi \, dx + \int_G \int_{G_x} F\Phi \, dy \, dx \ , \qquad \tilde{\varphi} \in V \ .$$

We will use the same notation to refer to the corresponding realizations of these operators on the spaces $\mathcal{H}$, $\mathcal{V}$ and $\mathcal{V}_h$.

Integrating (4.2) from 0 to T we obtain

$$\tilde{u} \in \mathcal{V} : \int_0^T \left(\frac{\partial}{\partial t}\tilde{u}, \tilde{\varphi} \right)_H dt + \int_0^T L\tilde{u}(\tilde{\varphi}) \, dt + \int_0^T \frac{1}{\delta}\mathcal{M}\tilde{u}(\tilde{\varphi}) \, dt \tag{4.5}$$

$$= \int_0^T \tilde{f}(\tilde{\varphi}) \, dt \ , \qquad \tilde{\varphi} \in \mathcal{V} \ .$$

Similarly from $(4.2)'$ we get

$$\tilde{u} \in \mathcal{V}_h : \int_0^T \left(\frac{\partial}{\partial t}\tilde{u}, \tilde{\varphi} \right)_H dt + \int_0^T L\tilde{u}(\tilde{\varphi}) \, dt = \int_0^T \tilde{f}(\tilde{\varphi}) \, dt \ , \qquad \tilde{\varphi} \in \mathcal{V}_h \ . \tag{4.5}'$$

In the situation described above, (4.5) and $(4.5)'$ have unique solutions and the solutions $\tilde{u}_\delta$ of (4.5) converge to the solution $\tilde{u}$ of $(4.5)'$ as $\delta \to 0$.

Theorem. *Given the spaces and operators as above, suppose that $\tilde{u}_0 = [u_0, U_0] \in H$ and $\tilde{f} = [f, F] \in \mathcal{V}'$. Then for every $\delta > 0$ there is a unique $\tilde{u}_\delta \in \mathcal{V}$ which satisfies* (4.5) *and $\tilde{u}_\delta(0) = \tilde{u}_0$. Also, there is a unique $\tilde{u} \in \mathcal{V}_h$ which satisfies* $(4.5)'$ *and where $\tilde{u}(0) = \tilde{u}_0$. Furthermore $\tilde{u}_\delta$ converges weakly to $\tilde{u}$ in $\mathcal{V}$ as $\delta \to 0$.*

The proof is a direct application of standard results on evolution equations in Hilbert space.

We have shown that (4.5) and $(4.5)'$ have unique solutions and that the two models which they represent are related. We remarked previously that allowing $\delta \to 0^+$ formally transformed the regularized model into the matched model. We have substantiated that observation by showing that the solutions u_δ converge to the solution of the matched model.

Note also that the variational form (in $(4.2)'$ for example) leads directly back to $(4.1)'$. This confirms that our choice of the exchange term in the physical model is the correct one.

Finally note that the models and results here could be generalized or extended in several ways. In (4.1.c) we might choose $\frac{1}{\delta}$ to be something other than a constant. If, for example, $\frac{1}{\delta}$ is assumed to be a monotone graph which is also a subgradient operator, an approach similar to that in [17] might be used to show existence of a solution. As stated earlier, Dirichlet boundary conditions on ∂G are not necessary, so some generalization is also possible in that respect. Finally, if additional assumptions about the differentiability of A and B and the smoothness of Γ_x and ∂G were made, then it might be possible to say more about the regularity of u and U.

5. Remarks

We describe first how microstructure models can arise by way of homogenization from partial differential equations with highly periodic coefficients or geometry *and* appropriately scaled coefficients. This provides the connection with the exact models described by classical equations. Then we close with some additional remarks and observations.

Homogenization. So far we have given a direct but only heuristic justification of the microstructure models. In order to employ them to simulate real phenomena, one must obtain realistic values for the coefficients, e.g., by matching with data. Here we briefly recall the derivation by homogenization of the distributed microstructure model of a totally fissured medium following [4],[6]. This provides simultaneously a justification of the model and a means to compute the *effective* coefficients in the microstructure model from known coefficients in the *exact* case.

We begin with the exact microscopic model of single phase flow in a fissured domain G, a bounded open subset of R^3. We assume that the geometry and the physical parameters of the problem have ε–periodic character. This implies that the solution to the problem exhibits periodic behavior. It has also some macroscopic (non–periodic) behavior which is seen on the scale of the whole region, G. The basic problem is to investigate the asymptotics of the solution as $\varepsilon \to 0$ in a family of properly scaled problems posed on domains G^ε formed by a lattice of copies of cells εY, where the unit reference cell is the cube $Y = (0,1)^3$. We use ε as a superscript or subscript on coefficients or variables to denote objects periodic with respect to εY; we omit this notation when $\varepsilon = 1$.

The reference cell Y defines the double component structure of the fissured domain, and we write $\overline{Y} = \overline{Y_1} \cup \overline{Y}_2$, with Y_1 and Y_2 denoting the fissure and matrix parts of the cell, respectively. Their respective boundaries are denoted by Γ_1 and Γ_2. The fissure–matrix interface is given by $\Gamma_{12} = \Gamma_1 \cap \Gamma_2$. Let Γ be that part of Γ_{12} which is contained in Y, and let Γ_{11} and Γ_{22} denote the respective intersections of ∂Y with Y_1 and Y_2. In the totally fissured case, we assume $\overline{Y}_2 \subset Y$, so the matrix interface Γ_{22} is empty and $\Gamma = \Gamma_{12}$. We then refer to Y_2 as a *block*. By ν we denote the normal unit vector to Γ which points in the direction *out* of Y_2.

The system of fissures and matrix blocks in G^ε are denoted by G_1^ε and G_2^ε, respectively. The exact (but singular) ε–model consists of a pair of differential equations, one on each of the subdomains G_1^ε and G_2^ε for the density, which will be denoted by u^ε. These equations are coupled by standard interface conditions on Γ_{12}^ε to insure conservation of mass and momentum across the fissure–matrix

interface $\Gamma_{12}^{\varepsilon}$. An exterior boundary condition and an initial condition must also be specified, but they do not enter into the derivation of the limit model. In order to preserve the magnitude of the flux crossing the interfaces contained within a fixed volume of the medium as $\varepsilon \to 0$, it is necessary to scale the permeability in the blocks by the factor ε^2. Thus, the ε–model of diffusion in a totally fissured medium has the form

$$\phi_1 \frac{\partial u^{\varepsilon}}{\partial t} - \nabla \cdot (\lambda_1 \nabla u^{\varepsilon}(x,t)) = 0, \qquad x \in G_1^{\varepsilon}, \tag{5.1.a}$$

$$\phi_2 \frac{\partial u^{\varepsilon}}{\partial t} - \nabla \cdot (\varepsilon^2 \lambda_2 \nabla u^{\varepsilon}(x,t)) = 0, \quad x \in G_2^{\varepsilon}, \tag{5.1.b}$$

$$u^{\varepsilon}|_{G_1^{\varepsilon}}(s,t) = u^{\varepsilon}|_{G_2^{\varepsilon}}(s,t), \qquad s \in \Gamma_{12}^{\varepsilon}, \tag{5.1.c}$$

$$\lambda_1 \nabla u^{\varepsilon}|_{G_1^{\varepsilon}} \cdot \nu = \varepsilon^2 \lambda_2 \nabla u^{\varepsilon}|_{G_2^{\varepsilon}} \cdot \nu, \qquad s \in \Gamma_{12}^{\varepsilon}. \tag{5.1.d}$$

If u^{ε} is expanded in powers of ε and the formal analysis of this expansion is carried out, it will be seen that the leading terms for the density $u^{\varepsilon}|_{G_1^{\varepsilon}}$ in the fractures and the density $u^{\varepsilon}|_{G_2^{\varepsilon}}$ in the matrix blocks will be a pair of functions, $u(x,t)$ and $U(x,y,t)$, $x \in G$, $y \in Y_2$, $t > 0$, respectively, which satisfy the system of equations

$$|Y_1| \frac{\partial u}{\partial t} - \nabla \cdot (\Lambda_1 \nabla u(x,t)) + q(x,t) = 0, \quad x \in G, \tag{5.2.a}$$

$$\phi_2 \frac{\partial U}{\partial t} - \nabla_y \cdot (\lambda_2 \nabla_y U(x,y,t)) = 0, \qquad y \in Y_2(x),\ x \in G, \tag{5.2.b}$$

$$U(x,s,t) = u(x,t), \qquad s \in \Gamma, \qquad x \in G, \tag{5.2.c}$$

$$q(x,t) = \int_{\Gamma} \lambda_2 \nabla_y U \cdot \nu \, dS, \qquad x \in G\ . \tag{5.2.d}$$

This is just the *matched* microstructure model (3.3). Here $|Y_1|$ denotes the ϕ_1–weighted volume of the reference set Y_1. The *effective* permeability tensor Λ_1 is given by

$$(\Lambda_1)_{ij} = \int_{Y_1} \lambda_1 \left(\delta_{i,j} |Y_1| + \frac{\partial \omega_i}{\partial y_j} \right) dy,$$

with the auxiliary functions ω_k, $k = 1,2,3$, being Y–periodic solutions of the *cell problem*

$$\begin{aligned} \nabla_y^2 \omega_k &= 0, && y \in Y_1, \\ \nabla_y \omega_k \cdot \nu &= -e_k \cdot \nu, && y \in \Gamma, \end{aligned}$$

where e_k is the unit vector in the direction of the k–axis.

Equation (5.2.a) is the macroscopic equation to be solved in G for the (macroscopic) density u. The distributed source term q accounts for the flux across the boundary Γ of the block Y_2; we denote it here by $Y_2(x)$ to emphasize that a copy of it is identified with each point $x \in G$. Blocks over different points in G are disconnected; thus, no flow can take place directly from one such block to another. It is this feature that limits this model to flow in a totally fissured medium. If the ε^2

scaling of the permeability in the blocks had been omitted, then the limit process would have led to a single diffusion equation with *effective* or averaged coefficient that fails to represent the desired delayed storage effects. Vogt [61] appears to have been the first to have recognized this idea in the development of a model for chromotography.

Further Remarks...

The basic distributed microstructure model (3.3) is obtained as the limit by homogenization of a corresponding exact but highly singular partial differential equation with rapidly oscillating coefficients. This provides not merely another derivation of the model equations, but shows also the relation with the classical but singular formulation as a single diffusion equation, and it provides a method for directly computing the coefficients in (3.3) which necessarily represent averaged material properties. A similar result of convergence of a classical system to (3.5) is to be expected. Also, the convergence of the solution of the well-posed ε-problem to a solution of this system provides a proof of existence for the distributed microstructure system. An alternative direct proof is independently available as in Section 4. Furthermore, the *parallel flow* model systems can be recovered as limiting cases of corresponding microstructure models. For example, the first order kinetic model (2.2) is the limiting case of (3.3) as the permeability coefficient B tends to infinity. In this limit the cell behaves as a single point, equivalently, the function U is independent of the local variable y, and the geometry of the cell is lost. The system (3.5) converges to (2.4) likewise as B tends to infinity. When δ tends to zero in (3.3) or (3.5), the solution converges to that of a limiting problem with the boundary conditions (3.3.c) or (3.5.c) replaced by the corresponding "matched" conditions of Dirichlet type. These results provide useful connections between the various classes of models [17].

An interesting open problem is the determination of the coefficients in the system (3.3) from measurements of data on the boundary of the global region. It would be particularly interesting to obtain information on the cell geometry from such boundary measurements.

The systems (2.1) and (2.2) comprise parabolic and degenerate parabolic dynamical systems, respectively, in the product space $L^2(G) \times L^2(G)$. The functional differential equations (3.4) and (3.6) lead to dynamical systems in $L^2(G)$, but these are governed by C_0 semigroups without regularizing effects, and the estimates and techniques for these are comparatively difficult. These equations lack the parabolic structure one seeks in such models. However the systems (3.3) and (3.5) do retain all of the parabolic structure and corresponding estimates and regularity of classical parabolic systems when they are posed on the spaces $L^2(G) \times L^2(G, L^2(G_x))$. The dynamics of these microstructure systems is given by an analytic semigroup. Thus, they are truly *parabolic* problems.

Interesting models of a *layered structure* are easily obtained from the examples above for partially fissured media. Rather than having a secondary flux in all three coordinate directions, the fissure pressure gradient induces transverse normal flow across the layers while the parallel flow within the layered matrix is driven by the internal pressure gradient. See [11], [10], [44] and [27] for alternative approaches.

All of the models and results we have presented here could be generalized or ex-

tended in several ways. In fact, most are special cases of what is already available in the literature. For example, the linear elliptic operators in the above examples can be replaced by quasilinear operators of divergence type, such as p-Laplace operators, and one can include semilinear operators such as the porous medium equation. We have not even mentioned results for equations of other types, such as hyperbolic. Furthermore, we have restricted discussion to problems with the simplest geometry, and we have not mentioned those involving, e.g., flow on boundaries with concentrated capacity, or, more generally, manifolds arising from periodic cells that have a non-flat geometry.

Experience suggests that the distributed microstructure models are conceptually easy to work with, they provide accurate models which include the fine scales and geometry appropriate for many problems, and their theory can be developed in a straightforward manner using conventional techniques. The numerical analysis of these systems provides a natural application of parallel methods.

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