Degenerate Evolution Equations and Applications

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1. Introduction. We shall consider the Cauchy problem for the equation

$$A(t)u(t) + D_tB(t)u(t) = f(t) \tag{1.1}$$

in which $\{A(t)\}$ and $\{B(t)\}$ are bounded and measurable families of linear operators between Hilbert spaces. In Section 2 we show that the problem is well-posed when the $\{B(t)\}$ are non-negative, $\{A(t) + \lambda B(t)\}$ are coercive, and the operators depend smoothly on t. The equation is degenerate since the coefficient operators are not necessarily invertible and may in fact vanish. Sections 3 and 4 contain some applications to various boundary value problems. The examples are chosen simply to indicate the large class of problems to which the abstract results can be applied, and so we do not attempt to present the most technically refined results in any sense.

The abstract problem which we shall formulate and resolve in Section 2 was initially motivated by boundary value problems for partial differential equations of the form

$$\frac{\partial}{\partial t}\{b_0(x, t)u(x, t) - b(x, t)\triangle u(x, t)\} - \triangle u(x, t) = f(x, t). \tag{1.2}$$

Such equations appear in various applications [1, 8, 9, 16] with non-negative (real) coefficients determined by material constants. The coefficient $b(x, t)$ usually has the dimensions of viscosity and distinguishes (1.2) from the standard equation of heat conduction. This equation is elliptic in that region where $b_0(x, t) = b(x, t) = 0$, parabolic where $b(x, t) = 0$ and $b_0(x, t) > 0$, and of Sobolev type where $b(x, t) > 0$. Since we permit our abstract equation (1.1) to degenerate, the coefficients in our model (1.2) may vanish identically on certain space-time regions. It became apparent that the abstract results obtained below would also give existence and uniqueness of weak solutions of boundary value problems of much more general type than was first anticipated, so we have attempted to illustrate some of these diverse applications in the examples.

In Section 3 we apply the results of Section 2 to the equation (1.2) in one space dimension. Of major interest here is the discussion of the allowed transitions from one type to another, since the type of the equation essentially determines which problems are well-posed. We note, in particular, that the hypotheses of our existence theorem are fulfilled when the time derivative of the leading coefficient $b_0(x, t)$ has a (possibly negative) lower bound, hence the coefficient itself may be negative over large regions.

Degenerate equations with smooth coefficients and space variables of higher dimension are studied in Section 4, but the emphasis here is on the types of boundary conditions that are permitted. The first example is an interface problem for an elliptic-parabolic equation with time-dependent interface and boundary conditions. Next we consider problems with time-differential constraints along a submanifold or portion of the boundary. These arise naturally in a region in which the equation is of Sobolev type, and they also are obtained by certain approximations of parabolic interface problems in a region in which a coefficient is singular. (Such problems were discussed, *e.g.*, in [4]; our Theorem 2 completes the uniqueness result claimed but not proved there.) Our final examples are the (non-local) boundary value problems of the fourth and fifth kind [0] for parabolic equations. These arise, *e.g.*, in problems of heat conduction in a body whose boundary is in contact with a (finite) reservoir of highly conductive material whose temperature is a function only of time and is affected by the heat flow across the boundary of the body.

We have used the abstract variational formulation of the Cauchy problem in Section 2 for the equation (1.1). Time dependent problems in which the leading operators are coercive (and hence non-degenerate) were considered by J. Lions [22], M. Visik [29], H. Levine [20], and this writer [27]. Similar results have been obtained by C. Bardos and H. Brezis [2, 3] for semi-linear but stationary problems. Related evolution equations of second order have been studied. See, *e.g.*, R. Carroll and C. Wang [6, 7] for degenerate equations and J. Lions [21] and M. Visik [29] for equations with operator coefficients in the term with highest derivative.

Some of our applications in Sections 3 and 4 can be related to previous works. In particular, the problems with elliptic-parabolic partial differential equations of second order and studied by G. Fichera [10], J. J. Kohn and L. Nirenberg [17], and O. Oleinik [25] are posed on regions much more general than the cylinders to which we are constrained. However, our techniques are elementary and much more direct than theirs, and they give comparable results in the cylindrical domains. The existence and uniqueness results of A. Friedman and Z. Schuss [14] and W. Ford [13] are contained in ours. These concern a weak form of the first initial-boundary value problem for equations like (1.2) with $b(x, t) \equiv 0$. The regularity theorem given in [14] is not comparable to ours and the writers assume that the coefficient operators $\{A(t)\}$ have a constant domain and satisfy certain conditions on their resolvents. Equations of Sobolev type have been treated by J. Lagnese [18, 19], T. W. Ting and this writer [26, 27, 28].

2. The Cauchy problem. Let V be a separable complex Hilbert space with norm $\|v\|$. V' is the antidual, and the antiduality is denoted by $\langle f, v\rangle$. W is a complex Hilbert space containing V and the injection is assumed continuous with norm $\leqq 1$. $\mathcal{L}(V, W)$ denotes the space of continuous linear operators from V into W. T is the unit interval $[0, 1]$ and $L^2(T, V)$ is the Hilbert space of (equivalence classes of) Lebesgue (weakly = strongly) measurable and square integrable functions from T to V; when V is the complex field, denote the above by $L^2(T)$. Finally, let $H^1(T, V)$ be the Hilbert space of those φ in $L^2(T, V)$ for which the (distribution) derivative φ' is in $L^2(T, V)$. This means that φ' is the (unique) function in $L^2(T, V)$ such that

$$\int_0^1 \varphi'(t)\psi(t)\,dt = -\int_0^1 \varphi(t)\psi'(t)\,dt$$

for all ψ in $\mathcal{D}(T)$, the infinitely differentiable functions with compact support in the interior of T. The integrals above have values in V; we refer to [5, 15, 23] for the calculus of vector-valued functions.

Assume that for each $t \in T$ we are given a continuous sesquilinear form $a(t; \cdot, \cdot)$ on V and that for each pair $x, y \in V$ the map $t \to a(t; x, y)$ is bounded and measurable. The Uniform Boundedness Principle then implies that there is a number $K_a > 0$ such that $|a(t; x, y)| \leqq K_a \|x\| \|y\|$ for all $x, y \in V$ and $t \in T$. Standard measurability arguments then show that for any pair $u, v \in L^2(T, V)$ the function $t \to a(t; u(t), v(t))$ is integrable [5, p. 168]. Similarly, we assume given for each $t \in T$ a continuous sesquilinear form $b(t; \cdot, \cdot)$ on W and that for each pair $x, y \in W$ the map $t \to b(t; x, y)$ is bounded and measurable. Let $\{V(t): t \in T\}$ be a family of closed subspaces of V and denote by $L^2(T, V(t))$ the Hilbert space consisting of those $\varphi \in L^2(T, V)$ for which $\varphi(t) \in V(t)$ *a.e.* on T. Finally, let $u_0 \in W$ and $f \in L^2(T, V')$ be given. A solution of the Cauchy problem (determined by the preceding data) is a $u \in L^2(T, V(t))$ such that

$$\int_0^1 a(t; u(t), v(t))\,dt - \int_0^1 b(t; u(t), v'(t))\,dt = \int_0^1 \langle f(t), v(t)\rangle\,dt + b(0; u_0, v(0)) \tag{2.1}$$

for all $v \in L^2(T, V(t)) \cap H^1(T, W)$ with $v(1) = 0$.

Definition. The family $\{a(t; \cdot, \cdot): t \in T\}$ of sesquilinear forms on V is *regular* if for each pair $x, y \in V$ the function $t \to a(t; x, y)$ is absolutely continuous and there is an $M(\cdot) \in L^1(T)$ such that for all $x, y \in V$ we have

$$|a'(t; x, y)| \leqq M(t)\,\|x\|\,\|y\|, \quad a.e.\ t \in T. \tag{2.2}$$

(The prime denotes the derivative with respect to t.)

Lemma 1. *Let $\{a(t; \cdot, \cdot): t \in T\}$ be a regular family on V. Then for each pair $u, v \in H^1(T, V)$ the function $t \to a(t; u(t), v(t))$ is absolutely continuous and its derivative is given by*

$$D_t a(t; u(t), v(t)) = a'(t; u(t), v(t)) + a(t; u'(t), v(t)) + a(t; u(t), v'(t)),$$

$$a.e.\ t \in T.$$

Proof. Define $\alpha(t) \in \mathcal{L}(V, V)$ by $(\alpha(t)x, y)_V = a(t; x, y)$, $x, y \in V$. Fix $x \in V$ and let $\{y_n : n \geqq 1\}$ be dense in V. For each $n \geqq 1$ define $(\dot{\alpha}(t)x, y_n)_V = D_t(\alpha(t)x, y_n)_V$, *a.e.* $t \in T$. (The estimate (2.2) shows $(\dot{\alpha}(t)x, y)_V$ is defined and continuous at every $y \in V$ and *a.e.* $t \in T$.) The map $\dot{\alpha}(\cdot)x$ is weakly, hence strongly, measurable and the estimate (2.2) shows it is in $L^1(T, V)$. The weak absolute continuity of $t \to \alpha(t)x$ then shows

$$\alpha(t)x = \alpha(0)x + \int_0^t \dot{\alpha}(s)x\, ds, \qquad t \in T.$$

Thus $\alpha(\cdot)x$ is strongly absolutely continuous and strongly differentiable *a.e.* on T.

Let $u \in H^1(T, V)$. For each $v \in V$ we have $(\alpha(t)u(t), v)_V = (u(t), \alpha^*(t)v)_V$ is absolutely continuous, since the above discussion applies as well to the adjoint $\alpha^*(t)$. Hence, $t \to \alpha(t)u(t)$ is weakly absolutely continuous. The strong differentiability of $\alpha(\cdot)$ from above implies

$$D_t[\alpha(t)u(t)] = \dot{\alpha}(t)u(t) + \alpha(t)u'(t),\ a.e.\ t \in T,$$

hence the indicated strong derivative is in $L^1(T, V)$. From this it follows that $\alpha(\cdot)u(\cdot)$ is strongly absolutely continuous and the desired result now follows easily. Q.E.D.

Remark. The conclusions of Lemma 1 hold if we assume only that $a(\cdot\ ; x, y)$ is absolutely continuous for each pair $x, y \in V$. Hence we need not assume an estimate like (2.2) in order to obtain the desired result. One can prove this assertion as follows: (1) use the closed-graph and uniform-boundedness theorems to obtain an estimate

$$\int_0^1 |a'(t; x, y)|\, dt \leqq K\, ||x||\, ||y||, \qquad x, y \in V;$$

(2) approximate u' in $L^2(T, V)$ by simple functions and use the Lebesgue theorem with (1) to obtain the result for the special case of constant $v \in V$; approximate v' by step functions and use the results of (1) and (2) to obtain the general result. The details are standard but involve some lengthy computations.

Our first two results concern, respectively, the existence of and an a-priori estimate on a solution, and the uniqueness of a solution. We compare our hypotheses and results with other works in the remarks below.

Theorem 1. *(Existence). Let the Hilbert spaces $V(t) \subset V \subset W$, sesquilinear forms $a(t; \cdot\,, \cdot)$ and $b(t; \cdot\,, \cdot)$ on V and W, respectively, $u_0 \in W$ and $f \in L^2(T, V')$*

be given as above. Assume that $\{b(t; \cdot\,, \cdot): t \in T\}$ *is a regular family of Hermitian forms on* W:

$$b(t; x, y) = \overline{b(t; y, x)}, \qquad x, y \in W, \quad t \in T,$$

and $b(0; x, x) \geqq 0$ *for* $x \in W$. *Assume that for some real* λ *and* $c > 0$

$$(2.3) \quad 2 \operatorname{Re} a(t; x, x) + \lambda b(t; x, x) + b'(t; x, x) \geqq c \, \|x\|_V^2, \quad x \in V(t), \text{ a.e. } t \in T.$$

Then there exists a solution u *of the Cauchy problem, and it satisfies*

$$\|u\|_{L^2(T,V)} \leqq \text{const.} \, (\|f\|^2_{L^2(T,V')} + b(0; u_0, u_0))^{\frac{1}{2}},$$

where the constant depends only on λ *and* c.

Proof. Note first that by a standard change-of-variable argument, we may replace $a(t; \cdot\,, \cdot)$ by $a(t; \cdot\,, \cdot) + (\lambda/2)b(t; \cdot\,, \cdot)$ in the equation (2.1). Hence we may assume that $\lambda = 0$ in (2.3):

$$2 \operatorname{Re} a(t; x, x) + b'(t; x, x) \geqq c \, \|x\|^2, \quad x \in V(t), \quad \text{a.e. } t \in T.$$

Define $H = L^2(T, V(t))$ with the norm $(\|u\|_H)^2 = \int_0^1 \|u(t)\|^2 \, dt$ and let $F \equiv \{\varphi \in H: \varphi' \in L^2(T, W), \varphi(1) = 0\}$ with the norm $(\|\varphi\|_F)^2 = (\|\varphi\|_H)^2 + b(0; \varphi(0), \varphi(0))$. For $u \in H$ and $\varphi \in F$ we define

$$E(u, \varphi) = \int_0^1 a(t; u(t), \varphi(t)) \, dt - \int_0^1 b(t; u(t), \varphi'(t)) \, dt$$

$$L(\varphi) = \int_0^1 \langle f(t), \varphi(t) \rangle \, dt + b(0; u_0, \varphi(0)).$$

Then $E: H \times F \to \mathbf{C}$ is sesquilinear and $L: F \to \mathbf{C}$ is antilinear, and we have the estimates

$$|E(u, \varphi)| \leqq K_a \|u\|_H \|\varphi\|_H + K_b \|u\|_H \|\varphi'\|_{L^2(T,W)},$$

$$\begin{aligned} |L(\varphi)| &\leqq \|f\|_{L^2(T,V')} \cdot \|\varphi\|_H + b(0; u_0, u_0)^{\frac{1}{2}} b(0; \varphi(0), \varphi(0))^{\frac{1}{2}} \\ &\leqq (\|f\|^2_{L^2(T,V')} + b(0; u_0, u_0))^{\frac{1}{2}} \|\varphi\|_F . \end{aligned}$$

These imply that $u \to E(u, \varphi)$ is continuous $H \to \mathbf{C}$ for each φ in F and that $L: F \to \mathbf{C}$ is continuous.

Finally, for φ in F we have from Lemma 1 and (2.3) (with $\lambda = 0$)

$$\begin{aligned} 2 \operatorname{Re} E(\varphi, \varphi) = &\int_0^1 2 \operatorname{Re} a(t; \varphi(t), \varphi(t)) \, dt \\ &+ \int_0^1 \{b'(t; \varphi(t), \varphi(t)) - D_t b(t; \varphi(t), \varphi(t))\} \, dt \\ &\geqq c \, \|\varphi\|_H^2 + b(0; \varphi(0), \varphi(0)) \geqq \min(c, 1) \, \|\varphi\|_F^2 . \end{aligned}$$

The above estimates show that a theorem of J. Lions is applicable [5, p. 169; 22, p. 37]. In particular, there is a $u \in H$ for which $E(u, \varphi) = L(\varphi)$ for all $\varphi \in F$, and $||u||_H \leqq (2/\min(c, 1)) \; ||L||_{F'}$. But then u is a solution of the Cauchy problem, and it satisfies the indicated estimate. Q.E.D.

Remarks. The assumption that (2.3) holds for some λ is weaker than requiring that for some $c > 0$ and $\alpha \geqq 0$ we have

$$b'(t; x, x) + \alpha b(t; x, x) \geqq 0, \; x \in V(t), \; a.e. \; t \in T, \tag{2.4}$$

$$\operatorname{Re} a(t; x, x) + \lambda b(t; x, x) \geqq c \; ||x||_V^2. \tag{2.5}$$

In particular, if (2.4) and (2.5) hold, then an easy estimate shows that (2.3) holds with λ replaced by $2\lambda + \alpha$. The pair of estimates above will hold in many of our applications below, *e.g.*, all those of Section 4. Note that (2.4) together with $b(0; x, x) \geqq 0$ implies that $b(t; x, x) \geqq 0$ for all $t \in T$. We give examples in Section 3 where $b(t; x, x)$ may be negative for some $t > 0$ in the situation where Theorem 1 holds. However, the solution will not be unique.

We give a uniqueness result below in which we assume the leading forms are non-negative on the diagonal, and the forms $a(t; \cdot, \cdot)$ are regular and Hermitian and satisfy an estimate like (2.5). All of these hypotheses are reasonable, as we can show by examples. However, we also add a rather severe restriction of monotonicity on the subspaces $\{V(t)\}$. This hypothesis is certainly "ad hoc", but we claim to justify it, first, by exhibiting certain examples of boundary value problems in which it is fulfilled, and, second, by noting the elementary character of our proofs. In the special case of the Cauchy problem obtained by setting $b(t; x, x) = \langle x, x \rangle$ in (1.1), our uniqueness and regularity theorems fall short of the best known results [5]. (We have obtained a uniqueness result in the situation of Theorem 1 where (2.5) holds and the subspaces $\{V(t)\}$ are constant, but the proof is technical and does not have the elementary character of those proofs presented here. Thus, we choose to omit it from this presentation.) Finally, we note that even if we assume the subspaces $\{V(t)\}$ are constant, the results are new, and they give new and worthwhile examples.

Theorem 2. (*Uniqueness*). *Let the Hilbert spaces* $V(t) \subset V \subset W$, *sesquilinear forms* $a(t; \cdot, \cdot)$ *and* $b(t; \cdot, \cdot)$ *on* V *and* W, *respectively,* $u_0 \in W$ *and* $f \in L^2(T, V')$ *be given as above. Assume that*

$$\operatorname{Re} b(t; x, x) \geqq 0, \quad x \in V(t), \quad a.e. \; t \in T,$$

that $\{a(t; \cdot, \cdot) : t \in T\}$ *is a regular family of Hermitian forms on* $V(t)$:

$$a(t; x, y) = \overline{a(t; y, x)}, \qquad x, y \in V(t), \; a.e. \quad t \in T, \tag{2.6}$$

and for some real λ *and* $c > 0$

$$a(t; x, x) + \lambda \operatorname{Re} b(t; x, x) \geqq c \; ||x||^2, \; x \in V(t), \; a.e. \; t \in T.$$

Finally, assume that the family of subspaces $\{V(t) : t \in T\}$ *is decreasing*:

$$t > \tau, \; t, \tau \in T \quad imply \quad V(t) \subset V(\tau). \tag{2.7}$$

Then there is at most one solution of the Cauchy problem.

Proof. Let $u(\cdot)$ be a solution of the Cauchy problem with $u_0 = 0$ and $f(\cdot) = 0$. By linearity it suffices to show that $u(\cdot) = 0$. Let $s \in (0, 1)$ and define $v(t) = -\int_t^s u(\tau)d\tau$ for $t \in [0, s]$ and $v(t) = 0$ for $t \in [s, 1]$. Then $v \in L^2(T, V)$ and (2.7) shows $v(t) \in V(t)$ for each $t \in T$. Also, $v'(t) = -u(t)$ for $t \in (0, s)$ and $v'(t) = 0$ for $s \in (0, 1)$, so we have $v' \in L^2(T, V(t) \subset L^2(T, W)$ and $v(1) = 0$. Since $u(\cdot)$ is a solution we have by (2.1)

$$\int_0^s a(t; v'(t), v(t))\, dt - \int_0^s b(t; u(t), u(t))\, dt = 0.$$

Lemma 1 and (2.6) then yield

$$\int_0^s 2 \operatorname{Re} b(t; u(t), u(t))\, dt = \int_0^s \{D_t a(t; v(t), v(t)) - a'(t; v(t), v(t))\}\, dt,$$

$$\int_0^s \{2 \operatorname{Re} b(t; u(t), u(t)) + a'(t; v(t), v(t))\}\, dt + a(0; v(0), v(0)) = 0.$$

As before, we may assume

$$a(t; x, x) \geqq c\, \|x\|^2, \quad x \in V(t),\, t \in T.$$

Define $W(t) = \int_0^t u(\tau)d\tau$; then $W(s) = -v(0)$ and $W(t) - W(s) = v(t)$ for $t \in (0, s)$ and we obtain the estimate

$$\begin{aligned} c\, \|W(s)\|^2 &\leqq a(0; W(s), W(s)) + \int_0^s 2 \operatorname{Re} b(t; u(t), u(t))\, dt \\ &= -\int_0^s a'(t; v(t), v(t))\, dt \leqq \int_0^s M(t)\, \|v(t)\|^2\, dt \\ &\leqq 2 \int_0^s M(t)\{\|W(t)\|^2 + \|W(s)\|^2\}\, dt. \end{aligned}$$

Choose $s_0 > 0$ so that $2 \int_0^{s_0} M(t)dt < c$. Then for $s \in [0, s_0]$ we have

$$\|W(s)\|^2 \leqq \left\{2 \Big/ \left(c - 2 \int_0^{s_0} M(t)\, dt\right)\right\} \int_0^s M(t)\, \|W(t)\|^2\, dt.$$

From the Gronwall inequality [5, p. 124] we conclude that $W(s) = 0$ for $s \in [0, s_0]$, hence $u(s) = 0$ for $s \in [0, s_0]$. Since $M(\cdot)$ is integrable on T we could use the absolute continuity of the integral $\int M(t)dt$ to choose $s_0 > 0$ in the above so that $\int_\tau^{\tau+s_0} M(t)dt < c/2$ for every $\tau \geqq 0$ with $\tau + s_0 \leqq 1$. Then we apply the above argument a finite number of times to obtain $u(\cdot) = 0$ on T. Q.E.D.

In the remainder of this section we shall examine some properties of solutions of the Cauchy problem which are related to initial conditions and regularity. Thus, let $u(\cdot)$ be such a solution and assume for the moment that the family $\{V(t): t \in T\}$ is decreasing as in (2.7). Let $\tau \in (0, 1)$, $v \in V(\tau)$ and $\varphi \in \mathfrak{D}(0, \tau)$. Define $v(t) = \varphi(t)v$ for $t \in (0, \tau)$ and $v(t) = 0$ for $t \in [\tau, 1]$. Then $v(\cdot) \in L^2(T, V(t)) \cap H^1(T, W)$ and $v(1) = 0$, so

$$\int_0^\tau a(t; u(t), v)\varphi(t)\,dt - \int_0^\tau b(t; u(t), v)\varphi'(t)\,dt = \int_0^\tau \langle f(t), v\rangle\varphi(t)\,dt.$$

That is, we have the identity

$$a(\cdot\,; u(\cdot), v) + D_t b(\cdot\,; u(\cdot), v) = \langle f(\cdot), v\rangle \tag{2.8}$$

in the space $\mathfrak{D}'(0, \tau)$ of distributions on $(0, \tau)$ for every $v \in V(\tau)$. The first and last terms of (2.8) are in $L^2(0, \tau)$ and so, then, is the second and we thus have pointwise values *a.e.* in (2.8) and hence the equation

$$\int_0^\tau a(t; u(t), v)g(t)\,dt + \int_0^\tau D_t b(t; u(t), v)g(t)\,dt = \int_0^\tau \langle f(t), v\rangle g(t)\,dt \tag{2.9}$$

for $g \in L^2(0, \tau)$ and $v \in V(\tau)$.

Suppose now that $\varphi \in H^1(0, \tau)$ is given with $\varphi(\tau) = 0$. Define $v(\cdot)$ as above so as to obtain from (2.1)

$$\int_0^\tau a(t; u(t), v)\varphi(t)\,dt - \int_0^\tau b(t; u(t), v)\varphi'(t)\,dt$$
$$= \int_0^\tau \langle f(t), v\rangle\varphi(t)\,dt + b(0; u_0, v)\varphi(0)$$

where $v \in V(\tau)$. Since (2.9) holds with $g(t) = \varphi(t)$, we obtain from these

$$\int_0^\tau \{D_t b(t; u(t), v)\varphi(t) + b(t; u(t), v)\varphi'(t)\}\,dt = -b(0; u_0, v)\varphi(0).$$

The integrand is the derivative of the function $t \to b(t; u(t), v)\varphi(t)$ in $H^1(0, \tau)$ and $\varphi(\tau) = 0$, so we have proved the following.

Theorem 3. *(Initial Conditions). Let $u(\cdot)$ be a solution of the Cauchy problem and assume the subspaces $\{V(t): t \in T\}$ are initially decreasing: there is a $t_0 \in (0, 1]$ such that (2.7) holds for $t, \tau \in [0, t_0]$. Then for every $\tau \in (0, t_0)$ and $v \in V(\tau)$ we have $t \to b(t; u(t), v)$ is in $H^1(0, \tau)$ (hence is absolutely continuous with distribution derivative in $L^2(0, \tau)$) and $b(0; u(0) - u_0, v) = 0$. Thus, if $\cup\,\{V(\tau): 0 < \tau < t_0\}$ is dense in W, then $b(0; u(0) - u_0, u(0) - u_0) = 0$.*

Another situation which arises frequently in applications (see Section 4) and to which a variation of the above technique is applicable is the following.

Theorem 4. *Let $u(\cdot)$ be a solution of the Cauchy problem and assume there is a closed subspace V_0 in V with $V_0 \subset \cap\,\{V(t): t \in T\}$. Define the two families of linear operators $\{A(t)\} \subset \mathfrak{L}(V, V_0')$ and $\{B(t)\} \subset \mathfrak{L}(W, V_0')$ by*

$$\langle A(t)x, y\rangle = a(t; x, y), \qquad x \in V, \quad y \in V_0, \quad t \in T$$
$$\langle B(t)x, y\rangle = b(t; x, y), \qquad x \in W, \quad y \in V_0, \quad t \in T.$$

Then, in the space $\mathfrak{D}'(V_0')$ of V_0'-valued distributions on T we have

$$A(t)u(t) + D_t(B(t)u(t)) = f(t) \tag{2.10}$$

and the function $t \to B(t)u(t)\colon T \to V_0'$ *is continuous.*

The proof of Theorem 4 follows from (2.8) with $v \in V_0$ and the remark that each term in (2.10), as well as $B(\cdot)u(\cdot)$, is in $L^2(T, V_0')$.

Our last result concerns variational boundary conditions. Suppose we have the situation of Theorem 4 and also that (2.7) holds. Let H be a Hilbert space in which V is continuously imbedded and V_0 is dense. We identify H' with H by the Riesz theorem and hence obtain $V_0 \to H \to V_0'$ and the identity $\langle h, v\rangle = (h, v)_H$ for $h \in H$, $v \in V_0$. Assume $f \in L^2(T, H)$, $\tau > 0$ and $v \in V(\tau)$. Then from (2.9) and (2.10) we obtain

$$\int_0^\tau a(t; u(t), v)\varphi(t)\,dt + \int_0^\tau D_t b(t; u(t), v)\varphi(t)\,dt$$
$$= \int_0^\tau (A(t)u(t) + D_t(B(t)u(t)), v)_H\varphi(t)\,dt$$

for each $\varphi \in \mathfrak{D}(0, \tau)$. This gives us the following.

Theorem 5. (*Variational Boundary Conditions*). *Assume the situation of Theorem* 4 *and that* (2.7) *holds. Let* H *be given as above and* $f \in L^2(T, H)$. *Then for each* $\tau > 0$ *and* $v \in V(\tau)$ *we have in* $L^2(0, \tau)$

$$a(t; u(t), v) + D_t b(t; u(t), v) = (A(t)u(t) + D_t B(t)u(t), v)_H\,. \tag{2.11}$$

3. Examples I. We shall apply the preceding results to a discussion of the first initial-boundary value problem for the partial differential equation (1.2) in one spatial dimension. The Dirichlet boundary conditions are realized by choosing $V_0 = V(t) = V = H^1(T)_0$, all $t \in T$, and we also set $H = L^2(T)$ in the following. Problems in higher dimension and with variable domain will be presented in the next section. In our first two examples, the coefficients in (1.2) are the characteristic functions of certain subsets of the xt-plane, and the third example is the case of a smooth $b_0(x, t) \geqq 0$ with $b(x, t) \equiv 0$. Our objective is to illustrate in these simple models some conditions on the coefficients sufficient to attain the hypotheses in our theorems of Section 2.

(*a*) *An elliptic-parabolic equation.* For our first example, choose $W = \{\phi \in H^1(T)\colon \phi(1) = 0\}$. If $\phi \in W$ and $x \in T$ we have

$$|\phi(x)| = \left|\int_x^1 \phi'\right| \leqq |x - 1|^{\frac{1}{2}}\,\|\phi'\|_{L^2(T)}$$

and, hence, the estimate

$$\sup\{|\phi(x)|\colon x \in T\} \leqq \|\phi'\|_{L^2(T)}\,,\ \phi \in W. \tag{3.1}$$

Let $\alpha\colon T \to T$ be absolutely continuous and define

$$b(t; \phi, \psi) = \int_0^{\alpha(t)} \phi(x)\overline{\psi(x)},dx, \qquad \phi, \psi \in W, \quad t \in T.$$

Then for each pair $\phi, \psi \in W$, the function $b(\cdot\ ; \phi, \psi)$ is absolutely continuous [24, p. 214] and

$$b'(t; \phi, \psi) = \phi(\alpha(t))\overline{\psi\alpha(t))}\alpha\ (t), \qquad a.e.\ t \in T. \tag{3.2}$$

Since $\alpha' \in L^1(T)$, (3.1) and (3.2) show that the family of Hermitian sesquilinear forms $\{b(t; \cdot\ , \cdot): t \in T\}$ is regular. Define a second family of forms by

$$a(t; \phi, \psi) = \int_0^1 \phi'(x)\overline{\psi'(x)}\, dx, \qquad \phi, \psi \in V, \quad t \in T.$$

This is (trivially) a regular family of Hermitian forms on V. The well-known estimate from the calculus of variations

$$\pi\ ||\phi||_{L^2(T)} \leqq ||\phi'||_{L^2(T)}\ , \qquad \phi \in H_0^1(T), \tag{3.3}$$

shows that (2.5) is satisfied with $\lambda = 0$. Finally, let $u_0 \in W$ and $F \in L^2(T \times T)$ be given, and define $f(t) = F(\cdot\ , t)$, $t \in T$. Then all of the hypotheses hold in our uniqueness theorem. Furthermore, if there is a number Σ, $0 < \Sigma < 2$, such that

$$\alpha'(t) \geqq \Sigma - 2,\ a.e.\ t \in T, \tag{3.4}$$

then (3.1) and (3.2) imply that

$$2a(t; \phi, \phi) + b'(t; \phi, \phi) \geqq (2 - \Sigma)\ ||\phi'||^2_{L^2(T)}\ , \qquad \phi \in V,$$

and, hence, by (3.3), it follows that (2.3) is satisfied with $\lambda = 0$. (Note that (2.4) holds if and only if $\alpha'(t) \geqq 0$, $a.e.\ t \in T$, whereas (3.4) allows α to decrease, but not "too fast".) The operators of Theorem 4 are given by $(B(t)\phi)(x) = b_0(x, t)\phi(x)$ (where $b_0(x, t) = 1$ for $0 < x < \alpha(t)$ and $b_0(x, t) = 0$ for $\alpha(t) < x < 1$) and $A(t)\phi = -\phi''$ (where the derivatives are taken as distributions on T.)

Let $u(\cdot\ , t) = u(t)$ be the unlque solution of the Cauchy problem. Then (2.10) gives a weak form of the partial differential equation (1.2) with $b(x, t) \equiv 0$, $u(t) \in V = H^1(T)_0$ gives the boundary conditions, $u(0, t) = u(1, t) = 0$, $a.e.\ t \in T$, and Theorem 3 gives the initial condition

$$u(x, 0) = u_0(x), \quad 0 < x < \alpha(0).$$

It follows from (2.10) that $u(t) \in H^2(T)$ at each $t \in T$, so $u(\cdot\ , t)$ and $u_x(\cdot\ , t)$ are continuous across the curve $x = \alpha(t)$. We note that this curve, determined by the smooth function α, is non-characteristic at $a.e.\ t \in T$, but ($e.g.$, $(x - 1/2)^3 = (t - 1/2)/4$) may actually be characteristic at certain points.

(b) *An elliptic-parabolic-Sobolev equation.* Our second example is similar but allows the equation to be of Sobolev type in portions of $T \times T$. Choose the spaces and define the forms $a(t; \cdot\ , \cdot)$ as above. Let α and β be absolutely continuous maps of T into itself and $b > 0$. For $\phi, \psi \in W$ define

$$b(t; \phi, \psi) = \int_0^{\alpha(t)} \phi(x)\overline{\psi(x)}\, dx + b\int_0^{\beta(t)} \phi'(x)\overline{\psi'(x)}\, dx, \qquad t \in T.$$

If we require that β be non-decreasing, then $b(\cdot\ ; \phi, \psi)$ is absolutely continuous [24, pp. 214–215], and we have

$$b'(t; \phi, \psi) = \phi(\alpha(t))\overline{\psi(\alpha(t))}\alpha'(t) + b\phi'(\beta(t))\overline{\psi'(\beta(t))}\, \beta'(t), \qquad a.e.\ t \in T.$$

By the Remark following Lemma 1, it follows that the result of Lemma 1 (and hence the proof of Theorem 1) holds for the family $\{b(t; \cdot\ , \cdot)\}$, even though it is not regular. Thus we need only check estimates to apply the theorems of Section 2. But $a(t; \cdot\ , \cdot)$ is coercive, as before, and $b(t; \cdot\ , \cdot)$ is non-negative, so we need only to check (2.3).

We shall show that (2.3) holds if $L \equiv \text{ess inf}\ \{\alpha'(t) : t \in T\} > -\infty$ and there is a number Σ, $0 < \Sigma < 2$, such that $\alpha'(t) \geqq \Sigma - 2$ whenever $\alpha(t) > \beta(t)$. First, note that for $a.e.\ t \in T$ and any $\lambda \geqq 0$

$$\text{(3.5)} \qquad 2 \operatorname{Re} a(t; \phi, \phi) + \lambda b(t; \phi, \phi) + b'(t; \phi, \phi)$$
$$\geqq 2 \int_0^1 |\phi'|^2 + \lambda b \int_0^{\beta(t)} |\phi'|^2 + |\phi(\alpha(t))|^2\, \alpha'(t).$$

(We have used the fact that $\beta'(t) \geqq 0$.) If $\alpha(t) \leqq \beta(t)$, then the estimate

$$\sup \{ |\phi(s)|^2 : 0 \leqq s \leqq \beta(t)\} \leqq \int_0^{\beta(t)} |\phi'|^2, \qquad \phi \in H^1(T)_0$$

shows that the right side of (3.5) is bounded from below by $2 \int_0^1 |\phi'|^2$, where we have chosen λ so large that $\lambda b \geqq |L|$. If $\alpha(t) > \beta(t)$, we drop the middle term in the right side of (3.5) and show as in (a) that the remaining terms are bounded from below by $(2 - \Sigma) \int_0^1 |\phi'|^2$. Thus (3.5) holds for $a.e.\ t \in T$, where our choice of λ depends on the essential infimum of $\alpha'(t)$.

Let $u_0 \in W$, $F \in L^2(T \times T)$ and set $f(t) = F(\cdot\ , t)$, $t \in T$. Theorems 1 and 2 assert the existence of a unique solution $u(t) = u(\cdot\ , t)$ of the Cauchy problem (2.1), and (2.10) is a weak form of (1.2) in which the operator $A(t)$ is given as above and

$$(B(t)\phi)(x) = b_0(x, t)\phi(x) - b(x, t)\phi''(x),$$

where $b_0(x, t)$ was given above and $b(x, t) = b$ for $0 < x < \beta(t)$ and $b(x, t) = 0$ for $\beta(t) < x < 1$. The inclusions $u(t) \in V(t) = H^1(T)_0$ give the Dirichlet boundary conditions as before, and Theorem 3 gives the initial condition

$$u(x, 0) = u_0(x), \quad 0 < x < \max\ \{\alpha(0), \beta(0)\}.$$

Equation (2.10) shows that $u(t) \in H^2(\beta(t), 1)$; examples can be given (see, *e.g.* [28]) to show that $u(t) \in H^1(0, \beta(t))$ is the best regularity we can expect in general. Note that the constraint that we needed on $\alpha(t)$ in this example is weaker than (3.4): the estimate in (3.4) was necessary only in that region in which $b(x, t) = 0$. The assumption that β be non-decreasing was used not only in estimates like (3.5) but also to obtain the absolute continuity of the functions $b(\cdot\ ; \phi, \psi)$. We can give examples (*e.g.*, set $\alpha(t) = 0$ in the above) to show that if β is permitted to decrease, then a compatibility condition is imposed on the initial data, u_0. Such behavior should be expected, since the lines "x = const." are characteristics for the third order equation (1.2) whenever $b(x, t) > 0$.

Before presenting the last example of this section, we prove a result which gives a large class of regular families of sesquilinear forms on $L^2(\Omega)$, and hence $H^k(\Omega)$, where Ω is any measurable set. This result will be used in all of the examples to follow.

Lemma 2. *Let $f\colon \Omega \times T \to \mathbf{C}$ be measurable and assume $f(\cdot\,, t) \in L^1(\Omega)$ for every $t \in T$, $f(x, \cdot)$ is absolutely continuous for a.e. $x \in \Omega$, and $|\partial f/\partial t| \leqq F$, where $F \in L^1(\Omega \times T)$. Then the function $t \to \int_\Omega f(x, t)dx$ is absolutely continuous and $|(\partial/\partial t) \int_\Omega f(x, t)dx| \leqq \int_\Omega F(x, t)dx$, a.e. $t \in T$.*

Proof. Using the Fubini Theorem and then the Fundamental Theorem of Calculus we have for $t \in T$

$$\int_0^t \int_\Omega (\partial f/\partial t)\, dx\, dt = \int_\Omega \int_0^t (\partial f/\partial t)\, dt\, dx = \int_\Omega (f(x, t) - f(x, 0))\, dx,$$

$$\int_\Omega f(x, t)\, dx = \int_\Omega f(x, 0)\, dx + \int_0^t \left\{ \int_\Omega (\partial f(\partial t)\, dx \right\} dt.$$

Since $\int_\Omega (\partial f(x, \cdot)/\partial t)dx$ is in $L^1(T)$, the result follows. Q.E.D.

Definition. A function $a(\cdot\,, \cdot) \in L^\infty(\Omega \times T)$ is *regular* if $a(x, \cdot)$ is absolutely continuous for *a.e.* $x \in \Omega$ and $|\partial a(x, t)/\partial t| \leqq M(t)$ for *a.e.* $t \in T$, where $M(\cdot) \in L^1(T)$.

Corollary. *Let $a(\cdot\,, \cdot)$ be regular and define a family of sesquilinear forms on $L^2(\Omega)$ by*

$$a(t; u, v) = \int_\Omega a(x, t)u(x)\overline{v(x)}\, dx, \qquad u, v \in L^2(\Omega), \quad t \in T.$$

Then $\{a(t; \cdot\,, \cdot) : t \in T\}$ is a regular family.

Proof. Use Lemma 2 with $f(x, t) = a(x, t)u(x)v(x)$ and $F(x, t) = M(t)\, |u(x)v(x)|$. Q.E.D.

(c) *Elliptic-(possibly backward) parabolic equation.* For our third example, we choose the spaces $V_0 = V(t) = V = H^1(T)_0$ and the forms $a(t; \cdot\,, \cdot)$ as in (a), and we also set $H = W = L^2(T)$. Let $b_0(\cdot\,, \cdot)$ be a regular real-valued function on $T \times T$ and define a (regular) family of sesquilinear forms on H by

$$b(t; \phi, \psi) = \int_0^1 b_0(x, t)\phi(x)\overline{\psi(x)}\, dx, \qquad \phi, \psi \in H, \quad t \in T.$$

Let $u_0 \in H$, $F \in L^2(T \times T)$ and set $f(t) = F(\cdot\,, t)$, $t \in T$. If $u(t)$ is a solution of the Cauchy problem (2.1), then it follows from Theorem 4 that the function defined by $u(\cdot\,, t) = u(t)$, $t \in T$, is a weak solution of the equation

$$\frac{\partial}{\partial t}(b_0(x, t)u(x, t)) - u_{xx}(x, t) = F(x, t), \qquad (x, t) \in T \times T.$$

Theorem 3 asserts that we have the initial condition

$$b_0(x, 0)\{u(x, 0) - u_0(x)\} = 0, \quad a.e.\ x \in T,$$

and the inclusions $u(t) \in H^1(T)_0$ give the null Dirichlet boundary conditions, $u(0, t) = u(1, t) = 0$, $t \in T$.

We shall consider separately the questions of uniqueness and existence of a solution. Note first that, as before, the forms $a(t; \cdot, \cdot)$, $t \in T$, are coercive over $H^1(T)_0$. Hence, in order to satisfy the hypotheses of Theorem 2 it suffices to assume in addition that

$$b_0(x, t) \geqq 0, \quad x, t \in T, \tag{3.6}$$

for this is equivalent to $b(t; \phi, \phi) \geqq 0$ for all $\phi \in H$ and $t \in T$. On the other hand, the hypotheses of Theorem 1 are satisfied if we assume

$$b_0(x, 0) \geqq 0,\ a.e. \quad x \in T, \quad \text{and} \quad \operatorname*{ess\,inf}_{T\times T} \{\partial b_0/\partial t\} > -2\pi^2. \tag{3.7}$$

If (3.7) holds, then there is a number Σ, $0 < \Sigma < 2\pi^2$, such that

$$(\partial/\partial t)b_0(x, t) \geqq \Sigma - 2\pi^2, \quad a.e.\ x, t \in T,$$

so we obtain for $\phi \in H^1(T)_0$, $a.e.\ t \in T$

$$b'(t; \phi, \phi) \geqq (\Sigma - 2\pi^2) \int_0^1 |\phi|^2.$$

From (3.3) we easily obtain (2.3) with $\lambda = 0$ and $c = \Sigma/2\pi^2$. Thus, (3.6) implies uniqueness of a solution and (3.7) is sufficient for existence.

Neither one of the conditions (3.6), (3.7) implies the other. In particular, (3.7) permits the coefficient $b_0(x, t)$ to attain negative values over large regions, and in such a region the partial differential equation may be backward parabolic. Such an example is given by

$$(\partial/\partial t)\{(\pi^2/4)(1 - 2t)^3 u(x, t)\} = \partial^2 u/\partial x^2.$$

For this example we have $\partial b_0/\partial t = (3\pi^2/4)(1 - 2t)^2 \geqq 0$ for all $t \in T$, so (3.7) is clearly satisfied, and Theorem 1 asserts the existence of a solution. Since b_0 is non-negative only on $[0, 1/2]$, we can use Theorem 2 to claim uniqueness only on this smaller interval. A solution of the first initial-boundary value problem with $u_0(x) = \sin(\pi x)$, $x \in T$, is given by

$$u(x, t) = (1 - 2t)^{-3} \exp\{1 - (1 - 2t)^{-2}\} \sin(\pi x).$$

To see that uniqueness does not hold beyond $t = 1/2$, note that a second solution can be obtained by changing the values of this function to zero for all $t > 1/2$.

Finally, we give an example in which (3.6) holds but (3.7) is violated. Let $N > 0$ and consider the first initial-boundary problem with u_0 given as above for the equation

$$(\partial/\partial t)\{(\pi^2/N)(1 - t)^{1/2} u(x, t)\} = \partial^2 u/\partial x^2.$$

The coefficient b_0 is non-negative so Theorem 2 asserts the uniqueness of a solution on T (or any subinterval). However, the time-derivative of b_0 is greater than $-2\pi^2$ only on the interval $[0, 1 - (4N)^{-2})$. Hence, for any $\Sigma > 0$, Theorem 1 asserts the existence of a solution of the above problem on the interval $[0, 1 - (4N)^{-2} - \Sigma]$. By choosing N large, we can make the corresponding interval of existence as close as desired to $[0, 1]$. However, it is easy to verify that the function

$$u(x, t) = (1 - t)^{-1/2} \exp \{2N(1 - t)^{1/2}\} \sin (\pi x)$$

is the unique solution on $[0, 1 - \Sigma]$ for any Σ, $0 < \Sigma < 1$, and this function is not $L^2(T \times T)$. Hence there is no solution on T.

4. Examples II. In this final section we apply our abstract results of Section 2 to four different types of boundary value problems. The necessary preliminary material on function spaces is given quickly, and one may consult [23] for additional details. The new points in our examples are not only that the coefficients appearing with time derivatives are not assumed to have positive uniform lower bounds, but also that we may add terms containing "traces" on a boundary or submanifold. Such terms are always degenerate in our sense, and they permit a much broader application of the theory than would be possible if one needed to assume coercivity estimates on them. Our first two examples exhibit a "monotone" variable domain for which Theorem 2 is applicable. Similar situations can easily be introduced in the remaining examples.

For each $i = 1, 2$, let Ω_i be a bounded open set in real Euclidean n-dimensional space and assume Ω_i is on one side of its piecewise continuously differentiable $(n - 1)$-dimensional boundary $\partial\Omega_i$ [12, 262–271]. Thus the Divergence Theorem holds on each Ω_i, and we denote by

$$n_i(s) = (n_i^1(s), n_i^2(s), \cdots, n_i^n(s))$$

the unit outward normal at points $s \in \partial\Omega_i$. Assume Ω_1 and Ω_2 are disjoint and their respective boundaries intersect in a submanifold Γ of dimension $n - 1$. Then $n_1(s) = -n_2(s)$ for $s \in \Gamma$. Let Ω be the interior of the closure of $\Omega_1 \cup \Omega_2$ so that the boundary of Ω is given by $\partial\Omega = \partial\Omega_1 \cup \partial\Omega_2 \sim (\Gamma \sim \partial\Gamma)$. We shall denote the Lebesgue measure in Ω by "dx" and that on $\partial\Omega_1 \cup \partial\Omega_2$ by "ds".

Let H be the Hilbert space $L^2(\Omega)$ of (equivalence classes of) square-integrable functions on Ω. This space is isometrically-isomorphic to the direct sum $L^2(\Omega_1) \oplus L^2(\Omega_2)$ in the natural way and we shall hereafter identify them. Let $H^1(\Omega_i)$ be the Sobolev space of those $\varphi \in L^2(\Omega_i)$ for which the distributional derivatives $D_j\varphi$ for $j = 1, 2, \cdots, n$ are in $L^2(\Omega_i)$. Then $H^1(\Omega_i)$ is a Hilbert space with inner-product

$$(\varphi, \psi)_{H^1(\Omega_i)} = \int_{\Omega_i} \left\{\varphi\bar{\psi} + \sum_{j=1}^{n} D_j\varphi D_j\bar{\psi}\right\} dx.$$

The trace operators $\gamma_i \in \mathfrak{L}(H^1(\Omega_i), L^2(\partial\Omega_i))$ are well-defined and coincide with the restriction to the boundary on those functions in $H^1(\Omega_i)$ which are continuous on the closure of Ω_i. (See [23] for these and later unsupported results on Sobolev spaces and trace.) Since $C^\infty(\bar{\Omega}_i)$ is dense in $H^1(\Omega_i)$ and the trace γ_i is continuous, we can extend the Divergence Theorem on Ω_i to obtain

$$\int_{\Omega_i} \{u(x)D_j v(x) + v(x)D_j u(x)\}\, dx = \int_{\partial\Omega_i} (\gamma_i u)(s)(\gamma_i v)(s) n_i^j(s)\, ds$$

for $u, v \in H^1(\Omega_i)$. Thus if $u_1 \in H^1(\Omega_1)$ and $u_2 \in H^1(\Omega_2)$ we have for each $\varphi \in C_0^\infty(\Omega)$

$$\int_\Omega \{uD_j)D_j\varphi + \varphi D_j u\}\, dx = \int_\Gamma (u_1(s) - u_2(s))\varphi(s) n_1^j(s)\, ds$$

where we have suppressed the trace operator and u is the element of $L^2(\Omega)$ which we identify as above with the pair (u_1, u_2) in $L^2(\Omega_1) \oplus L^2(\Omega_2)$. Thus, $u \in H^1(\Omega)$ if and only if $u_1(s) = u_2(s)$ for *a.e.* $s \in \Gamma$. Finally, we remark that the kernel of γ_i is the closure in $H^1(\Omega_i)$ of $\mathfrak{D}(\Omega_i)$, the infinitely differentiable functions with compact support in Ω_i, and we denote it by $H^1(\Omega_i)_0$. The dual of $H^1(\Omega_i)_0$ can be identified with a subspace of distributions on Ω_i [23, p. 78].

(*a*) *Degenerate elliptic-parabolic interface problem.* Let a_0, a and b be regular functions on $\Omega \times T$ and assume that, for each $i = 1, 2$ and each $t \in T$, the restriction of $a(\cdot\,, t)$ $(b(\cdot\,, t))$ to Ω_i is uniformly continuous; we denote this restriction by $a_i(\cdot\,, t)$ (respectively, $b_i(\cdot\,, t)$). Define the Hilbert spaces $H = W = L^2(\Omega)$, $V = H^1(\Omega_1) \oplus H^1(\Omega_2)$ and $V_0 = H^1(\Omega_1)_0 \oplus H^1(\Omega_2)_0$. Let $\{\Gamma_t : t \in T\}$ be a family of measurable subsets of Γ and suppose $\Gamma_s \subset \Gamma_t$ when $t > s$. Similarly, $\{\Sigma_t : t \in T\}$ is a family of measurable subsets of $\partial\Omega_1$ with $\Sigma_s \subset \Sigma_t$ when $t > s$. Define $V(t) = \{\varphi \in V : \gamma_1\varphi(s) = \gamma_2\varphi(s)$ for *a.e.* $s \in \Gamma_t$; $\gamma_1\varphi(s) = 0$ for *a.e.* $s \in \Sigma_t\}$ for $t \in T$. Then (2.7) holds and each $V(t)$ is closed in V and contains V_0.

Denote by $\nabla\varphi$ the gradient of $\varphi \in H^1(\Omega)$; that is $\nabla\varphi = (D_1\varphi, D_2\varphi, \cdots, D_n\varphi)$. We now define the two families of sesquilinear forms

$$a(t; \varphi, \psi) = \int_\Omega a(x, t)(\nabla\varphi(x)\cdot\nabla\overline{\psi(x)})\, dx + \int_\Omega a_0(x, t)\varphi(x)\overline{\psi(x)}\, dx, \qquad \varphi, \psi \in V,$$

$$b(t; \varphi, \psi) = \int_\Omega b(x, t)\varphi(x)\overline{dx,} \qquad \varphi, \psi \in W.$$

Since the coefficients in these forms are regular functions, Lemma 2 shows the forms are regular. The operators $\{A(t)\}$ defined in Theorem 4 are given by

$$A(t)\varphi = -\sum_{j=1}^n D_j(a(\cdot\,, t)D_j\varphi(\cdot)) + a_0(\cdot\,, t)\varphi(\cdot), \qquad t \in T, \quad \varphi \in V \tag{4.1}$$

where the derivatives are taken as distributions on $\Omega_1 \cup \Omega_2$, and we have (formally) by the Divergence Theorem

$$a(t; \varphi, \psi) - (A(t)\varphi, \psi)_H = \int_{\partial\Omega_1} a_1(s, t)\frac{\partial\varphi}{\partial n_1}\bar{\psi}\, ds + \int_{\partial\Omega_2} a_2(s, t)\frac{\partial\varphi}{\partial n_2}\bar{\psi}\, ds \tag{4.2}$$

when $A(t)\varphi \in H$ and $\varphi \in H^2(\Omega_1) \oplus H^2(\Omega_2)$. (The latter inclusion follows from the first when $A(t)$ is regular elliptic.) The directional derivatives in (4.2) are given by $\partial\varphi/\partial n_i = \nabla\varphi\cdot n_i$, and we suppress the trace operator for simplicity. Similarly,

$$B(t)\varphi = b(\cdot\,,t)\varphi(\cdot), \quad t \in T, \quad \varphi \in W. \tag{4.3}$$

Assume the following: $u_0 \in L^2(\Omega)$ and $F \in L^2(\Omega \times T)$ are given; the coefficients in (4.1) and (4.3) are regular real-valued functions and satisfy

$$\begin{aligned} &b(x,0) \geqq 0, \qquad (\partial/\partial t)b(t,x) + \lambda b(x,t) \geqq 0, \\ &a_0(x,t) \geqq 0, \qquad a_0(x,t) + \lambda b(x,t) \geqq c, \qquad a(x,t) \geqq c \end{aligned} \tag{4.4}$$

for *a.e.* $(x,t) \in \Omega \times T$, where $\lambda \geqq 0$ and $c > 0$ are constants. Then the estimates (2.4) and (2.5) are satisfied, so Theorems 1 and 2 imply the existence and uniqueness of a solution $u(t)$ of the Cauchy problem. Setting $u(\cdot\,,t) = u(t)$, we have by Theorem 4 the partial differential equation

$$a_0(x,t)u(x,t) - \sum_{j=1}^{n} D_j(a(x,t)D_ju(x,t)) + D_t(b(x,t)u(x,t)) = F(x,t) \tag{4.5}$$

in $\mathcal{D}'((\Omega_1 \cup \Omega_2) \times T)$ and by Theorem 3 the initial condition $b(x,0)(u(x,0) - u_0(x)) = 0$ in $L^2(\Omega)$, hence

$$u(x,0) = u_0(x), \quad x \in \Omega: b(x,0) \neq 0.$$

Since $u(t) \in V(t)$ *a.e.* in T we have the boundary conditions (*a.e.*)

$$\begin{aligned} \gamma_1 u(s,t) &= \gamma_2 u(s,t), && s \in \Gamma_t\,, \\ \gamma_1 u(s,t) &= 0, && s \in \Sigma_t\,, \end{aligned} \tag{4.6}$$

and from (2.11) and (4.2) we obtain (formally) for $\tau > 0$

$$\int_{\partial\Omega_1} a_1 \frac{\partial u}{\partial n_1}\gamma_1 v\, ds + \int_{\partial\Omega_2} a_2 \frac{\partial u}{\partial n_2}\gamma_2 v\, ds = 0 \tag{4.7}$$

for *a.e.* $t \in (0,\tau)$ and each $v \in V(\tau)$. If the subsets Σ_t and Γ_t vary "smoothly from below" at the point τ, then from the variational boundary condition (4.7) we obtain

$$\frac{\partial u(s,\tau)}{\partial n_1} = 0, \qquad s \in \partial\Omega_1 \sim \Gamma_\tau \sim \Sigma_\tau\,; \frac{\partial u(s,\tau)}{\partial n_2} = 0, \qquad s \in \partial\Omega_2 \sim \Gamma_\tau\,,$$

and

$$a_1(s,\tau)\frac{\partial u(s,\tau)}{\partial n_1} = a_2(s,\tau)\frac{\partial u(s,\tau)}{\partial n_1}\,, \qquad s \in \Gamma_\tau \sim \Sigma_\tau\,.$$

The new thing in this example is that the coefficient $b(x,t)$ need not have a (strictly) positive lower bound, so (4.5) is a non-uniformly parabolic equation [13]. Our technique is a direct extension of the case $b(x,t) \equiv 1$ given in [22, pp. 103–105]. When our problem describes a heat conduction or diffusion process, then the monotonicity conditions on $\{\Gamma_t\}$ and $\{\Sigma_t\}$ require that these respective surfaces of contact between the two media and between the medium Ω_1 and the outside, respectively, be non-decreasing.

(b) *Degenerate elliptic-parabolic-Sobolev equation.* Let $a_0, a, b_0, b \in L^\infty(\Omega \times T)$, $\beta_1 \in L^\infty(\partial\Omega_1 \times T)$ and $\beta_2 \in L^\infty(\partial\Omega_2 \times T)$ be real-valued regular functions, and assume the restrictions $a_i(\cdot\,, t)$ and $b_i(\cdot\,, t)$ of $a(\cdot\,, t)$ and $b(\cdot\,, t)$, respectively, are uniformly continuous on Ω_i for $i = 1, 2$. Let $H = L^2(\Omega)$, $V = W = H^1(\Omega_1) \oplus H^1(\Omega_2)$, and $V_0 = H^1(\Omega_1)_0 \oplus H^1(\Omega_2)_0$. Let the subsets Γ_t and Σ_t determine subspaces defined as above. Define the regular families

$$a(t; \varphi, \psi) = \int_\Omega a(x, t)(\nabla\varphi(x)\cdot\nabla\overline{\psi(x)})\, dx + \int_\Omega a_0(x, t)\varphi(x)\overline{\psi(x)}\, dx,$$

$$b(t; \varphi, \psi) = \int_\Omega b(x, t)(\nabla\varphi(x)\cdot\nabla\overline{\psi(x)})\, dx + \int_\Omega b_0(x, t)\varphi(x)\overline{\psi(x)}\, dx$$
$$+ \int_{\partial\Omega_1} \beta_1(s, t)\gamma_1\varphi(s)\overline{\gamma_1\psi(s)}\, ds + \int_{\partial\Omega_2} \beta_2(s, t)\gamma_2\varphi(s)\overline{\gamma_2\psi(s)}\, ds$$

for $\varphi, \psi \in V = W$. The operators of Theorem 4 are given by (4.1) and

$$B(t)\varphi = -\sum_{j=1}^{n} D_j(b(\cdot, t)D_j\varphi) + b_0(\cdot, t)\varphi,$$

and from the Divergence Theorem we have (4.2) and

(4.8)
$$b(t; \varphi, \psi) - (B(t)\varphi, \psi)_H = \int_{\partial\Omega_1} \left(b_1 \frac{\partial\varphi}{\partial n_1} + \beta_1\varphi\right)\bar{\psi}\, ds + \int_{\partial\Omega_2} \left(b_2 \frac{\partial\varphi}{\partial n_2} + \beta_2\varphi\right)\bar{\psi}\, ds$$

for smooth φ and $\psi \in W$.

Assume the coefficients satisfy

$$b(x, 0) \geqq 0,\ (\partial/\partial t)b(x, t) + \lambda b(x, t) \geqq 0;$$

the above with b_0, β_1, β_2 in place of b;

$$a(x, t) \geqq 0,\ a_0(x, t) \geqq 0;\ a(x, t) + \lambda b(x, t) \geqq c,$$
$$a_0(x, t) + \lambda b_0(x, t) \geqq c$$

for $(x, t) \in \Omega \times T$, where $\lambda \geqq 0$ and $c > 0$ are real. Let $u_0 \in W$ and $F \in L^2(\Omega \times T)$ be given and set $f(t) = F(\cdot\,, t)$ for $t \in T$. Then, there is a unique solution of the Cauchy problem of Section 2 which satisfies in $\mathfrak{D}'((\Omega_1 \cup \Omega_2) \times T)$ the partial differential equation

$$a_0(x, t)u(x, t) - \sum_{j=1}^{n} D_j(a(x, t)D_ju(x, t))$$
$$+ D_t\left[b_0(x, t)u(x, t) - \sum_{j=1}^{n} D_j(b(x, t)D_ju(x, t))\right] = F(x, t),$$

the initial condition

$$b(0; u(0) - u_0, v) = 0, \quad \tau > 0, \quad v \in V(\tau) \tag{4.9}$$

and the boundary conditions

$$u(t) \in V(t)$$

$$\int_{\partial\Omega_1} \left[a_1 \frac{\partial u}{\partial n_1} + D_t\left(b_1 \frac{\partial u}{\partial n_1} + \beta_1 u\right)\right] v\, ds + \int_{\partial\Omega_2} \left[a_2 \frac{\partial u}{\partial n_2} + D_t\left(b_2 \frac{\partial u}{\partial n_2} + \beta_2 u\right)\right] v\, ds = 0 \tag{4.10}$$

for *a.e.* $t \in (0, \tau)$ and each $v \in V(\tau)$, where the second part of (4.10) is obtained from (4.2), (4.8) and (2.11). From the initial condition (4.9) we have

$$B(0)(u(0) - u_0) = 0 \quad \text{in} \quad H^{-1}(\Omega_1) \oplus H^{-1}(\Omega_2)$$

$$b_1(s, 0) \frac{\partial}{\partial n_1} (u(s, 0) - u_0(s)) + b_2(s, 0) \frac{\partial}{\partial n_2} (u(s, 0) - u_0(s)) + (\beta_1(s, 0) + \beta_2(s, 0))(u(s, 0) - u_0(s)) = 0, \qquad s \in \partial\Omega_1 \cup \partial\Omega_2 \sim \Sigma_0,$$

with the understanding that $b_1 = \beta_1 = 0$ on $\partial\Omega_2 \sim \Gamma$ and $b_2 = \beta_2 = 0$ on $\partial\Omega_1 \sim \Gamma$ in the above. We can interpret the boundary conditions (4.10) as follows:

$$\gamma_1 u(s, t) = 0, \qquad s \in \Sigma_t ;$$

$$\gamma_1 u(s, t) = \gamma_2 u(s, t), \quad \text{and} \quad (a_1 - a_2) \frac{\partial u}{\partial n_1} + D_t\left[(b_1 - b_2) \frac{\partial u}{\partial n_1} + (\beta_1 + \beta_2)u\right] = 0, \qquad s \in \Gamma_t ;$$

$$a_1 \frac{\partial u}{\partial n_1} + D_t\left(b_1 \frac{\partial u}{\partial n_1} + \beta_1 u\right) = 0, \qquad s \in \partial\Omega_1 \sim \Gamma_t \sim \Sigma_t ;$$

$$a_2 \frac{\partial u}{\partial n_2} + D_t\left(b_2 \frac{\partial u}{\partial n_2} + \beta_2 u\right) = 0, \qquad s \in \partial\Omega_2 \sim \Gamma_t .$$

Such boundary conditions can appear in applications to fluid flow problems [1] in which they represent the specification of linear combinations of pressure and flow rate.

(*c*) *Problems with time derivatives on interface or boundary.* Let the functions a_0, a and b in $L^\infty(\Omega \times T)$, $\alpha \in L^\infty(\Sigma)$ and α_0 and β in $L^\infty(\Sigma \times T)$ be given, where Σ is a proper open subset of $\partial\Omega_1$. Assume that each of these is a real-valued regular function on its corresponding domain, (4.4) holds, and that

$$\beta(s, 0) \geqq 0, \qquad (\partial/\partial t)\beta(s, 0) + \lambda\beta(s, 0) \geqq 0, \qquad \alpha_0(s, t) + \lambda\beta(s, t) \geqq 0, \qquad \alpha(s) \geqq 0 \tag{4.11}$$

for *a.e.* $(s, t) \in \Sigma \times T$. Let the restrictions $a_i(\cdot\,, t)$ and $b_i(\cdot\,, t)$ of $a(\cdot\,, t)$ and $b(\cdot\,, t)$, respectively, be uniformly continuous on Ω_i, $i = 1, 2$. Define the Hilbert spaces $H = L^2(\Omega)$, $W = H^1(\Omega_1) \oplus L^2(\Omega_2)$, and

$$V = \{\varphi \in H^1(\Omega) : (\alpha(\cdot))^{\frac{1}{2}}\nabla'\gamma_1\varphi(\cdot) \in L^2(\Sigma)\}$$

where ∇' is the gradient differential operator in local coordinates on Σ. We note that the condition defining V can be expressed in terms of differential forms, since

$$\int_\Sigma \alpha\, d(\gamma_1\varphi) \wedge_* d(\gamma_1\psi) = \int_\Sigma \alpha(s)\nabla'\gamma_1\varphi\cdot\nabla'\gamma_1\psi\, ds, \tag{4.12}$$

and is independent of the choice of coordinates [11, 12]. V is a Hilbert space with the inner product

$$(\varphi, \psi)_V = (\varphi, \psi)_{H^1(\Omega)} + (4.12).$$

We take $V(t) \equiv V$, all $t \in T$, and define the sesquilinear forms

$$a(t; \varphi, \psi) = \int_\Omega a(x, t)\nabla\varphi(x)\cdot\overline{\nabla\psi(x)}\, dx + \int_\Omega a_0(x, t)\varphi(x)\overline{\psi(x)}\, dx$$
$$+ \int_\Sigma \alpha(s)\nabla'\gamma_1\varphi(s)\cdot\nabla'\overline{\gamma_1\psi(s)}\, ds + \int_\Sigma \alpha_0(s, t)\varphi(s)\overline{\psi(s)}\, ds, \qquad \varphi, \psi \in V,$$

$$b(t; \varphi, \psi) = \int_\Omega b(x, t)\varphi(x)\overline{\psi(x)}\, dx + \int_\Sigma \beta(s, t)\gamma_1\varphi(s)\overline{\gamma_1\psi(s)}\, ds, \qquad \varphi, \psi \in W.$$

Lemma 2 shows that these are regular families of sesquilinear forms, and we also find that the operators of Theorem 4 are given by (4.1) and (4.3). We obtain (formally) from the Divergence Theorem

(4.13)

$$a(t; \varphi, \psi) - (A(t)\varphi, \psi)_H = \int_{\partial\Omega_1} a_1(s, t)\frac{\partial\varphi}{\partial n_1}\bar{\psi}\, ds + \int_{\partial\Omega_2} a_2(s, t)\frac{\partial\varphi}{\partial n_2}\bar{\psi}\, ds$$
$$+ \int_\Sigma \alpha(s)\nabla'(\gamma_1\varphi)\cdot\nabla'(\gamma_1\bar{\psi})\, ds + \int_\Sigma \alpha_0(s, t)\varphi\bar{\psi}\, ds$$
$$= \int_{\partial\Omega_1} a_1\frac{\partial\varphi}{\partial n_1}\bar{\psi} + \int_{\partial\Omega_2} a_2\frac{\partial\varphi}{\partial n_2}\bar{\psi}$$
$$+ \int_\Sigma \{\alpha_0(\gamma_1\varphi) - \nabla'(\alpha\nabla'(\gamma_1\varphi))\}\bar{\psi} + \int_{\partial\Sigma} \alpha\frac{\partial(\gamma_1\varphi)}{\partial n_\Sigma}\overline{\gamma_1\psi}$$

where $\varphi, \psi \in V$, φ is smooth, and the term in brackets contains distribution derivatives on Σ. Finally we note the identity

$$b(t; \varphi, \psi) - (B(t)\varphi, \psi)_H = \int_\Sigma \beta(s, t)\gamma_1\varphi\overline{\gamma_1\psi}\, ds, \qquad B(t)\varphi \in H, \quad \psi \in W. \tag{4.14}$$

Let $u_0 \in W$ and $F \in L^2(\Omega \times T)$ be given. Then $f(t) = F(\cdot\,, t)$ defines $f \in L^2(T, H)$ and the results of Section 2 imply the existence and uniqueness of the generalized solution $u(t) = u(\cdot\,, t)$ of the partial differential equation (4.5) in $(\Omega_1 \cup \Omega_2) \times T$ and (Theorem 5)

$$\alpha_0(s, t)u(s, t) - \nabla'(\alpha(s)\nabla' u(s, t)) + a_1(s, t)\frac{\partial u(s, t)}{\partial n_1} + a_2(s, t)\frac{\partial u(s, t)}{\partial n_2} + D_t\{\beta(s, t)u(s, t)\} = 0 \tag{4.15}$$

in $\mathfrak{D}'(\Sigma \times T)$, where $u(s, t) = \gamma_1 u(t)(s)$ and $a_2(s, t) = 0$ when $s \notin \Gamma$. V is clearly dense in W, so Theorem 3 implies initial conditions

$$b(x, 0)\{u(x, 0) - u_0(x)\} = 0, \qquad a.e.\ x \in \Omega,$$

$$\beta(s, 0)\{u(s, 0) - u_0(s)\} = 0, \qquad a.e.\ s \in \Sigma.$$

The boundary conditions arise from the requirement that $u(t) \in V$, hence

$$\gamma_1 u(t)(s) = \gamma_2 u(t)(s), \quad s \in \Gamma, \tag{4.16}$$

and from (2.11), which with (4.5) and (4.13) gives

$$a_1(s, t)\frac{\partial u(s, t)}{\partial n_1} + a_2(s, t)\frac{\partial u(s, t)}{\partial n_2} = 0, \qquad s \in \partial\Omega_1 \cup \partial\Omega_2 \sim \Sigma,$$

where we set $a_1 = 0$ on $\partial\Omega_2 \sim \Gamma$ and $a_2 = 0$ on $\partial\Omega_1 \sim \Gamma$, and remember that $n_1(s) = -n_2(s)$ for $s \in \Gamma$. Finally, from (2.11), (4.13) and (4.15) we obtain by the Divergence Theorem on Σ

$$\alpha(\xi)\cdot\frac{\partial u(\xi, t)}{\partial n_\Sigma} = 0, \qquad \xi \in \partial\Sigma$$

where n_Σ is the unit outward normal to $\partial\Sigma$, the $(n - 2)$-dimensional boundary of Σ.

Boundary value problems like our example arise from the consideration of fluid flow problems in a region containing a narrow fracture Σ which is characterized by very high permeability. In such a fracture, the flow across the fracture is negligible compared to that in tangential directions, and this accounts for the appearance of terms like (4.12) in the problem. See [4] for further discussion and references.

(*d*) *Boundary value problems of fourth and fifth type.* For our last example, we illustrate how our results of Section 2 give existence and uniqueness of solutions of some new boundary value problems [0]. Although variable domain models can be easily obtained, we omit this here. For simplicity, we let $\Omega_2 = \varnothing$, hence $\Omega = \Omega_1$, and let Σ be a measurable subset of $\partial\Omega$. Suppose we are given a function $\alpha \in L^\infty(\partial\Omega)$ with $\alpha(s) \geqq 0$, *a.e.* $s \in \partial\Omega$. Then we define Hilbert spaces $H = L^2(\Omega)$, $W = \{\phi \in H^1(\Omega): (\alpha)^{1/2}(\partial\phi/\partial n) \in L^2(\partial\Omega)\}$, and $V(t) = V = \{\phi \in W:$

$\phi + \alpha(\partial\phi/\partial n) = \text{constant on } \Sigma\}$. The inner product on V and W is taken to be $(\phi, \psi)_V = (\phi, \psi)_{H^1(\Omega)} + \int_{\partial\Omega} \alpha(\partial\phi/\partial n)\overline{(\partial\psi/\partial n)}\, ds$. Suppose we are given the regular functions $a(\cdot\,,\,\cdot)$, $b(\cdot\,,\,\cdot)$ and $a_0(\cdot\,,\,\cdot)$ on $\Omega \times T$ which satisfy (4.4), and a non-negative and absolutely continuous real-valued function β on T. Then we can give two regular families of sesquilinear forms on V by

$$a(t; \phi, \psi) = \int_\Omega (a(x, t)\nabla\phi\cdot\overline{\nabla\psi} + a_0(x, t)\phi(x)\overline{\psi(x)})\, dx + \int_{\Omega\partial} \overline{\alpha(s)}a(s, t)\, \frac{\partial\phi}{\partial n}\, \overline{\frac{\partial\psi}{\partial n}}\, ds,$$

$$b(t; \phi, \psi) = \int_\Omega b(x, t)\phi(x)\overline{\psi(x)}\, dx + \beta(t)\int_{\partial\Omega} \left(\phi + \alpha\frac{\partial\phi}{\partial n}\right)\overline{\left(\psi + \alpha\frac{\partial\psi}{\partial n}\right)}\, ds.$$

The operators of Theorem 4 are given by (4.1) and (4.3), and we have the identities

$$a(t; \phi, \psi) - (A(t)\phi, \psi)_H = \int_{\partial\Omega} a(s, t)\, \frac{\partial\phi}{\partial n}\, \overline{\left\{\psi + \alpha\frac{\partial\psi}{\partial n}\right\}}\, ds,$$

$$b(t; \phi, \psi) - (B(t)\phi, \psi)_H = \beta(t)\int_{\partial\Omega} \left(\phi + \alpha\frac{\partial\phi}{\partial n}\right)\overline{\left(\psi + \alpha\frac{\partial\psi}{\partial n}\right)}\, ds,$$

for appropriate $\phi, \psi \in V$. If $u_0 \in W$ and $F \in L^2(\Omega \times T)$ are given, we define $f \in L^2(T, V)$ by $f(t) = F(\cdot\,, t)$, $t \in T$. The assumptions (4.4) imply the estimates (2.4) and (2.5), so there is a unique solution $u(t)$ of the Cauchy problem (2.1). Then the function $u(\cdot\,, t) = u(t)$ is the weak solution of the partial differential equation (4.5). The inclusions $u(t) \in V$ imply that for each $t \in T$,

$$u(s, t) + \alpha(s)\, \frac{\partial u(s, t)}{\partial n} = g(t), \qquad s \in \Sigma. \tag{4.16}$$

That is, the indicated combination is independent of s on Σ, hence gives a function of t as shown. This is a non-local boundary condition. We obtain from Theorem 5 (formally) the additional boundary conditions

$$\begin{aligned} &\int_\Sigma a(s, t)\, \frac{\partial u(s, t)}{\partial n}\, ds + D_t(\beta(t)g(t))\cdot\int_\Sigma ds = 0, \qquad t \in T \\ &a(s, t)\, \frac{\partial u(s, t)}{\partial n} + D_t\left(\beta(t)\left[u(s, t) + \alpha(s)\, \frac{\partial u(s, t)}{\partial n}\right]\right) = 0, \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad s \in \partial\Omega \sim \Sigma, \quad t \in T. \end{aligned} \tag{4.17}$$

Theorem 3 gives the initial conditions

$$b(x, 0)\{u(x, 0) - u_0(x)\} = 0, \qquad x \in \Omega$$

$$\beta(0)\left\{g(0)\int_\Sigma ds - \int_\Sigma \left(u_0(s) + \alpha(s)\, \frac{\partial u_0(s)}{\partial s}\right) ds\right\} = 0,$$

$$\beta(0)\left\{(u(s, 0) - u_0(s)) + \alpha(s)\, \frac{\partial}{\partial n}\, (u(s, 0) - u_0(s)\right\} = 0, \qquad s \in \partial\Omega \sim \Sigma.$$

The boundary conditions (4.16), (4.17) are called the conditions of the *fourth type* where $\alpha(s) = 0$ and the conditions of the *fifth type* wherever $\alpha(s) > 0$ [0].

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