DIFFUSION IN DEFORMABLE MEDIA

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Abstract. We begin with the initial-boundary-value problem for a coupled system of partial differential equations which describes the Biot consolidation model in poro-elasticity. Existence, uniqueness and regularity theory is developed for the quasi-static case as an application of the theory of linear degenerate evolution equations in Hilbert space, and this leads to a precise description of the dynamics of the system. Current work on the foundations of the model and appropriate extensions to models with elastic-viscous-plastic media or nonhomogeneous media will be briefly described.

Key words. Poro-elasticity, deformable porous media, thermo-elasticity, Biot consolidation problem, coupled quasi-static, secondary consolidation, degenerate evolution equations, initial-boundary-value problems, existence-uniqueness theory, regularity.

AMS(MOS) subject classifications. 35D05, 35D10, 35K50, 35K65, 76S05.

1. Introduction. Analysis of the quasi-static deformation and associated pressure distribution in a porous fluid-saturated elastic structure is generally based on *poroelasticity theory*. This consists of the mathematical description of the dynamics of the pore-fluid pressure and the solid stress fields of the structure formulated by coupling the partial differential equations of the diffusion process with those of the elasticity theory for the structure. Any model of fluid flow through a deformable solid matrix must account for this coupling between the mechanical behavior of the matrix and the fluid dynamics. For example, compression of the medium leads to increased pore pressure, if the compression is fast relative to the fluid flow rate. Conversely, an increase in pore pressure induces a dilation of the matrix in response to the added stress. This coupled pressure-deformation interraction is the basis of the development of poro-elasticity starting with the work of Terzaghi (1925) [39], (1943) [40]. The concept of *total stress* is the essence of coupled deformation-flow behavior within porous media and sets it apart from the theory of flow through a rigid structure. The first detailed studies of the coupling between the pore-fluid pressure and solid stress fields were described by Biot (1941) [9]. The basic constitutive equations relate the total stress to both the *effective stress* given by the strain of the structure and to the *pressure* arising from the pore-fluid. Time dependent fluid flow is incorporated by combining the fluid mass conservation with Darcy's law, and the displacement of the structure is described by combining Hooke's law for elastic deformation with the momentum balance equations. The transient flow and deformation behavior in a deformable porous medium may result from changes in either the fluid pressure, flux, displacements, or traction conditions applied to the bound-

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ary of the medium. The model for consolidation requires the *quasi static* assumption that the dynamic momentum equations are replaced by the corresponding equilibrium equations.

We briefly recall this classical model of diffusive flow in a porous deformable medium. Let Ω be a smoothly bounded region which represents the porous and permeable elastic matrix with density ρ, and assume it is saturated by a slightly compressible and viscous fluid which diffuses through it. The *displacement* of the solid matrix is denoted by $\mathbf{u}(x,t)$ for each point $x \in \Omega$ and time $t > 0$. Let ρ_s be the density of the solid and ϕ the *porosity* of the medium, i.e., the volume fraction available to the fluid. Let ρ_f be the density of the fluid and $\mathbf{w}$ be the *fluid velocity.* The Darcy *relative bulk velocity* of the fluid is defined by $\mathbf{v} \equiv \phi(\mathbf{w} - \dot{\mathbf{u}})$. For each subdomain $B \subset \Omega$, the momentum of the corresponding portion of the matrix is given by $\int_B (\rho\dot{\mathbf{u}}(x,t) + \rho_f \mathbf{v}(x,t))\,dx$. Here $\rho = \phi\rho_f + (1-\phi)\rho_s$ is the total density, and so $\rho\dot{\mathbf{u}} + \rho_f\mathbf{v} = (1-\phi)\rho_s\dot{\mathbf{u}} + \phi\rho_f\mathbf{w}$ is the combined momentum of solid and fluid. The forces acting on the body B consist of the traction forces applied by the complement of B across its boundary ∂B with normal $\mathbf{n}$. These are given by $\int_{\partial B} \sigma_{ij}(x,t)\,n_j\,dS$, where the *stress* σ_{ij} is the symmetric tensor that represents the internal forces on surface elements. Thus we obtain the equation for *balance of momentum*

$$\frac{\partial}{\partial t}\int_B (\rho\frac{\partial \mathbf{u}(x,t)}{\partial t} + \rho_f\mathbf{v}(x,t))\,dx = \int_{\partial B} \sigma(\cdot,t,\mathbf{n})\,dS + \int_B \mathbf{f}(x,t)\,dx$$

for each subdomain B, where $\mathbf{f}(\cdot,t)$ denotes the volume-distributed external forces. The components of the *normal stress* $\sigma(\cdot,t,\mathbf{n})$ are given by $\sigma(\cdot,t,\mathbf{n})_i = \sigma_{ij}(\cdot,t)\,n_j$. With the divergence theorem this gives the *momentum equations*

$$\frac{\partial}{\partial t}\left(\rho\frac{\partial u_i(x,t)}{\partial t} + \rho_f v_i(x,t)\right) - \partial_j\sigma_{ij}(x,t) = f_i(x,t)\,, \qquad 1 \le i \le 3\,.$$

The mass of fluid in each such subdomain B is $\int_B \eta(x,t)\,dx$, and this defines the *fluid content* $\eta(x,t)$ of the medium. The *flux* is the mass flow rate $\mathbf{q}(x,t)$ of fluid relative to the matrix, so the rate at which fluid moves across the boundary ∂B is given by $\int_{\partial B} \mathbf{q}(x,t)\cdot\mathbf{n}\,dS$. Then the *conservation of mass* of fluid takes the integral form

$$\frac{\partial}{\partial t}\int_B \eta(x,t)\,dx + \int_{\partial B} \mathbf{q}\cdot\mathbf{n}\,dS = \int_B \rho_f h(x,t)\,dx\ , \qquad B \subset \Omega\ ,$$

in which $h(\cdot,t)$ denotes any volume distributed *source density*. When the flux and content are differentiable, we obtain the equations of *mass balance* in the differential form

$$\frac{\partial}{\partial t}\eta(x,t) + \nabla\cdot\mathbf{q}(x,t) = \rho_f h(x,t)\ , \qquad x \in \Omega\ .$$

For the corresponding *constitutive equations*, we assume the total stress and fluid content are given respectively by

$$\sigma_{ij} = \lambda\delta_{ij}\varepsilon_{kk}(\mathbf{u}) + 2\mu\varepsilon_{ij}(\mathbf{u}) - \alpha\delta_{ij}\,p\,,$$
$$\eta = \rho_f(c_0\,p + \alpha\,\nabla\cdot\mathbf{u})\,,$$

where $p(x,t)$ denotes the *pressure* distribution within the medium Ω and the small local *strain* of the solid is denoted by $\varepsilon_{kl}(\mathbf{u}) \equiv \frac{1}{2}(\partial_k u_l + \partial_l u_k)$. The positive Lamé constants λ and μ are the *dilation* and *shear* moduli of elasticity, respectively. The coefficient $\alpha > 0$ is the *Biot-Willis constant* that accounts for the *pressure-deformation coupling*; it is a measure of the fluid volume forced out of the solid skeleton by a dilation. The coefficient $c_0 \geq 0$ is the combined *porosity* of the medium and *compressibility* of the fluid and solid. We also assume the *flux* $\mathbf{q}$ is given by *Darcy's law*

$$\mathbf{q} = \rho_f\mathbf{v}\,, \quad \mathbf{v} = -k\nabla p\,,$$

for the laminar flow through the medium. We ignore the effects of gravity, as the corresponding term does not affect the structure of the problem. The momentum balance equations for the displacement of the medium and the mass balance equation for the pressure distribution are then given by the (fully dynamic) *classical Biot system*

$$\frac{\partial}{\partial t}(\rho\frac{\partial\mathbf{u}}{\partial t}) - (\lambda+\mu)\nabla(\nabla\cdot\mathbf{u}) - \mu\Delta\mathbf{u} + \alpha\nabla p = \mathbf{f}(x,t)\,, \tag{1.1}$$

$$\frac{\partial}{\partial t}(c_0\,p + \alpha\nabla\cdot\mathbf{u}) - \nabla\cdot k\nabla p = h(x,t) \text{ in } \Omega\,. \tag{1.2}$$

Here the inertia of the Darcy velocity is assumed to be relatively negligible, so the variation of $\rho_f\mathbf{v}$ has been deleted. This additional simplification results in a system of mixed wave-parabolic type for the solid displacement and fluid pressure. The small deformations of the matrix are described by the Navier equations of linear elasticity, and the diffusive fluid flow is described by Duhamel's equation. We shall consider such diffusion and deformation processes in the case for which the remaining inertia effects are negligible, so the first term in this system is deleted. This *quasi-static* assumption arises naturally in the classical Biot model of consolidation.

We note finally that the Biot system is formally equivalent to the classical coupled *thermo-elasticity system* which describes the flow of heat through an elastic structure. In that context, $p(x,t)$ denotes the *temperature*, $c_0 > 0$ is the *specific heat* of the medium, and $k > 0$ is the *conductivity.* Then $\alpha\nabla p(x,t)$ arises from the *thermal stress* in the structure, and the term $\alpha\nabla\cdot\frac{\partial\mathbf{u}(x,t)}{\partial t}$ corresponds to the *internal heating* due to the *dilation rate.* We have *not* made the uncoupling assumption in which this term is deleted from the diffusion equation. See Norris (1992) [30] for the static case.

2. Remarks on Literature. For a small sample of fundamental work on the storage equation and its application in reservoir simulation, see Bear (1972) [4], Collins (1961) [19], Peaceman (1977) [31], and Huyakorn-Pinder (1983) [26]. For a history of developments in soil science, see the recent book of de Boer (2000) [15].

The *fully dynamic* system with $\rho > 0$ was developed by Biot (1956) [11, 12], (1962) [13], (1972) [14] to describe (higher frequency) deformation in porous media. For the theory of this system in the context of thermoelasticity, see the fundamental work of Dafermos (1968) [20], the exhaustive and complementary accounts of Carlson (1972) [17] and Kupradze (1979) [28], and the development in the context of strongly elliptic systems by Fichera (1974) [24]. By contrast, very few references are to be found in the thermoelasticity literature for the mathematical well-posedness of even the simplest linear problem for the *coupled quasi-static* case in which the system degenerates to a mixed elliptic-parabolic type. Such a system in one spatial dimension was developed by classical methods in the book of Day [21]. According to a scaling argument in Boley-Wiener [16], it appears that the reasons for taking $\rho = 0$ apply as well to simultaneously delete the term $\alpha\nabla\cdot\dot{\mathbf{u}}(t)$ and thereby uncouple the system, so these two assumptions are frequently taken together. This may explain in part the limited attention given to this case in the thermoelasticity literature. Although this decoupling assumption is appropriate in many thermoelasticity applications, it is *never* permissible for the consolidation problems of poroelasticity [32, 47].

The consolidation model of Biot requires the *quasi-static* case, $\rho = 0$; see Biot (1941) [9] and (1955) [10], Rice and Cleary (1976) [32], Zienkiewicz *et al.* (1980) [47]. An additional degeneracy occurs in the *incompressible* case in which we have also $c_0 = 0$, and then the system is formally of elliptic type. The mathematical issues of well-posedness for the quasi-static case were first studied in the fundamental work of J.-L Auriault and Sanchez-Palencia (1977) [1]. They derived a non-isotropic form of the Biot system by homogenization and then proved existence and uniqueness of a *strong* solution for which the equations hold in $L^2(\Omega)$. In the later paper of Zenisek (1984) [46], a *weak* solution is obtained in the first order Sobolev space $H^1(\Omega)$, so the equations hold in the dual space, $H^{-1}(\Omega)$ (see below). Additional issues of analysis and approximation of this case are developed in [13, 14, 29, 33, 45, 48]. A complete development of the existence, uniqueness, and regularity theory for the Biot system together with extensions to include the possibility of viscous terms arising from secondary consolidation and the introduction of appropriate boundary conditions at both closed and drained interfaces were recently given in [36].

3. The Differential Operators. We shall formulate the system (1.1, 1.2) together with appropriate boundary and initial conditions in the abstract form of evolution equations in Hilbert space. In order to carry this out, we construct the relevant stationary operators within the system.

3.1. The Elasticity Operator. We recall the Navier system of partial differential equations which describes the small displacements of a purely elastic structure and the variational formulation of the associated boundary-value problem in Sobolev spaces. Let Ω be a smoothly bounded domain in R^3, and denote by Γ_0 and Γ_t two complementary parts of a partition of the boundary, $\partial\Omega$. The general stationary elasticity system is given by the equations of equilibrium

$$-\partial_j \sigma_{ij} = f_i \quad in \;\; \Omega \tag{3.1}$$

$$u_i = 0 \; on \; \Gamma_0 \,, \qquad \sigma_{ij} n_j = g_i \; on \; \Gamma_t \tag{3.2}$$

for each $1 \leq i \leq 3$. Thus the boundary condition on Γ_0 is a constraint on displacement, and on Γ_t it involves the surface density of forces or *traction* $\sigma(\mathbf{n})$ with i-th component given by $\sigma_{ij} n_j$ and value determined by the unit outward normal vector $\mathbf{n} = (n_1, n_2, n_3)$ on Γ_t. In order to obtain the weak formulation of this boundary value problem, we define the Sobolev space

$$\mathbf{V} = \left\{ \mathbf{v} \in \mathbf{H}^1(\Omega) : \;\; \mathbf{v} = \mathbf{0} \;\; on \;\; \Gamma_0 \right\}$$

of *admissable displacements* in $H^1(\Omega)^3$. We shall assume that Γ_0 has strictly positive measure. Thus, we write the elasticity system (3.1, 3.2) in the form

$$\mathbf{u} \in \mathbf{V} : \quad \mathcal{E}(\mathbf{u})(\mathbf{v}) = \mathbf{h}(\mathbf{v}) \,, \qquad \mathbf{v} \in \mathbf{V}, \tag{3.3}$$

where the *elasticity operator* $\mathcal{E} : \mathbf{V} \longrightarrow \mathbf{V}'$ and the conjugate linear functional $\mathbf{h}(\cdot) \in \mathbf{V}'$ are defined by

$$\mathcal{E}(\mathbf{u})(\mathbf{v}) = \int_\Omega (\lambda (\partial_k u_k) \, \overline{(\partial_i v_i)} + 2\mu \varepsilon_{ij}(\mathbf{u}) \overline{\varepsilon_{ij}(\mathbf{v})) } \, d\mathbf{x} \,,$$

$$\mathbf{h}(\mathbf{v}) = \int_\Omega f_i \overline{v_i} \, d\mathbf{x} + \int_{\Gamma_1} g_i \overline{v_i} \, ds \,, \quad \mathbf{v} \in \mathbf{V} \,.$$

For $\mathbf{u} \in \mathbf{V}$ we define the restriction of $\mathcal{E}(\mathbf{u}) \in \mathbf{V}'$ to $\mathbf{C}_0^\infty(\Omega)$ by $\mathcal{E}_0(\mathbf{u})$; this is the distribution $\mathcal{E}_0(\mathbf{u}) \equiv -(\lambda + \mu)\nabla(\nabla \cdot \mathbf{u}) - \mu \Delta \mathbf{u}$. Then the weak form of the boundary-value problem (3.1, 3.2) is just (3.3). If the closures of Γ_0 and Γ_t do not intersect, and if the boundary is sufficiently smooth, then the regularity theory for strongly elliptic systems shows that whenever $\mathcal{E}_0(\mathbf{u}) \in \mathbf{L}^2(\Omega)$ we have $\mathbf{u} \in \mathbf{H}^2(\Omega) \cap \mathbf{V}$, and then from Stokes' theorem there follows

$$\mathcal{E}(\mathbf{u})(\mathbf{v}) = (\mathcal{E}_0(\mathbf{u}), \mathbf{v})_{\mathbf{L}^2(\Omega)} + (\sigma_{ij}(\mathbf{u}) n_j, v_i)_{L^2(\Gamma_t)}, \quad \mathbf{v} \in \mathbf{V}.$$

This shows how $\mathcal{E}$ decouples into the sum of its *formal* part $\mathcal{E}_0$ on Ω and its *boundary* part $\sigma(\mathbf{n})$ on Γ_t. From Korn's inequality and Poincare's theorem it follows that $\mathcal{E}$ is $\mathbf{V}$ coercive, so $\mathcal{E}$ is an isomorphism. (See Duvaut-Lions [22] or Ciarlet [18].

3.2. The Diffusion Operator. Suppose we are given the function $k \in L^\infty(\Omega)$ satisfying $k(x) \geq k_0 > 0,\ x \in \Omega$. This determines the Neumann problem

$$-\nabla \cdot (k\nabla p) = h_1 \text{ in } \Omega\,, \tag{3.4}$$

$$k\frac{\partial p}{\partial n} = h_2 \text{ in } \Gamma\,. \tag{3.5}$$

Let $V = H^1(\Omega)$ and define the conjugate linear functional $h(\cdot)$ and the symmetric and monotone operator $A : V \longrightarrow V^{'}$ by

$$Ap(q) = \int_\Omega k\nabla p \cdot \overline{\nabla q}\, dx, \qquad p,\ q \in V\,,$$
$$h(q) = \int_\Omega h_1 \overline{q}\, d\mathbf{x} + \int_\Gamma h_2 \overline{q}\, ds\,, \quad q \in V\,.$$

Then the Neumann problem (3.4), (3.5) is given by

$$p \in V: \quad A(p)(q) = h(q) \quad q \in V. \tag{3.6}$$

The restriction to $C_0^\infty(\Omega)$ of $A(p)$ is the formal part in $H^{-1}(\Omega)$ given by the elliptic operator $A_0(p) = -\nabla\cdot k\nabla p$. If $p \in V$, $A_0 p \in L^2(\Omega)$, and if $k(\cdot)$ is smooth, then the elliptic regularity theory implies that $p \in V \cap H^2(\Omega)$, and we obtain the decoupling of A

$$Ap(q) = (A_0 p, q)_{L^2(\Omega)} + (k\frac{\partial p}{\partial n}, \gamma q)_{L^2(\partial\Gamma)}\,, \quad q \in V\,.$$

into a formal part A_0 on Ω and a boundary part $k\frac{\partial p}{\partial n}$ on Γ.

3.3. The Pressure-Dilation Operators. Let the function $\beta(\cdot) \in L^\infty(\Gamma_t)$ be given; we shall assume that $0 \leq \beta(s) \leq 1,\ s \in \Gamma_t$. Then define the corresponding *gradient* operator, $\vec{\nabla} : V \to \mathbf{L}^2(\Omega) \oplus \mathbf{L}^2(\Gamma_t)$, by

$$\langle \vec{\nabla} p,\ [\mathbf{f}, \mathbf{g}] \rangle \equiv \int_\Omega \partial_j p\, \overline{f}_j\, dx - \int_{\Gamma_t} \beta\, p n_j\, \overline{g_j}\, ds\,,$$
$$p \in V,\ [\mathbf{f}, \mathbf{g}] \in \mathbf{L}^2(\Omega) \oplus \mathbf{L}^2(\Gamma_t)\,.$$

This consists explicitly of a *formal* part ∇p in Ω and a *boundary* part $-\beta\, p\, \mathbf{n}$ on Γ_t, and we denote this representation by

$$\vec{\nabla} p = [\nabla p,\ -\beta\, p\, \mathbf{n}]\,. \tag{3.7}$$

Define $\vec{\nabla}\cdot : \mathbf{L}^2(\Omega) \oplus \mathbf{L}^2(\Gamma_t) \to V'$ to be the negative of the corresponding dual operator. This is the *divergence* operator $\vec{\nabla}\cdot = -\vec{\nabla}'$ given by

$$\langle \vec{\nabla}\cdot[\mathbf{f}, \mathbf{g}],\ p \rangle \equiv -\overline{\langle \vec{\nabla} p,\ [\mathbf{f}, \mathbf{g}] \rangle}\,, \quad [\mathbf{f}, \mathbf{g}] \in \mathbf{L}^2(\Omega) \oplus \mathbf{L}^2(\Gamma_t),\ p \in V\,.$$

The trace map gives a natural identification $\mathbf{v} \mapsto [\mathbf{v},\, \gamma(\mathbf{v})|_{\Gamma_t}]$ of

$$\mathbf{V} \subset \mathbf{L}^2(\Omega) \oplus \mathbf{L}^2(\Gamma_t)\,,$$

and this identification will be employed throughout the following. It also gives the identification $p \mapsto [p,\, \gamma(p)|_{\Gamma_t}]$ of

$$V \subset L^2(\Omega) \oplus L^2(\Gamma_t)\,.$$

We note that both of these identifications have dense range, and so the corresponding duals can be identified. That is, we have

$$\mathbf{L}^2(\Omega) \oplus \mathbf{L}^2(\Gamma_t) \subset \mathbf{V}', \qquad L^2(\Omega) \oplus L^2(\Gamma_t) \subset V'.$$

For smoother functions $\mathbf{v} \in \mathbf{V} \subset \mathbf{L}^2(\Omega) \oplus \mathbf{L}^2(\Gamma_t)$ we have the *Stokes' Formula*

$$\begin{aligned}\langle \vec{\nabla}\cdot\mathbf{v},\, p\rangle &= -\int_\Omega \overline{\partial_j p}\, v_j\, dx + \int_{\Gamma_t} \beta\, \overline{p} v_j n_j\, ds \\ &= \int_\Omega \partial_j v_j \overline{p}\, dx - \int_{\Gamma_t} (1-\beta)\mathbf{v}\cdot\mathbf{n}\, \overline{p}\, ds\,, \quad p \in V.\end{aligned}$$

This shows the restriction satisfies

$$\vec{\nabla}\cdot :\ \mathbf{V} \to L^2(\Omega) \oplus L^2(\Gamma_t)$$

and that the divergence operator has a *formal* part in Ω as well as a *boundary* part on Γ_t. We denote the part in $L^2(\Omega)$ by $\nabla\cdot$, that is, $\nabla\cdot\mathbf{v} = \partial_j v_j$, and the identity above is indicated by

$$\vec{\nabla}\cdot\mathbf{v} = [\nabla\cdot\mathbf{v},\, -(1-\beta)\mathbf{v}\cdot\mathbf{n}] \in L^2(\Omega) \oplus L^2(\Gamma_t), \quad \mathbf{v} \in \mathbf{V}. \tag{3.8}$$

Now we can extend the definition of $\vec{\nabla}$ from V up to $L^2(\Omega) \oplus L^2(\Gamma_t)$. This extension is obtained as $-(\vec{\nabla}\cdot)'$, the negative of the dual of the restriction to $\mathbf{V}$ of the divergence. This dual operator

$$(\vec{\nabla}\cdot)' :\ L^2(\Omega) \oplus L^2(\Gamma_t) \to \mathbf{V}'$$

is defined for each $[f,g] \in L^2(\Omega) \oplus L^2(\Gamma_t)$ by

$$\begin{aligned}\langle (\vec{\nabla}\cdot)'[f,g],\, \mathbf{v}\rangle &= \overline{(\vec{\nabla}\cdot\mathbf{v},\, [f,g])}_{L^2(\Omega)\oplus L^2(\Gamma_t)} \\ &= \overline{(\partial_j v_j,\, f)}_{L^2(\Omega)} - \overline{((1-\beta)\mathbf{v}\cdot\mathbf{n},\, g)}_{L^2(\Gamma_t)} \\ &= (f,\, \nabla\cdot\mathbf{v})_{L^2(\Omega)} - (g,\, (1-\beta)\mathbf{v}\cdot\mathbf{n})_{L^2(\Gamma_t)}\,, \quad \mathbf{v} \in \mathbf{V}\,.\end{aligned}$$

For the smoother case of $[f,\, g] = [w,\, w|_{\Gamma_t}]$, with the indicated $w \in V$ identified as a function on Ω and its trace on Γ_t, the Stokes' formula shows that

$$\begin{aligned}-\langle (\vec{\nabla}\cdot)'[w,\, w|_{\Gamma_t}],\, \mathbf{v}\rangle &= -(w,\, \nabla\cdot\mathbf{v})_{L^2(\Omega)} + (w,\, (1-\beta)\mathbf{v}\cdot\mathbf{n})_{L^2(\Gamma_t)} \\ &= (\partial_j w,\, v_j)_{L^2(\Omega)} - (\beta w,\, \mathbf{v}\cdot\mathbf{n})_{L^2(\Gamma_t)} \\ &= (\vec{\nabla} w, \mathbf{v})_{L^2(\Omega)\oplus L^2(\Gamma_t)}\,, \quad w \in V,\ \mathbf{v} \in \mathbf{V}\,,\end{aligned}$$

and this shows that $-\vec{\nabla}\cdot'$ provides the desired extension of $\vec{\nabla}$ from V to $L^2(\Omega)\oplus L^2(\Gamma_t)$. Note that by taking $[f,\, g] = \vec{\nabla}\cdot\mathbf{v} = [\nabla\cdot\mathbf{v},\, -(1-\beta)\mathbf{v}\cdot\mathbf{n}]$ above, we obtain

$$
\begin{aligned}
&\langle(\vec{\nabla}\cdot)'\vec{\nabla}\cdot\mathbf{v},\, \mathbf{w}\rangle = (\vec{\nabla}\cdot\mathbf{v},\, \vec{\nabla}\cdot\mathbf{w})_{L^2(\Omega)\oplus L^2(\Gamma_t)}\\
&= (\nabla\cdot\mathbf{v},\, \nabla\cdot\mathbf{w})_{L^2(\Omega)} + ((1-\beta)\mathbf{v}\cdot\mathbf{n},\, (1-\beta)\mathbf{w}\cdot\mathbf{n})_{L^2(\Gamma_t)} \quad \mathbf{v},\ \mathbf{w}\in\mathbf{V}\,.
\end{aligned}
$$

The preceding constructions are summarized in the following diagram.

$$
\begin{array}{ccccc}
\mathbf{L}^2(\Omega)\oplus\mathbf{L}^2(\Gamma_t) & \xrightarrow{\vec{\nabla}\cdot=-\vec{\nabla}'} & V' & & \\
\bigcup & & \bigcup & & \\
\mathbf{V} & \xrightarrow{\vec{\nabla}\cdot} & L^2(\Omega)\oplus L^2(\Gamma_t) & \xrightarrow{\vec{\nabla}=-(\vec{\nabla}\cdot)'} & \mathbf{V}' \\
 & & \bigcup & & \bigcup \\
 & & V & \xrightarrow{\vec{\nabla}} & \mathbf{L}^2(\Omega)\oplus\mathbf{L}^2(\Gamma_t)
\end{array}
$$

4. The Quasi-static Biot System . Using the notation introduced in the previous section, we first display an initial-boundary-value problem for the system of partial differential equations (1.1), (1.2) and then discuss the relation of these boundary conditions to the Biot consolidation problem. This problem is written as an evolution equation in Hilbert space. The Cauchy problem for this abstract Biot evolution system has a unique solution in two situations. With L^2-type data prescribed, it has a strong solution, and when H^{-1}-type data is prescribed, it has a weak solution. These results will appear in [36], and we provide here a summary of that work.

4.1. The Initial-Boundary-Value Problem. Denote the *characteristic function* of the traction boundary, Γ_t by χ_t. The first objective is a study of initial boundary value problems of the form

$$\mathcal{E}_0\mathbf{u}(t)) + \nabla p(t) = \mathbf{0} \text{ and} \tag{4.1}$$

$$\frac{\partial}{\partial t}(c_0 p(t) + \nabla\cdot\mathbf{u}(t)) + A_0(p(t)) = h_0(t) \text{ in } \Omega\,, \tag{4.2}$$

$$\mathbf{u}(t) = \mathbf{0} \text{ on } \Gamma_0, \tag{4.3}$$

$$\sigma_{ij}(\mathbf{u}(t))n_j - p(t)\, n_i\, \beta\chi_S = 0,\ 1\le i\le 3,\ \text{ on } \Gamma_t\,, \tag{4.4}$$

$$-\frac{\partial}{\partial t}(\mathbf{u}(t)\cdot\mathbf{n})\,(1-\beta)\chi_t + k\frac{\partial p(t)}{\partial n} = h_1(t)\,\chi_t \text{ on } \Gamma\,, \tag{4.5}$$

$$\lim_{t\to 0^+}(c_0 p(t) + \nabla\cdot\mathbf{u}(t)) = v_0 \text{ in } L^2(\Omega)\,, \tag{4.6}$$

$$\lim_{t\to 0^+}(1-\beta)(\mathbf{u}(t)\cdot\mathbf{n}) = v_1 \text{ in } L^2(\Gamma_t)\,. \tag{4.7}$$

The partial differential equations (4.1), (4.2) are just the Biot system (1.1), (1.2). We discuss the meaning of the boundary conditions in the context of the poroelasticity model. The boundary conditions (4.3), (4.4) consist of the complementary pair requiring null displacement on the *clamped boundary*, Γ_0, and a balance of forces on the *traction boundary*, Γ_t. The boundary condition (4.5) requires a balance of fluid mass. The function $\beta(\cdot)$ is defined on that portion of the boundary Γ_t which is not (drained or) clamped, and it specifies the surface fraction of the pores which are *sealed* along Γ_t. For these the hydraulic pressure contributes to the total stress within the structure. The remaining portion $1-\beta(\cdot)$ of the pores are *exposed* along Γ_t, and these contribute to the flux. On any portion of Γ_t which is completely exposed, that is, where $\beta = 0$, only the *effective* or elastic component of stress is specified, since there the fluid pressure does not contribute to the support of the matrix. On the entire boundary there is a transverse flow that is given by the input $h_1(\cdot)$ and the relative normal displacement of the structure. This input could be specified in the form $h_1(t) = -(1-\beta)\mathbf{v}(t)\cdot\mathbf{n}$, where $\mathbf{v}(t)$ is the given velocity of fluid or boundary flux on Γ_t. The first term and right side of this flux balance is null where $\beta = 1$, so the same holds for the second terms in (4.5), that is, we have the *impermeable* conditions $k\frac{\partial p(t)}{\partial n} = 0$ on a completely sealed portion of Γ_t. We also note that in (4.5) the first term on the left side and the right side of the equation are null on Γ_0, so the same necessarily holds for the second term on the left side. That is, we always have the *null flux* condition $k\frac{\partial p}{\partial n} = 0$ on Γ_0.

4.2. The Strong Solution. We show that the quasi-static system (4.1 – 4.7) is essentially a *parabolic* system which has a *strong* solution under minimal smoothness requirements on the initial data and source $h(\cdot)$. Let $P : (L^2(\Omega)\oplus L^2(\Gamma_t))^2 \longrightarrow (L^2(\Omega)\oplus\{0\})^2$ be the indicated *projection* operator onto the first components. In terms of the operators constructed in Section 3, the quasi-static system (4.1) – (4.7) is equivalent to

$$\mathcal{E}(\mathbf{u}(t)) + \vec{\nabla}p(t) = \mathbf{0}\,, \tag{4.8}$$

$$\frac{d}{dt}(c_0 P p(t) + \vec{\nabla}\cdot\mathbf{u}(t)) + A(p(t)) = h(t)\,, \tag{4.9}$$

$$c_0 P p(0) + \vec{\nabla}\cdot\mathbf{u}(0) = [v_0,\, -v_1]\,. \tag{4.10}$$

The first system (4.8) corresponds to the *equilibrium system* for momentum and the second system (4.9) consists of the *mass balance* for double-diffusion. The first equation holds in the space $\mathbf{V}'$ and the second in V'. The first system is elliptic, and the second equation is of mixed elliptic-parabolic type with $c_0 \geq 0$. The forcing term $h(t)$ represents any external sources. Note that we can assume without loss of generality that first system is homogeneous by a simple translation, since $\mathcal{E}$ os surjective.

Note that (4.9) requires that $p(t) \in V$, so both terms of (4.8) are necessarily in $(\mathbf{L}^2(\Omega)\oplus\mathbf{L^2}(\Gamma_t))^3$, and this forces additional regularity on the displacement $\mathbf{u}(t)$. By a *strong solution*, we mean that equation (4.9) holds in the smaller space $L^2(\Omega) \oplus L^2(\Gamma_t) \subset V'$, so this solution has the additional regularity necessary to decouple the partial differential equations and the boundary conditions implicit in (4.9).

The fundamental point is the following.

LEMMA 4.1. *The operator $B = -\vec{\nabla}\cdot\mathcal{E}^{-1}\vec{\nabla} : L^2(\Omega) \oplus L^2(\Gamma_S) \to L^2(\Omega) \oplus L^2(\Gamma_S)$ is continuous, monotone and self-adjoint with $Ker(B) = Ker(\vec{\nabla})$, and each of the Sobolev spaces $(H^m(\Omega) \cap V) \oplus H^{m-\frac{1}{2}}(\Gamma_S)$ is invariant under B.*

The system (4.8), (4.9) can be written as a single equation

$$\frac{d}{dt}\,(c_0P + B)p(t)) + A(p(t)) = h(t)\,,$$

for which we can show the dynamics is described by an *analytic semigroup*. This gives the following.

THEOREM 4.1. *Let $T > 0$, $v_0 \in L^2(\Omega)$, $v_1 \in L^2(\Gamma_S)$, and the pair of Hölder continuous functions $h_0(\cdot) \in C^\alpha([0,T], L^2(\Omega))$, $h_1(\cdot) \in C^\alpha([0,T], L^2(\Gamma_S))$ be given with*

$$\int_\Omega v_0(x)\,dx - \int_{\Gamma_S} v_1(s)\,ds = 0, \tag{4.11}$$

$$\int_\Omega h_0(x,t)\,dx + \int_{\Gamma_S} h_1(s,t)\,ds = 0, \quad t \in [0,T]\,. \tag{4.12}$$

Then there exists a pair of functions $p(\cdot) : (0,T] \to V$ and $\mathbf{u}(\cdot) : (0,T] \to \mathbf{V}$ for which $c_0p(\cdot) + \nabla\cdot\mathbf{u}(\cdot) \in C^0([0,T], L^2(\Omega)) \cap C^1((0,T], L^2(\Omega))$ and $\mathbf{u}(\cdot)\cdot\mathbf{n} \in C^0([0,T], L^2(\Gamma_S)) \cap C^1((0,T], L^2(\Gamma_S))$, and they satisfy the initial-boundary-value problem (4.8 – 4.10) with $t \mapsto tA(p(t))$ belonging to the space $L^\infty([0,T], L^2(\Omega) \oplus L^2(\Gamma_S)) \cap C^0((0,T], L^2(\Omega) \oplus L^2(\Gamma_S))$ and

$$\int_\Omega (c_0p(t) + \nabla\cdot\mathbf{u}(t))\,dx - \int_{\Gamma_S} (1-\beta)\,\mathbf{u}(t)\cdot\mathbf{n}\,ds = 0\,, \quad t \in (0,T]\,.$$

The function $\mathbf{u}(\cdot)$ is unique. When $Ker(c_0P+B+A) = \{0\}$, $p(\cdot)$ is unique, and if $Ker(c_0P+B) = \{0\}$ we delete the integral constraints (3) and (4).

When the data $h_0(\cdot)$, $h_1(\cdot)$ is smooth, we can show that the solution $p(\cdot)$ is $C^\infty(\Omega \times (0,T])$. Thus, the system is *parabolic*, even if $c_0 = 0$.

4.3. The Weak Solution. For another approach, we differentiate the first equation to obtain the system

$$\frac{d}{dt}\begin{pmatrix} \mathcal{E} & \vec{\nabla} \\ \vec{\nabla}\cdot & c_0P \end{pmatrix}\begin{bmatrix} \mathbf{u}(t) \\ p(t) \end{bmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & A \end{pmatrix}\begin{bmatrix} \mathbf{u}(t) \\ p(t) \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ h(t) \end{bmatrix}$$

The *holomorphic* case for the weak solution is given by

THEOREM 4.2. *Let $T > 0$, $v_0 \in V_a'$, and $h(\cdot) \in C^\alpha([0,T], V_a')$ be given. Then there exists a pair of functions $p(\cdot) : (0,T] \to V$ and $\mathbf{u}(\cdot) : (0,T] \to \mathbf{V}$*

for which $c_0 Pp(\cdot) + \vec{\nabla}\cdot\mathbf{u}(\cdot) \in C^0([0,T], V_a') \cap C^1((0,T], V_a')$, *and they satisfy the initial-value problem*

$$\mathcal{E}(\mathbf{u}(t)) + \vec{\nabla} p(t) = \mathbf{0}, \tag{4.13}$$

$$\frac{d}{dt}(c_0 Pp(t) + \vec{\nabla}\cdot\mathbf{u}(t)) + A(p(t)) = h(t), \quad t \in (0,T], \tag{4.14}$$

$$\lim_{t\to 0^+}(c_0 Pp(t) + \vec{\nabla}\cdot\mathbf{u}(t)) = v_0 \ \textit{in}\ V_a'. \tag{4.15}$$

The function $\mathbf{u}(\cdot)$ *is unique. When* $Ker(c_0 P + B + A) = \{0\}$, *the function* $p(\cdot)$ *is unique.*

Related problems arise in the modeling of clays, and there one finds an additional term to represent the *secondary consolidation* effects. A typical system is given by

$$-\mu^* \vec{\nabla}\frac{d}{dt}(\vec{\nabla}\cdot\mathbf{u}(t)) + \mathcal{E}(\mathbf{u}(t)) + \vec{\nabla} p(t) = \mathbf{h}(t)\,, \tag{4.16}$$

$$\frac{d}{dt}c_0 Pp(t) + A(p(t)) + \frac{d}{dt}\vec{\nabla}\cdot\mathbf{u}(t) = h(t)\,. \tag{4.17}$$

The solution of this degenerate *viscous* system is even *less regular* than the weak solution of Theorem 4.2. Specifically, not only is the diffusion equation (4.17) in V_a', but the momentum equation (4.17) is in $\mathbf{V}'$, so neither of them has the appropriate regularity to be decoupled into a system of partial differential equations and boundary conditions. Finally, we note that many of the above results for quasi-static systems are extensions of related and somewhat easier results for the *fully dynamic models* such as

$$\rho\ddot{\mathbf{u}}(t) - \mu^*\nabla(\nabla\cdot\dot{\mathbf{u}}(t)) + \mathcal{E}(\mathbf{u}(t)) + \nabla p(t) = \mathbf{h}(t)\,, \tag{4.18}$$

$$c_0\dot{p}(t) + A(p(t)) + \nabla\cdot\dot{\mathbf{u}}(t) = h(t)\,. \tag{4.19}$$

This is a *coupled wave-parabolic system.*

5. Projects. Here we briefly describe various systems that are being developed in order to model less restrictive and more realistic situations.

5.1. Non-Darcy flow, plastic deformation. More general constitutive equations are required for many applications. We indicate such an extension of the theory with the following non-Darcy flow model with plasticity. An additional momentum equation for the velocity of the pore fluid, $\mathbf{w}(t)$, and an elementary plasticity model of Prandtl-Reuss type are included in the system

$$c_0\dot{p}(t) + \nabla\cdot\mathbf{w}(t) + \nabla\cdot\dot{\mathbf{u}}(t) = h(t)\,, \tag{5.1}$$

$$\rho_f\dot{\mathbf{w}}(t) + \rho_f\ddot{\mathbf{u}}(t) + K^{-1}\mathbf{w}(t) + \nabla p(t) \ = \mathbf{0}\,, \tag{5.2}$$

$$\rho_f\dot{\mathbf{w}}(t) + \rho\ddot{\mathbf{u}}(t) + \mathcal{F}(\dot{\mathbf{u}}(t)) + \mathcal{E}^{1/2}\sigma_1(t) + \nabla(\sigma_2(t) + p(t)) = \mathbf{f}(t)\,, \tag{5.3}$$

$$\sigma_1(t) + \mathcal{E}^{1/2}\mathbf{u}(t) = 0\,, \tag{5.4}$$

$$\dot{\sigma}_2(t) + \nabla\cdot\dot{\mathbf{u}}(t) + \partial\varphi(\sigma_2(t)) = 0\,. \tag{5.5}$$

This *fully dynamic coupled system* is of mixed parabolic-hyperbolic type. The third equation is the momentum balance for the solid-fluid structure, and the second equation is momentum balance for the fluid. The matrix K is the *permeability*, so the term $K^{-1}\mathbf{w}(t)$ is the resistance of the solid structure to the diffusing fluid. In the case of a rigid solid, this equation takes the form

$$\rho_f \dot{\mathbf{w}}(t) + K^{-1}\mathbf{w}(t) + \nabla p(t) = \mathbf{0}\,,$$

which is a Darcy law with momentum. If we ignore the fluid momentum, i.e., if we set $\rho_f = 0$, then this is the classical Darcy law. The last equation is the *plastic* component of the total stress. Here $\varphi(\cdot)$ is the *indicator function* of a convex set which represents the yield surface for the plastic flow. Plastic behavior is prescribed in terms of the relative change of stress with respect to strain, and thereby it permits a dynamic formulation which is *rate independent* and contains *hysteresis* effects [38]. Additional nonlinear problems such as *partially saturated flow and deformation* are currently under investigation.

5.2. Composite media. The representation of porosity and permeability in naturally occuring materials often requires several distinct spatial scales. Thus the need arises for more general models incorporating qualitatively different characteristics. We briefly mention some ongoing work on two classes of models of composite media.

5.2.1. Parallel models. In problems of fluid flow in subsurface reservoirs and aquifers, the simplest and most frequently used model is the dual-porosity/dual-permeability medium which consists of two distinct components, both of which occur locally in any representastive volume element and behave as independent diffusion processes which are coupled by a distributed exchange term. In order to describe the flow of a single phase, slightly compressible fluid in a *composite medium*, that is, a porous medium composed of two interwoven (and possibly connected) components, we introduce at each point in space a density, pressure or concentration for each component, each being obtained by averaging in the respective medium over a generic neighborhood sufficiently large to contain a representative sample of each component. This construction and its application to the description of composite diffusion processes are generally attributed to Barenblatt *et al.* (1960) [3]. A straightforward unification of the models of Barenblatt and Biot is the system

$$\mathcal{E}(\mathbf{u}(t)) + \alpha_1 \nabla p_1(t) + \alpha_2 \nabla p_2(t) = \mathbf{0}\,, \tag{5.6}$$

$$c_1 \dot{p}_1(t) + A_1 \nabla p_1(t) + \alpha_1 \nabla\cdot\dot{\mathbf{u}}(t) + \gamma(p_1(t) - p_2(t)) = h_1(t)\,, \tag{5.7}$$

$$c_2 \dot{p}_2(t) + A_2 \nabla p_2(t) + \alpha_2 \nabla\cdot\dot{\mathbf{u}}(t) + \gamma(p_2(t) - p_1 t)) = h_2(t)\,, \tag{5.8}$$

where $\mathbf{u}$ is the displacement of the solid skeleton, and the pressures p_1 and p_2 have the meaning described above. For the case of a fractured medium, the first component is a matrix of porous and somewhat permeable material, and the second component is a system of highly permeable fractures, so both dual-porosity and dual-permeability characteristics are exhibited. The common characteristics of fractured media are that the solid matrix occupies a much larger volume than the fractures and that it is relatively much more resistant to fluid flow than is the fracture system. As a consequence, most of the flow passes through the system of fractures, while the bulk storage of fluid takes place primarily inside the porous matrix. The flow in the composite is enhanced by the exchange of fluid which takes place on the matrix–fracture interface. The theory described above for the Biot system (1.1), (1.2) has been recently extended to the Barenblatt-Biot system (5.6), (5.7), (5.8). For the development of such models, see [2], [6], [7], [8], [37], [41], [42], [43], [44].

5.2.2. Distributed microstructure Models. The introduction of *distributed microstructure* models represents an attempt to recognize the geometry and the multiple scales in the problem as well as to better quantify the exchange of fluid and momentum across the intricate interface between the components. Such models are frequently obtained as the limit by *homogenization* of corresponding exact but highly singular partial differential equations with rapidly oscillating coefficients. This provides not only a derivation of the model equations, but shows also the relation with the classical but singular problem on the microscale, and it provides a method for directly computing the *effective coefficients* which represent averaged material properties. For example, one can start on the microscale with Darcy flow models for each component, possibly with scaled permeability parameters, and obtain in the limit as the spatial scale goes to zero such a model for the macroscale behavior. This technique was used to derive the Biot system [1]. There the Navier elasticity system was coupled to a Stokes flow system on the microscale to obtain the Biot system in the limit as the macroscale model of the deforming porous medium. See the book [25] for a survey and perspectives.

We have investigated the limiting behavior of various combinations in the micromodel of a deforming porous medium at the mesoscale. For a fractured medium model, for example, we have used a Biot system for the matrix and Darcy flow for the fissures. One can use Biot systems for each component and scale the parameters for each component appropriately for the situation. Also, one can start with a Biot system for the structure coupled to a fluid flow model either of Stokes type or of slightly compressible flow type and then investigate the limiting form of the composite for various scalings of the parameters. An important technical aspect for each case is the appropriate set of boundary conditions to use at the interface between the porous medium and the fluid. Experience suggests that the distributed microstructure models provide accurate models which include the fine scales and geometry appropriate for many situations.

REFERENCES

[1] J.- L. Auriault, E. Sanchez-Palencia, *Étude du comportement macroscopique d'un milieu poreux saturé déformable*, Journal de Mécanique **16** (1977), pp. 575–603.

[2] M. Bai, D. Elsworth, and J. Roegiers, *Multiporosity/multipermeability approach to the simulation of naturally fractured reservoirs,* Water Resources Research **29** (1993), pp. 1621-1633.

[3] G. I. Barenblatt, I.P. Zheltov, and I.N. Kochina, *Basic concepts in the theory of seepage of homogeneous liquids in fissured rocks,* Prikl. Mat. Mekh.**24** (1960), pp. 852-864. (English translation: J. Appl. Mech. **24** (1960), pp. 1286–1303.)

[4] J. Bear, *Dynamics of Fluids in Porous media,* American Elsevier, New York, 1972.

[5] J. Bear, *Modeling Flow and Contaminant Transport in Fractured Rocks,* in Flow and Contaminant Transport in Fractured Rock, Bear, J., C.-F. Tsang, and G. de Marsily (editors), Academic Press, New York, 1993.

[6] J. Berryman, and H. F. Wang, *The elastic coefficients of double-porosity models for fluid transport in jointed rock,* J. Geophys. Res. **100** (1995), pp. 24611–24627.

[7] D. E. Beskos and E. C. Aifantis, *On the theory of consolidation with double porosity,* Int. J. Engng. Sci. **20** (1982), pp. 1079–1094.

[8] D. E. Beskos and E. C. Aifantis, *On the theory of consolidation with double porosity, II,* Int. J. Engng. Sci. **24** (1986), pp. 1697–1716.

[9] M. Biot, *General theory of three-dimensional consolidation,* J. Appl. Phys. **12** (1941), pp. 155–164.

[10] M. Biot, *Theory of elasticity and consolidation for a porous anisotropic solid*, J. Appl. Phys. **26** (1955), pp. 182–185.

[11] M. Biot, *Theory of propagation of elastic waves in a fluid-saturated porous solid, I. Low frequence range, II. Higher frequency range* J. Acoust. Soc. Amer. **28** (1956) pp. 168–178, 179–191.

[12] M. Biot, *Theory of deformation of a porous viscoelastic anisotropic solid,* Appl. Phys. **27** (1956), pp. 459–467.

[13] M. Biot, *Mechanics of deformation and acoustic propagation in porous media,* J. Appl. Phys. **33** (1962), pp. 1482–1498.

[14] M. Biot, *Theory of finite deformations of porous solids*, Indiana Univ. Math. J. **21** (1972), pp. 597–620.

[15] R. de Boer, *Theory of Porous Media–Highlights in the historical development and current state*, Springer-Verlag, Berlin, 2000.

[16] B.A. Boley and J.H. Weiner, *Theory of Thermal Stresses*, Wiley, New York, 1960.

[17] D.E. Carleson, *Linear Thermoelasticity*, in "Handbuch der Physik", VIa/2, Springer, New York, 1972.

[18] Ph. G. Ciarlet, *Mathematical elasticity. Vol. I. Three-dimensional elasticity*, North Holland, Amsterdam, 1988.

[19] R. E. Collins, *Flow of fluid through porous materials,* Research & Engineering Consultants, Inc. Englewood, Colorado, 1961.

[20] C.M. Dafermos, *On the existence and asymptotic stability of solutions to the equations of linear thermoelasticity*, Arch. Rational Mech. Anal. **29** (1968), pp. 241–271.

[21] W.A. Day, *Heat conduction within linear thermoelasticity*, Springer, New York, 1985.

[22] G. Duvaut and J.-L. Lions, *Inequalities in Mechanics and Physics*, Springer-Verlag, Berlin, 1976.

[23] G. Fichera, *Existence theorems in elasticity,* Handbuch der Physik, vol. VIa/2, Springer, New York, 1972.

[24] G. Fichera, *Uniqueness, existence and estimate of the solution in the dynam-*

ical problem of thermodiffusion in an elastic solid, Archiwum Mechaniki Stosowanej **26** (1974), pp. 903–920.

[25] Ulrich Hornung, editor, *Homogenization and Porous Media*, Interdisciplinary Applied Mathematics Series, vol. 6, Springer, New York, 1996.

[26] P. S. Huyakorn and G. F. Pinder, *Computational Methods in Subsurface Flow*, Academic Press, New York, 1983.

[27] T. Kato, *Perturbation Theory for Linear Operators*, Springer, New York 1966.

[28] V.D. Kupradze, editor, *Three-Dimensional Problems of the Mathematical theory of Elasticity and Thermoelasticity*, Applied Mathematics and Mechanics series, North-Holland, Amsterdam, 1979.

[29] A. Mourits and F. M. Sattari, *Coupling of geomechanics and reservoir simulation models*, Computer Methods and Advances in Geomechanics, 1994, pp. 2151–2158.

[30] Norris, *On the correspondence between poroelasticity and thermoelasticity*, J. Appl. Phys. **71** (1992), pp. 1138–1141.

[31] D. W. Peaceman, *Fundametals of numerical reservoir simulation*, Elsevier Scientific Publishing Company, Amsterdam-Oxford-New York, 1977.

[32] J. R. Rice and M. P. Cleary, *Some basic stress diffusion solutions for fluid-saturated elastic porous media with compressible constituents*, Rev. Geophysics and Space Phy. **14** (1976), pp. 227–241.

[33] A.P.S. Selvadurai, editor, *Mechanics of Poroelastic Media,* Kluwer Acad. Pub., Dordrecht, 1996.

[34] R. E. Showalter, *Hilbert space methods for partial differential equations*, Pitman, 1977, and Electronic Monographs in Differential Equations, http://ejde.math.swt.edu//mono-toc.html.

[35] R. E. Showalter, *Monotone operators in Banach space and nonlinear partial differential equations,* Mathematical Surveys and Monographs; no. 49. American Mathematical Society, Providence, (1997).

[36] R. E. Showalter, *Diffusion in poro-elastic media*, Jour. Math. Anal. Appl. **251** (2000), pp. 310–340.

[37] B. Momken and R.E. Showalter, *Single-phase Flow in Composite Poroelastic Media*, Math. Methods in the Applied Sciences, to appear.

[38] R.E. Showalter and Peter Shi, *Plasticity models and nonlinear semigroups*, Jour. Math. Anal. Appl. **216** (1997), pp. 218–245.

[39] K. von Terzaghi, *Erdbaumechanic auf Bodenphysikalischer Grundlage,* Franz Deuticke, Leipzig, 1925.

[40] K. von Terzaghi, *Theoretical Soil Mechanics,*, Wiley, New York, 1943.

[41] K. Tuncay and M.Y. Corapcioglu, *Wave propogation in fractured porous media*, Transport in Porous Media **23** (1996), pp. 237–258.

[42] K. Tuncay and M.Y. Corapcioglu, *Body waves in fractured porous media saturated by two immiscible Newtonian fluids*, Transport in Porous Media **23** (1996), pp. 259–273.

[43] S. Valliappan and N. Khalili-Naghadeh, *Flow through fissured porous media with deformable matrix*, Int. J. Numerical Methods in Engineering **29** (1990), pp. 1079–1094.

[44] R. K. Wilson and E. C. Aifantis, *On the theory of consolidation with double porosity*, Int. J. Engineering Sciences **20** (1982), pp. 1009–1035.

[45] Y. Yoloo, K. Yamagata and H. Nagoaka, *Finite element method applied to Biot's consolidation theory*, Soils and foundations **11** (1971), pp. 29–46.

[46] A. Zenisek, *The existence and uniqueness theorem in Biot's consolidation theory*, Apl. Mat. **29** (1984), pp. 194–211.

[47] O.C. Zienkiewicz, C.T. Chang, P. Battess, *Drained, undrained, consolidating, and dynamic behaviour assumptions in soils, limits of validity*, Geotechnique **30** (1980), pp. 385–395.

[48] O.C. Zienkiewicz, A.H.C. Chan, M. Pastor, B.A. Schrefler, T. Shiomi, *Computational Geomechanics*, Wiley, Chichester, 1999.