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PORO-VISCO-ELASTIC COMPACTION IN SEDIMENTARY BASINS*

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Abstract. The porosity of a visco-elastic medium is shown to satisfy a nonlinear pseudoparabolic partial differential equation of the form $u' + A(u)(\alpha(u) + \eta u') = G(t, u)$ in which u' denotes the time derivative, $A(v) = -\boldsymbol{\nabla} \cdot \kappa(v)\boldsymbol{\nabla}$ is a linear second order elliptic operator in divergence form with coefficient depending on a function $v(x)$, $\alpha(\cdot)$ is affine-bounded and $\alpha(\cdot) + kI$ is monotone for some $k \in \mathbb{R}$, $G(t, u)$ is a linear first order operator in u, and $\eta > 0$. The third order nonlinear term $A(u)u'$ distinguishes this equation from the classical porous medium equation. The solvability of an elliptic boundary-value problem for $(I + \eta A(v))u = f$ for $\eta > 0$ and the continuous dependence of the solution u on the function v is used to establish existence of the solution of the initial-boundary-value problem for the pseudoparabolic equation. We establish bounds on the solution that prevent degeneracy of the coefficient $\kappa(\cdot)$ and prove regularity properties of the solution. These results are obtained by methods of monotonicity and compactness.

Key words. pseudoparabolic equations, visco-elastic rheology, monotone operators, evolution equations

AMS subject classifications. Primary, 47J35, 47H05, 35F61; Secondary, 35Q35, 76S05

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1. Introduction.

Consolidaton is the process by which a load on a deformable saturated porous medium such as soil will decrease its volume by means of the rearrangement and more efficient packing of the solid grains with the corresponding expulsion of pore fluid and reduction of porosity. The load may be an overburden stress applied to the structure or its own weight. Small deformations of the medium can be partially reversed by a reduction of the load or an increase in the fluid pressure. *Compaction* includes additional processes that may occur on larger scales and at greater depth, such as the viscous flow of the medium due to higher pressures and chemical compaction processes of pressure solution. Such coupled poromechanical processes are of central importance in the description of subsurface flow and pressurization by fluid injection that occur in oil well drilling operations, geothermal energy production, and the removal of groundwater. Here we study the compaction of a sedimentary basin as indicated by the *porosity* $\phi(x, t)$, the local volume fraction of pore space between the grains of the medium. We begin by reviewing conservation laws and constitutive assumptions that lead to a single partial differential equation that determines the evolution of porosity and pressure during the compaction process as the grains of the medium rearrange and expel pore water. The mathematical model will depend on the specific poromechanical constitutive assumptions on the medium. Here we consider visco-elastic assumptions on the medium by which the porosity satisfies an initial-boundary-value problem for the partial differential equation

$$\frac{\partial \phi}{\partial t} - \boldsymbol{\nabla} \cdot \kappa(\phi) \boldsymbol{\nabla} \left(\alpha(\phi) + \eta \frac{\partial \phi}{\partial t} + P \right) = \boldsymbol{\nabla} \cdot (1 - \phi) \boldsymbol{\nabla} \Delta^{-1} F \,. \tag{1}$$

In the following sections, we shall develop the existence and regularity theory for this problem. Equation (1) is of *pseudoparabolic* type, due to the third order term with $\eta >$

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0, and we shall show that the properties of the solution reflect remarkably well those expected of the porosity. In particular, any discontinuity in the value or derivative of the solution in the normal direction across a lower dimensional surface does not disappear instantly, as it would for a parabolic equation ($\eta = 0$ and α monotone) but rather decays at a rate determined by $\alpha(\phi)$ and η. This is a pointwise manifestation of the general fact that for pseudoparabolic equations the spatial regularity is not improved with advancing time but is *preserved*.

There is an enormous literature of the applications and mathematical development of pseudoparabolic equations. They arose in a range of classical applications including radiation with time delay [36], soil mechanics [54], degenerate cases of double-diffusion and heat-conduction models of composite systems [45, 6] such as fissured media [11], flow of second order fluids [52], long waves [39], and the regularization of ill-posed backward parabolic problems [47, 22, 29]. More recent applications include level set methods [16] and models of lightning propagation [3].

Pseudoparabolic equations were first analyzed in [49, 53]; see Chapter 3 of [19] for an extensive review and bibliography up to 1976 and [20] with its bibliography for early extensions to doubly nonlinear equations. The more recent book [2] contains an extensive bibliography of work on pseudoparabolic equations and the more general class of partial differential equations of *Sobolev type*, i.e., equations in which the time derivative of highest order is implicit in a differential operator in spatial variables. Various regularizations of forward-backward parabolic equations [38, 37, 5] have led to extensive current and emerging studies of doubly nonlinear pseudoparabolic equations; see [21, 46, 8, 9] and their references. Similar regularizations of Navier–Stokes or Euler systems [10, 31] and of long-wave equations [13, 30] have been analyzed.

Equations of the form (1) describe two-phase flow with *dynamic capillary pressure* [28, 41, 40] as well as the compaction models [56, 51, 33] to be described below. If the η-term is linear, e.g., a positive-definite tensor and $\kappa(\phi) = 1$, that term dominates the others and existence of a solution follows as in [11, 37, 42, 43, 23]. For the non-degenerate case $\kappa(\phi) \geq \kappa_0 > 0$ with $\alpha(\cdot)$ Lipschitz and monotone, there are various formulations [24] and uniqueness [17]; existence was established in [12]. The two-phase flow model requires the degenerate case of (1) for which $\kappa(\phi) \geq 0$ and $\kappa(0) = 0$, while $\alpha(\cdot)$ is monotone but singular at 0. Large time existence was proved for this situation by Mikelić [34]; the entropy method was previously used for a similar degenerate pseudoparabolic problem [35]. This approach was used in [18] to prove existence for a degenerate system which extends a single equation like (1).

We provide here an independent study of (1) as a model of compaction described below. The existence results are most closely related to those in [12], but here we develop them directly from elementary methods of monotonicity and compactness without recourse to interpolation theory. Also we extend the coverage to let $\alpha(\cdot)$ be the *sum* of a monotone function and a Lipschitz function rather than requiring *both* properties. This permits $\alpha(\cdot)$ to be singular (with infinite slope) at isolated points and includes the regularized forward-backward equations. (These are ill-posed if $\eta = 0$.) However, we do assume $\kappa(\cdot)$ is nondegenerate; this is appropriate for the compaction model, since the porosity is observed to be limited to the interval $[0.1, 0.5]$, even at depths up to 3 kilometers. We show that solutions will stay constrained by the limits of the initial porosity, and thus will not lead to degeneracy of the elliptic coefficient. This eliminates the need for the use of entropy estimates. These simpler proofs used here should make the material accessible to a broader group of readers. The development of (1) as a model of compaction of a sedimentary basin is the content of the remainder of this introduction. In section 2 we will show that $I + \eta A_u$ is invertible and develop continuity properties of the resolvent $J(u,f) = (I + \eta A_u)^{-1} f$ that will be used in section 3 to establish existence of a solution to the initial-boundary-value problem for (1). When $u_0 \in L^2(G)$, a solution is obtained with $u(t) \in L^2(G)$ for $0 \le t \le T$; see Theorem 23. If additionally $\nabla u_0 \in L^2(G_0)$ for some measurable $G_0 \subset G$, then $\nabla u(t) \in L^2(G_0)$ for $0 \le t \le T$, and if u_0 has a jump along a submanifold in G, the jump is maintained there by the solution $u(t)$ at every $t \in [0,T]$; see section 5.

1.1. The rheology. A sedimentary basin is a saturated granular deformable porous medium consisting of the particles of sand or silt that accumulate between the sea bottom and (possibly) a confining bedrock. Such basins are ubiquitous and frequently extend to depths of kilometers and widths of 10s or 100s of kilometers. As sediments are deposited from above, they are subjected to increasing stress under which the lower regions compact, expelling pore water by increasing pore pressures well above hydrostatic levels.

First we recall the construction of mathematical models for the study of compaction of a sedimentary basin [4, 26]. The primary simplifying assumption in the mechanics of the model is that the ratio of shear to bulk modulus of the granular porous medium is extremely low, so the shear forces between grains will be ignored. This holds well for sedimentary basins, since they are often well approximated by one-dimensional models due to their wide extent, and this assumption holds also on a smaller scale for media with a loose sediment containing substantial organic matter, and likewise for deforming incompressible soft granular media, even on laboratory scales. For such media, the *effective stress* is given by the tensor $\sigma_{eff} = \sigma\delta$, where the scalar-valued $p_s = -\sigma$ is *effective pressure* within the sediment matrix of incompressible particles. Since the sediment is cohesionless, it cannot support any tensile stress, so the effective pressure is necessarily nonnegative. In the study of soil consolidation, the porosity is related to the effective pressure by a *normal consolidation curve* which represents nonlinear elastic compression and irreversible damage. This leads to a nonlinear parabolic partial differential equation so long as the effective pressure is *increasing*. Slow and small variations of effective pressure below this curve can be described well by reversible elastic relations between porosity and effective pressure in sediments at depths up to at most a kilometer. Such poroelastic consolidation due to grain packing and rearrangement provides an adequate description of the rheology only in these upper shallower regions. See section 7.10 in [26]. Somewhat faster variations follow small hysteretic loops that can be modeled by local visco-elastic rheology.

At greater depths and effective pressures, a viscous effect known as *pressure solution* becomes prevelant. This is the result of dissolution of the grains at contact points due to the excessive local effective pressure followed by precipitation in adjacent pore space, where pressure is essentially the lower fluid pressure. It is a consequence of the enhanced solubility of minerals due to increased stress at the grain contacts. This chemical compaction process results in an effective viscous creep of the granular structure and leads to an increase in local grain contact area which distributes the intergranular stress over a larger surface. The process is referred to as viscous compaction, viscous creep, or pressure solution and is described by a relation between effective stress and *strain rate*, e.g., the rate of change of porosity. (See Chapter 5 of [7].) It constitutes a fundamental deformation process in the upper crust of the Earth, not only for sediment basins but also for the dynamics of faults, ground subsidence due to fluid withdrawal, and even the propogation of magma through the Earth's mantle. Eventually the chemical compaction leads to a thermodynamic equilibrium at the grain contacts. This corresponds to a balance of all the forces acting at these contacts and is determined by the limiting value of porosity. See section 7.11 in [26] and [25, 44, 55].

Here we extend the rheology of the medium to include a nonlinear visco-elastic model of Kelvin–Voigt type for the granular porous medium. This will apply not only to small variations in effective pressure but also to larger variations so long as the medium is not irreversibly consolidated. Such viscous contributions to the consolidation curves are observed as small hysteretic cycles at all depths. They become increasingly important at depths below 500 m where grain creep within the medium is already substantial, and they may be the predominant deformation mechanism a kilometer lower in sedimentary basins. As noted above, the rheology model should have two asymptotic limits, one for the viscous behavior at short time scales and a second for the elastic behavior at long time equilibrium limits. These are characteristics of the Kelvin–Voigt visco-elastic model

$$\sigma = \alpha(\phi) + \eta \frac{\partial \phi}{\partial t} \tag{2}$$

in which the effective stress is the sum of a nonlinear strain component and a strain rate component. The first is the bulk modulus of the medium and η is the viscosity. This is a realistic model of material response (creep) to constant stress. We shall assume that the function $\alpha(\cdot)$ is continuous and affine-bounded and that $\alpha(\cdot) + kI$ is monotone for some $k \in \mathbb{R}$. For the last condition, it suffices that $\alpha(\cdot)$ is either Lipschitz or monotone.

The added viscous forces change the corresponding partial differential equation to one of *pseudoparabolic* type, and consequently the properties of the solution are very different from those of the usual diffusion type problem. In particular, spatial regularity of the solution is preserved in time, and any discontinuity in the solution (or a derivative of the solution) does not disappear but remains at that location and decreases at a rate which is directly computed from (2). This localization and perseverence of smoothness are properties typical of materials.

1.2. The mathematical model. We recall the basic two-phase model for a sedimentary basin [7, 4, 26]. Assume a fully saturated cohesionless sediment for which *porosity* $\phi(x,t)$ denotes the volume fraction of the medium occupied by the fluid, *fluid velocity* is $\mathbf{v}_f(x,t)$, *solid velocity* is $\mathbf{v}_s(x,t)$, and their respective densities ρ_f, ρ_s are constant. The grains are heavier than the fluid, so $\rho_s > \rho_f$. *Permeability* of the matrix of solid particles depends on porosity and is denoted by $K(\phi)$. *Pressure* of the fluid is $p(x,t)$ and μ is its viscosity. Since shear stress in the sediment matrix is assumed to be negligible, the *effective stress* is given by the tensor $\sigma_{eff} = -p_s\delta$, where p_s is *effective pressure* within the sediment matrix of solid particles.

Assume the domain G in $\mathbb{R}^N$ is contained in the sediment basin. Let $T > 0$ and set $G_T = G \times (0,T)$. The compaction model is given by the system

$$\frac{\partial \rho_f \phi}{\partial t} + \boldsymbol{\nabla} \cdot (\rho_f \phi \mathbf{v}_f) = \rho_f F, \tag{3a}$$

$$\frac{\partial \rho_s (1-\phi)}{\partial t} + \boldsymbol{\nabla} \cdot (\rho_s (1-\phi) \mathbf{v}_s) = 0, \text{ and} \tag{3b}$$

$$\phi(\mathbf{v}_f - \mathbf{v}_s) = -\frac{K(\phi)}{\mu} \boldsymbol{\nabla} p \text{ in } G_T. \tag{3c}$$

The conservation of fluid is (3a), the conservation of solid is (3b), and (3c) is *Darcy's law* for the flow of fluid relative to the solid. Adding the two conservation equations gives

$$\nabla \cdot (\phi \mathbf{v}_f + (1-\phi)\mathbf{v}_s) = F \text{ in } G_T \tag{4}$$

for the *composite flow rate* $\mathbf{v} \equiv \phi \mathbf{v}_f + (1-\phi)\mathbf{v}_s$. Assume this is *irrotational.* Then it can be determined from (4) by solving a Neumann problem for $\Delta f = F$ with normal composite flux $\nabla f \cdot \mathbf{n} = \mathbf{v} \cdot \mathbf{n}$ known on the boundary to get $\mathbf{v} = \nabla f$. This yields the composite *volume flow rate*

$$\phi \mathbf{v}_f + (1-\phi)\mathbf{v}_s = \nabla \Delta^{-1} F. \tag{5}$$

From (3c) and (5) we obtain

$$\mathbf{v}_s = \frac{K(\phi)}{\mu} \nabla p + \nabla \Delta^{-1} F,$$

and inserting this into (3b) yields the *porosity-pressure equation*

$$\frac{\partial \phi}{\partial t} - \nabla \cdot (1-\phi) \frac{K(\phi)}{\mu} \nabla p = \nabla \cdot (1-\phi) \nabla \Delta^{-1} F\,. \tag{6}$$

(See equation 7.250 in [26].) The effective pressure of the solid shares the load of the *overburden pressure* P with the fluid pressure, so we have

$$P = p + p_s = p - \sigma. \tag{7}$$

Following [44], we substitute the Kelvin–Voigt visco-elastic *stress-porosity* constitutive relation (2) with (7) into (6) to obtain the partial differential equation (1) for the evolving porosity in the sedimentary basin. See [27] for discussion of elastic compaction, [25] for viscous compaction, and [55] for a model with visco-elastic rheology of *Maxwell* type. The one-dimensional case of (1) was obtained for slow compaction in [56]. A further approximation was made there by which (1) was replaced by a weakly damped wave equation.

We shall denote by $\kappa(\phi) = (1-\phi)\frac{K(\phi)}{\mu}$ the coefficient arising in the porosity-pressure equation (6). Since the grains cannot fit together perfectly, the porosity has a positive lower bound, and it is bounded well below 1 since we are in the region of sediment. We show that the solution remains within an interval $[\underline{k}, \bar{k}] \subset (0,1)$ when $\nabla P = 0$ and $F(\cdot) = 0$. Thus we assume in the following that $\kappa(\cdot)$ is bounded to an interval (κ_0, κ_1) with $0 < \kappa_0 \le \kappa_1$.

1.3. Laboratory scale models. It remains a challenge to design experiments to study poroelastic deformations in the laboratory. When the fluid pressure exceeds intergranular stresses in the solid, the stored energy is negligible. Since fluid pressure must be comparable to the elastic modulus of the solid structure in order to store elastic energy, the experiment requires either very high pressures or a very soft medium. This limitation was circumvented in [51] by using soft open-cell polymer foams. For the one-dimensional case these were modeled as poroelastic ($\eta = 0$ in (2)) and numerically simulated. The results were in agreement with the experiment when small initial compactions were used. Since the model predictions were less accurate for larger initial compactions, it was expected that omitted viscous lubrication forces had become significant during the more rapid compaction. More recently, the visco-elastic deformations of a soft granular structure have been remarkably recorded by injecting fluid into a layer of spherical particles confined between glass plates. These experiments exhibit striking poroelastic phenomena at low pressures [33]. The high resolution imaging and particle tracking enabled experimental observation of the full deformation field, illustrating visco-elastic response as well as hysteresis. A linear continuum model of the same type as (1) captures major macroscopic features of the deformation.

1.4. The plan. We provide below an independent study particularly directed toward (1). The results here are most closely related to those of [12], but here we develop them directly without recourse to interpolation theory. In section 2 we will show that $I+\eta A_u$ is invertible, and in section 3 we develop continuity properties of the resolvent $J(u,f) = (I+\eta A_u)^{-1}f$ that will be used in section 4 to establish existence of a solution to the initial-boundary-value problem for (1). When $u_0 \in L^2(G)$, a solution is obtained with $u(t) \in L^2(G)$ for $0 \le t \le T$; see Theorem 23. If additionally $\boldsymbol{\nabla} u_0 \in L^2(G_0)$ for some measurable $G_0 \subset G$, then $\boldsymbol{\nabla} u(t) \in L^2(G_0)$ for $0 \le t \le T$, and if u_0 has a jump along a submanifold in G, the jump is maintained there by the solution $u(t)$ at every $t \in [0,T]$; see section 5.

2. The elliptic operator. We shall start by fixing some notation. Let G be a smoothly bounded region in $\mathbb{R}^3$ and denote its boundary by $\Gamma = \partial\Omega$. We will make use of the Lebesgue space $L^p(G)$ of pth power summable functions, the corresponding Sobolev space $W^{m,p}(G)$ of functions which with derivatives up to order m belong to $L^p(G)$ and the spaces $C(\bar{G})$, $C^1(\bar{G})$ of uniformly continuous functions or functions with derivatives in $C(\bar{G})$, respectively. The reader is referred to any of [1, 2, 48] for definitions and properties of these spaces. Mostly we use only the Hilbert spaces $L^2(G)$, $H^1(G)$, $H_0^1(G)$ and the dual space $H^{-1}(G) = H_0^1(G)'$. We shall denote corresponding spaces of vector-valued functions taking values in the Hilbert space $\mathbb{H}$ by $C([0,T],\mathbb{H})$, $L^2((0,T),\mathbb{H})$, $H^1((0,T),\mathbb{H})$. In particular, we will use the notation $(\cdot,\cdot)$ for the scalar product in $L^2(G)$. We use $\|\cdot\|_{L^2}$ to represent the norm of $L^2(G)$ as well as the product space $(L^2(G))^N$. It should be clear from context which is being used.

Let $\kappa : \mathbb{R} \to \mathbb{R}$ be a continuous function such that $\kappa_0 \le \kappa(s) \le \kappa_1$ with $\kappa_0 > 0$. For a measurable function u, consider the bilinear form

$$a_u(v,w) = \int_G \kappa(u(x))\nabla v(x)\cdot\nabla w(x)dx,\ \ v,w \in H_0^1(G).$$

Since κ is continuous and bounded, we have $\kappa(u)$ is in $L^\infty(G)$ for each measurable function u. Thus, the bilinear form $a_u(\cdot,\cdot)$ is continuous on $H_0^1(G)\times H_0^1(G)$. From the definition of the weak derivative, the corresponding linear operator $A_u : H_0^1(G) \to H^{-1}(G)$ given by $A_u v(w) = a_u(v,w)$ is the elliptic operator in divergence form

$$A_u v = -\nabla\cdot\kappa(u)\nabla v$$

that arises in the partial differential equation (1). Equation (1) corresponds to $\kappa(\phi) = (1-\phi)\frac{K(\phi)}{\mu}$. We note that $I+\eta A_u$ is invertible and develop continuity properties of the resolvent $J(u,f) \equiv (I+\eta A_u)^{-1}f$.

LEMMA 1. *Let the measurable function u on G and $\eta > 0$ be given. The map $I+\eta A_u : H_0^1(G) \to H^{-1}(G)$ is an isomorphism, bounded in norm by* $\max\{1,\eta\kappa_1\}$, *and the norm of $(I+\eta A_u)^{-1}$ is at most* $1/\min\{1,\eta\kappa_0\}$ *in $\mathcal{L}(H^{-1}(G),H_0^1(G))$. In particular, the bounds are independent of the choice of u. Moreover, $(I+\eta A_u)^{-1} : L^2(G) \to L^2(G)$ is a contraction.*

Proof. From the calculation

$$((I+\eta A_u)v,w)=(v,w)+\eta a_u(v,w)\le \max\{1,\eta\kappa_1\}\,\|v\|_{H^1}\,\|w\|_{H^1}$$

we get the upper bound, and the lower bound follows from

$$\min\{1,\eta\kappa_0\}\,\|v\|_{H^1}^2\le \|v\|_{L^2}^2+\eta\kappa_0\,\|\nabla v\|_{L^2}^2\le (v,v)+\eta a_u(v,v).$$

It follows from the Lax–Milgram theorem that the operator is an isomorphism. The upper bound on the norm of $(1+\eta A_u)^{-1}$ follows directly. Finally, $(I+\eta A_u)^{-1}$ is a contraction on $L^2(G)$ as

$$\|v\|_{L^2}^2\le (v,v)+\eta a_u(v,v)=((I+\eta A_u)v,v)\le \|(I+\eta A_u)v\|_{L^2}\,\|v\|_{L^2}$$

for all $v\in H_0^1(G)$. The result extends to all of $L^2(G)$ by density. ∎

We turn to the dependence of $J(u,f)$ on u.

LEMMA 2. *Fix f in $H^{-1}(G)$ and u_1, u_2 measurable. Then we have*

$$\|J(u_1,f)-J(u_2,f)\|_{H^1}\le C\,\|(\kappa(u_1)-\kappa(u_2))\nabla J(u_i,f)\|_{L^2}\,,\ i=1,2,$$

where C depends only on η and κ_0.

Proof. Let $v_1=J(u_1,f)$, $v_2=J(u_2,f)$. For all w in $H_0^1(G)$ we have

$$(v_1-v_2,w)+\eta\,(\kappa(u_1)\nabla v_1-\kappa(u_2)\nabla v_2,\nabla w)=0.$$

Adding $\eta(\kappa(u_2)\nabla v_1-\kappa(u_2)\nabla v_1,\nabla w)$ and rearranging show

$$(v_1-v_2,w)+\eta\,(\kappa(u_2)\nabla(v_1-v_2),\nabla w)=\eta\,((\kappa(u_2)-\kappa(u_1))\nabla v_1,\nabla w)\,.$$

Set $w=v_1-v_2$ and note that $\kappa_0\le\kappa(u_2)$ everywhere to get

$$\|v_1-v_2\|_{L^2}^2+\eta\kappa_0\,\|\nabla(v_1-v_2)\|_{L^2}^2\le \eta\,((\kappa(u_2)-\kappa(u_1))\nabla v_1,\nabla(v_1-v_2))\,.$$

By applying the Cauchy–Schwarz inequality and $ab\le\frac12\left(\varepsilon a^2+\varepsilon^{-1}b^2\right)$ for a, b, ε positive, we obtain

$$\begin{aligned}&\eta\,((\kappa(u_2)-\kappa(u_1))\nabla v_1,\nabla(v_1-v_2))\\ &\quad\le \tfrac{\eta}{2\varepsilon}\,\|(\kappa(u_2)-\kappa(u_1))\nabla v_1\|_{L^2}^2+\tfrac{\eta\varepsilon}{2}\,\|\nabla(v_1-v_2)\|_{L^2}^2\,,\end{aligned}$$

and combining this with the above and rearranging gives

$$\|v_1-v_2\|_{L^2}^2+\eta(\kappa_0-\tfrac{\varepsilon}{2})\,\|\nabla(v_1-v_2)\|_{L^2}^2\le \tfrac{\eta}{2\varepsilon}\,\|(\kappa(u_2)-\kappa(u_1))\nabla v_1\|_{L^2}^2\,.$$

Now take $\varepsilon=\kappa_0$ and set

$$C^2=\tfrac{\eta}{2\kappa_0\min\{1,\eta\kappa_0/2\}}=\max\left\{\tfrac{\eta}{2\kappa_0},\tfrac{1}{\kappa_0^2}\right\}$$

to get the desired result for $i=1$. The case $i=2$ is identical. ∎

This leads to an additional estimate and a continuity result.

Proposition 3.

1. *For f_1, f_2 in $H^{-1}(G)$ and u_1, u_2 measurable, we have*

$$\begin{aligned}&\|J(u_1,f_1)-J(u_2,f_2)\|_{H^1}\\&\qquad\le C\,\|(\kappa(u_1)-\kappa(u_2))\nabla J(u_i,f_i)\|_{L^2}+\|J(u_i,f_1-f_2)\|_{L^2}\,,\ i=1,2,\end{aligned}$$

with C as in Proposition 2. In particular, if f_1, f_2 are in $L^2(G)$, we then have

$$\|J(u_1,f_1)-J(u_2,f_2)\|_{H^1}\le C\,\|(\kappa(u_1)-\kappa(u_2))\nabla J(u_i,f_i)\|_{L^2}+\|f_1-f_2\|_{L^2}$$

for $i = 1, 2$.

2. *For each f in $H^{-1}(G)$, the map $J(\cdot, f) : L^2(G) \to H_0^1(G)$ is continuous. More generally, if u_n converges to u in measure, then $J(u_n, f)$ converges to $J(u, f)$ in $H_0^1(G)$*

Proof. The first follows from the estimate

$$J(u_1,f_1)-J(u_2,f_2)=J(u_1,f_1)-J(u_2,f_1)+J(u_2,f_1)-J(u_2,f_2)$$

with Lemmas 1 and 2.

Since convergence in L^2 implies convergence in measure, we only need to show the latter to prove (2). Thus, we suppose u_n converges to u in measure and define $v = J(u, f)$, $v_n = J(u_n, f)$. From Proposition 2, we have the estimate

$$\|v-v_n\|_{H_0^1}\le C\,\|(\kappa(u_n)-\kappa(u))\nabla v\|_{L^2}\,.$$

Recall that convergence in measure guarantees that for any subsequence $\{u_{n_k}\}$ of $\{u_n\}$, we can find a further subsequence $\{u_{n_{k_j}}\}$ such that $u_{n_{k_j}}(x) \to u(x)$ almost everywhere. By the continuity of κ, we can conclude that $\kappa(u_{n_{k_j}}(x))$ converges to $\kappa(u(x))$ almost everywhere. Further, we have

$$\left|\kappa(u(x))-\kappa(u_{n_{k_j}}(x))\right|\,\|\nabla v(x)\|_{\mathbb{R}^N}\le 2\kappa_1\,\|\nabla v(x)\|_{\mathbb{R}^N}\,,$$

where the right-hand side is a square-integrable function. By applying Lebesgue dominated convergence, we can conclude that

$$\lim_{k\to\infty}\left\|v-v_{n_{j_k}}\right\|_{H^1}\le C\lim_{k\to\infty}\left\|(\kappa(u)-\kappa(u_{n_{k_j}}))\nabla v\right\|_{L^2}=0.$$

As this can be done for every subsequence, we can thus conclude that v_n converges to v in H^1 norm. That is, $J(u_n, f)$ converges to $J(u, f)$ in $H_0^1(G)$. ◻

Next we show that $J(\cdot,\cdot) : L^2(G) \times H^{-1}(G) \to H_0^1(G)$ is jointly continuous.

Proposition 4. *Suppose u_n, f_n converge to u, f in $L^2(G)$ and $H^{-1}(G)$, respectively. Then $J(u_n, f_n)$ converges to $J(u, f)$ in $H^1(G)$.*

Proof. We have

$$\begin{aligned}\|J(u,f)-J(u_n,f_n)\|_{H^1}&=\|J(u,f)-J(u_n,f)+J(u_n,f)-J(u_n,f_n)\|_{H^1}\\&\le\|J(u,f)-J(u_n,f)\|_{H^1}+\|J(u_n,f-f_n)\|_{H^1}\\&\le\|J(u,f)-J(u_n,f)\|_{H^1}\\&\quad+\|J(u_n,\cdot)\|_{\mathcal{L}(H^{-1},H_0^1)}\,\|f-f_n\|_{H^{-1}}\,.\end{aligned}$$

Note that the first term is known to go to zero by Proposition 3. Further, the second term goes to zero, since $\|J(u_n, \cdot)\|$ is uniformly bounded by some constant C'. Thus we have

$$\lim_{n\to\infty} \|J(u,f) - J(u_n, f_n)\|_{H^1} = 0,$$

which is the desired result. $\square$

COROLLARY 5. *Fix f in $H^{-1}(G)$. The range of $J(\cdot, f) : L^2(G) \to L^2(G)$ is precompact. More generally, the set $J(L^2(G), B)$ is precompact in $L^2(G)$ for any set B bounded in $H^{-1}(G)$.*

Proof. We know that $J(L^2(G), B)$ is a bounded set in $H_0^1(G)$. As the inclusion map from $H_0^1(G)$ to $L^2(G)$ is compact, this set is precompact in $L^2(G)$. $\square$

We recall the Schauder fixed point theorem. We note in passing that it implies the map $J(\cdot, f)$ has a fixed point for each f.

THEOREM 6 (Schauder). *Let X be a Hausdorff topological vector space with nonempty, closed, convex subset K. If $T : K \to K$ is continuous and $T(K)$ is contained in a compact subset of K, then T has a fixed point.*

COROLLARY 7. *For every f in $H^{-1}(G)$, there exists a u_0 in $L^2(G)$ such that $u_0 = J(u_0, f)$. This solution is characterized by*

$$\begin{gathered} u_0 - \eta\nabla\cdot\kappa(u_0)\nabla u_0 = f \text{ in } G, \\ u_0 = 0 \text{ on } \partial G. \end{gathered}$$

Proof. For f in $H^{-1}(G)$, the map $u \mapsto J(u, f)$ is a continuous map on $L^2(G)$ with range contained in a compact set. Schauder's fixed point theorem then gives the result. $\square$

Of course, this result is not at all optimal. In fact, one can resolve uniquely the more general boundary-value problem

$$\beta(u_0) \in H_0^1(G), \quad u_0 - \Delta\beta(u_0) \ni f \text{ in } H^{-1}(G),$$

where $\beta(\cdot)$ is a maximal monotone graph [15, 48]. For the special case in which β is also Lipschitz continuous, we have $\Delta\beta(u_0) = \nabla\cdot\beta'(u_0)\nabla u_0$, and this recovers Corollary 7.

3. The integral equation. We shall write the initial-boundary-value problem for (1) in the form

$$u'(t) + A_{u(t)}(\eta u'(t) + \alpha(u(t)) + P(t)) = \nabla\cdot(1-u(t))\nabla\Delta^{-1}F(t), \quad u(0) = u_0, \tag{8}$$

as an equation in $H^{-1}(G)$. At almost every $t \in (0,T)$ we have a Dirichlet boundary condition on $\eta u'(t) + \alpha(u(t)) + P(t) \in H_0^1(G)$, the fluid pressure, whereas the initial value is prescribed for the porosity, $u(t)$. By adding $\frac{1}{\eta}(\alpha(u(t)) + P(t))$ to both sides, applying the resolvent, and then integrating, we obtain the integral equation

$$\begin{aligned} &u(t) + \tfrac{1}{\eta}\int_0^t [\alpha(u(s)) + P(s)]\,ds \\ &\quad = u_0 + \int_0^t (I + \eta A_{u(s)})^{-1}\left(\tfrac{1}{\eta}\left(\alpha(u(s)) + P(s)\right) + \nabla\cdot(1-u(s))\nabla\Delta^{-1}F(s)\right) ds. \end{aligned} \tag{9}$$

When $u_0 \in L^2(G)$, we shall show that there exists a function $u \in H^1((0,T); L^2(G))$ that satisfies (9), and we will henceforth refer to such a function as a *solution* to the integral equation. It is also a solution of (8) obtained from (9) by differentiation. The advantage of (9) is that the right side contains the elliptic resolvent operator in spatial variables and the left side involves an ordinary differential equation in time. This separation will be exploited throughout this section. Note, however, that the ordinary differential equation and the elliptic resolvent operator are independently nonlinear.

Our assumptions are as follows

- (A1) $\kappa : \mathbb{R} \to \mathbb{R}$ is continuous, and there are constants κ_0 and κ_1 such that $0 < \kappa_0 \leq \kappa(\xi) \leq \kappa_1$ for all ξ in $\mathbb{R}$.
- (A2) $\alpha : \mathbb{R} \to \mathbb{R}$ is continuous with affine bound and there exists $k \in \mathbb{R}$ such that $\alpha + kI$ is monotone.
- (A3) $\left\|\boldsymbol{\nabla}\Delta^{-1}F(t)\right\|_{L^\infty} \leq C(t)$, where $C(t)$ is integrable.
- (A4) $\int_0^T \|P(t)\|_{L^2}^2 \, dt < \infty$.

The growth assumptions in (A2) are equivalent to requiring that α satisfy

$$|\alpha(\xi)| \leq K_\alpha(|\xi| + 1), \quad \xi \in \mathbb{R}, \text{ and} \tag{10a}$$

$$(\alpha(\xi_1) - \alpha(\xi_2))(\xi_1 - \xi_2) \geq -k(\xi_1 - \xi_2)^2, \quad \xi_1, \xi_2 \in \mathbb{R}. \tag{10b}$$

The second condition (10b) follows if either α is monotone (k=0) or if α is Lipschitz (with Lipschitz constant $k > 0$). In particular, it gives that α is a sum of a monotone function and a Lipschitz function. Furthermore, it is readily checked that assumption (A3) gives $\boldsymbol{\nabla} \cdot (1-u)\boldsymbol{\nabla}\Delta^{-1}F(t) \in H^{-1}(G)$ for all $0 \leq t \leq T$ and u in $L^2(G)$.

3.1. The elliptic resolvent operator. First we consider the integral operator $\mathcal{Q}$ with integrand Q defined respectively by

$$\mathcal{Q}(v)(t) = \int_0^t Q(v)(s)ds,$$
$$Q(v)(s) = J\left(v(s), \tfrac{1}{\eta}(\alpha(v(s)) + P(s)) + \boldsymbol{\nabla} \cdot (1 - v(s)\boldsymbol{\nabla}\Delta^{-1}F(s)\right)$$

for v in $L^2((0,T); L^2(G))$.

PROPOSITION 8. *The map* $Q : L^2((0,T); L^2(G)) \to L^2((0,T); H_0^1(G))$ *is bounded and continuous.*

Proof. Recall that the norm of the linear operator $J(v, \cdot) : H^{-1}(G) \to H_0^1(G)$ is uniformly bounded by a constant C'. From the definition of J, we have

$$\begin{aligned}
\|Q(u)(t)\|_{H_0^1} &\leq \left\|J(u(t), \tfrac{1}{\eta}\left(\alpha(u(t)) + P(t)\right) + \nabla \cdot (1 - u(t))\nabla\Delta^{-1}F(t))\right\|_{H_0^1} \\
&\leq C'\left[\tfrac{1}{\eta}\left\|\alpha(u(t)) + P(t)\right\|_{L^2} + \left\|\nabla \cdot (1-u(t))\nabla\Delta^{-1}F(t)\right\|_{H^{-1}}\right] \\
&\leq C'\left[\tfrac{1}{\eta}\left(\|P(t)\|_{L^2} + K_\alpha\left(\|u(t)\|_{L^2} + m(G)^{1/2}\right)\right) + C(t)\left\|1 - u(t)\right\|_{L^2}\right].
\end{aligned}$$

That is, $\|Q(u)(t)\|_{H_0^1}$ is dominated by a square-integrable function.

Further, we have that if $u_n(t)$ converges to $u(t)$ for each t, then $Q(u_n)(t)$ converges to $Q(u)(t)$ for each t. This follows as a consequence of the continuity of $v \mapsto \alpha(v) : L^2(G) \to L^2(G)$, $\boldsymbol{\nabla}\cdot : (L^2(G))^N \to H^{-1}(G)$ and the joint continuity of J. By applying dominated convergence in $L^2((0,T); H^1(G))$, we have that $Q(u_n)$ converges to $Q(u)$. $\square$

COROLLARY 9. *The integral operator* $\mathcal{Q} : L^2((0,T); L^2(G)) \to H^1((0,T); H_0^1(G))$ *is bounded and continuous.*

This is immediate from $\mathcal{Q}(v)(t) = \int_0^t Q(v)(s)ds$.

3.2. The pointwise integral equation. We begin the second step by considering the remaining part of the integral equation (9), i.e., we replace $\mathcal{Q}(u)$ with a prescribed v. This leads to the integral equation

$$u(t) + \tfrac{1}{\eta}\int_0^t (\alpha(u(s)) + P(s))\, ds = u_0 + v(t), \quad 0 \le t \le T, \tag{11}$$

where u_0 is in $L^2(G)$, v is in $H^1((0,T); H_0^1(G))$, and P satisfies assumption (A4). Then $u = W(v)$ defines the function

$$W : H^1((0,T); L^2(G)) \to H^1((0,T); L^2(G)).$$

Since $\alpha + kI$ is continuous and monotone and $v \in H^1((0,T); L^2(G))$, the integral equation (11) is equivalent to the initial-value problem

$$u'(t) + \tfrac{1}{\eta}(\alpha(u(t)) + P(t)) = v'(t), \quad u(0) = u_0 + v(0), \tag{12}$$

for which there exists a unique solution $u \in H^1((0,T); L^2(G))$. This follows much more generally by Brezis' theorem for evolution equations with operators which are subgradients of convex functions. (See Theorem III.3.6 and Proposition 3.12 in [14] or Theorem IV.4.3 in [48].)

The first properties of W will follow from elementary *Grönwall inequalities.*

LEMMA 10. *Suppose that* $g(t)$ *satisfies* $g(t) \le \int_0^t a(s)g(s)ds + h(t)$ *with* $a(t)$ *positive. Then* $g(t)$ *also satisfies the estimate*

$$g(t) \le h(t) + \int_0^t h(s)a(s)e^{\int_s^t a(\tau)d\tau} ds$$

for all t. *Additionally, if* $a(t)$ *and* $h(t)$ *are bounded, then*

$$g(t) \le \|h\|_{L^\infty(0,t)} \exp\left(t\, \|a\|_{L^\infty(0,t)}\right).$$

COROLLARY 11. *The map* W *satisfies the bounds*

$$\|W(v)(t)\|_{L^2} \le e^{K_\alpha t/\eta}\left(\|v\|_{C([0,T];L^2)} + H(t)\right),$$

where $H(t) = \frac{1}{\eta}K_\alpha m(G)^{1/2}t + \|u_0\|_{L^2} + \frac{1}{\eta}\int_0^t \|P(s)\|_{L^2} ds$. *Consequently,* W *is a bounded map on* $C([0,T]; L^2(G))$.

Proof. Repeated use of the triangle inequality together with assumption (A2) gives

$$\|W(v)(t)\|_{L^2} \le \tfrac{K_\alpha}{\eta}\int_0^t \|W(v)(s)\|_{L^2} ds$$
$$+ \|v(t)\|_{L^2} + \tfrac{1}{\eta}K_\alpha m(G)^{1/2}t + \|u_0\|_{L^2} + \tfrac{1}{\eta}\int_0^t \|P(s)\|_{L^2} ds.$$

The result follows from Lemma 10. ∎

The following is a special case of Lemma IV.4.1 in [48].

LEMMA 12. *Let $k \in \mathbb{R}$, $b(\cdot) \in L^1(0,T)$ with $b(t) \geq 0$, and let the absolutely continuous $w : [0,T] \to \mathbb{R}$ satisfy*

$$\tfrac{1}{2} w'(t) \leq k\, w(t) + b(t)\, w^{\frac{1}{2}}(t), \quad t \in [0,T].$$

Then

$$w^{\frac{1}{2}}(t) \leq w^{\frac{1}{2}}(0) e^{kt} + \int_0^t e^{k(t-s)} b(s)\, ds, \quad t \in [0,T].$$

We obtain fundamental local estimates for (12).

THEOREM 13. *For v_1, v_2 in $H^1((0,T);L^2(G))$ and P_1, P_2 in $L^2((0,T);L^2(G))$, let u_1, u_2 in $H^1((0,T);L^2(G))$ be corresponding solutions of* (12). *Then for each measurable $G_0 \subset G$ we have*

$$\begin{aligned}(13)\quad &\|u_1(t) - u_2(t)\|_{L^2(G_0)} \leq e^{\frac{k}{\eta} t} \|u_1(0) - u_2(0)\|_{L^2(G_0)} \\ &+ \int_0^t e^{\frac{k}{\eta}(t-s)} (\|v_1'(s) - v_2'(s)\|_{L^2(G_0)} + \frac{1}{\eta} \|P_1(s) - P_2(s)\|_{L^2(G_0)})\, ds, \quad 0 \leq t \leq T.\end{aligned}$$

Proof. Taking the difference of the equations (12) gives

$$u_1'(t) - u_2'(t) + \tfrac{1}{\eta} \left(\alpha(u_1(t)) - \alpha(u_2(t))\right) = \tfrac{1}{\eta}(P_2(t) - P_1(t)) + v_1'(t) - v_2'(t)$$

pointwise a.e. in G. Multiplying the difference above by $u_1(t) - u_2(t)$ and using (10b) lead to

$$\begin{aligned}(u_1'(t) - u_2'(t))(u_1(t) - u_2(t)) &\leq \tfrac{k}{\eta}(u_1(t) - u_2(t))^2 \\ &+ \left(\tfrac{1}{\eta}(P_2(t) - P_1(t)) + v_1'(t) - v_2'(t), u_1(t) - u_2(t)\right),\end{aligned}$$

and we integrate this over G_0 and apply the Cauchy–Schwarz inequality to get

$$\begin{aligned}\tfrac{1}{2} \frac{d}{dt} \|u_1(t) - u_2(t)\|^2_{L^2(G_0)} &\leq \tfrac{k}{\eta} \|u_1(t) - u_2(t)\|^2_{L^2(G_0)} \\ &+ \left\| \tfrac{1}{\eta}(P_2(t) - P_1(t)) + v_1'(t) - v_2'(t) \right\|_{L^2(G_0)} \|u_1(t) - u_2(t)\|_{L^2(G_0)}.\end{aligned}$$

The estimate (13) follows from Lemma 12. ∎

The following theorem lists consequences of Theorem 13 on the map W, which uses the case where $P_1(t) = P_2(t) = P(t)$ and $G_0 = G$.

THEOREM 14. *The map W satisfies the following:*
- *W is a bounded map on $H^1((0,T);L^2(G))$,*
- *$W : H^1((0,T);L^2(G)) \to C([0,T];L^2(G))$ is continuous, and*
- *for v_1, v_2 in $H^1((0,T);L^2(G))$, we have*

$$\begin{aligned}(14)\quad \|W(v_1)(t) - W(v_2)(t)\|_{L^2} &\leq \|v_1(0) - v_2(0)\|_{L^2}\, e^{\frac{k}{\eta} t} \\ &+ \int_0^t e^{\frac{k}{\eta}(t-s)} \|v_1'(s) - v_2'(s)\|_{L^2}\, ds, \quad 0 \leq t \leq T.\end{aligned}$$

Proof. This follows much more generally for evolution equations with maximal monotone operators as indicated above. For the special case considered here, we give the estimates independently.

That W is bounded from $C([0,T];L^2(G))$ to itself follows from Corollary 11, and so W is bounded from $H^1((0,T);L^2(G))$ to $L^2((0,T);L^2(G))$. Since $u = W(v)$ satisfies (12), we have

$$\|u'(t)\|_{L^2} \leq \tfrac{1}{\eta}\left(K_\alpha(\|u(t)\|_{L^2} + m(G)^{1/2}) + \|P(t)\|_{L^2}\right) + \|v'(t)\|_{L^2(G)}\,.$$

Equation (14) follows from Lemma 13. Continuity follows from the estimate (14). ◻

We shall establish conditions that prove that W is a compact map when the initial data belongs to $H^1(G)$. To this end, for each $\delta > 0$ define $G_\delta \equiv \{x \in G : \operatorname{dist}(x, \partial G) > \delta\}$. Let v be a function in $L^2(G)$, and for each $h \in \mathbb{R}$ with $|h| < \delta$ define a *translate* of v by

$$(\tau_h v)(x_1, x_2, \ldots, x_n) \equiv v(x_1 + h, x_2, \ldots, x_n),\ x \in G_\delta, \tag{15a}$$

and the corresponding *difference quotient* by

$$\nabla_h v \equiv \tfrac{1}{h}(\tau_h v - v) \tag{15b}$$

if $h \neq 0$. The following two results are well known and are used to establish compactness of the operator.

LEMMA 15. *For $\delta > 0$, we have for all $v \in C^1(\bar{G})$*

$$\|\nabla_h v\|_{L^2(G_\delta)} \leq \|\partial_1 v\|_{L^2(G)}\,,\ 0 < |h| < \delta.$$

COROLLARY 16. *For each v in $H^1(G)$, $\lim_{h\to 0}(\nabla_h v) = \partial_1 v$ in $L^2(G_\delta)$ for every $\delta > 0$.*

LEMMA 17. *Let $u_0 \in H^1(G)$, $P \in L^2((0,T);H^1(G))$, and $v \in H^1((0,T);H_0^1(G))$. Then $u = W(v) \in H^1((0,T);L^2(G))$ with $\partial_1 u \in C([0,T];L^2(G)$ and satisfies the estimate*

$$\begin{aligned}\|\partial_1 u(t)\|_{L^2(G)} &\leq \|u(0)\|_{H^1(G)}\, e^{\frac{k}{2}t} \\ &\quad + \int_0^t e^{\frac{k}{2}(t-s)} \left(\|v'(s)\|_{H^1(G)} + \frac{1}{\eta}\|P(s)\|_{H^1(G)}\right) ds, \quad 0 \leq t \leq T.\end{aligned} \tag{16}$$

Proof. Apply Theorem 13 to $\tau_h u(x,t)$ and $u(x,t)$ and divide by $h > 0$ to obtain

$$\begin{aligned}\|\nabla_h u(t)\|_{L^2(G_\delta)} &\leq \|\nabla_h u(0)\|_{L^2(G_\delta)}\, e^{\frac{k}{2}t} \\ &\quad + \int_0^t e^{\frac{k}{2}(t-s)}\left(\|\nabla_h v'(s)\|_{L^2(G_\delta)} + \frac{1}{\eta}\|\nabla_h P(s)\|_{L^2(G_\delta)}\right) ds \\ &\leq \|u(0)\|_{H^1(G)}\, e^{\frac{k}{2}t} \\ &\quad + \int_0^t e^{\frac{k}{2}(t-s)}\left(\|v'(s)\|_{H^1(G)} + \frac{1}{\eta}\|P(s)\|_{H^1(G)}\right) ds.\end{aligned}$$

Hence, for each $t > 0$, $\{\nabla_h u(\cdot,t) : |h| < \delta\}$ is bounded in the Hilbert space $L^2(G_\delta)$. There is a sequence $h_n \to 0$ for which $\nabla_{h_n} u(t)$ converges weakly in $L^2(G_\delta)$. But

$\nabla_{h_n} u(t)$ converges weakly to $\partial_1 u(t)$, so the uniqueness of weak limits implies that $\partial_1 u(t) \in L^2(G_\delta)$ and the original sequence converges weakly to $\partial_1 u(t)$. Take the lim inf as h goes to 0 to get $\|\partial_1 u(t)\|_{L^2(G_\delta)}$ bounded by the right side of (16). Then let $\delta \to 0$. ∎

THEOREM 18. *Suppose that u_0 is in $H^1(G)$ and $P \in L^2((0,T); H^1(G))$. Then $W : H^1((0,T); H_0^1(G)) \to C([0,T]; H^1(G))$ satisfies the estimate*

$$\|\boldsymbol{\nabla} W(v)(t)\|_{L^2(G)} \leq K \|u_0\|_{H^1(G)} e^{\frac{k}{2}t} + K \int_0^t e^{\frac{k}{2}(t-s)} \left(\|v'(s)\|_{H^1(G)} \, ds + \frac{1}{\eta} \|P(s)\|_{H^1(G)} \right) ds, \tag{17}$$

so it is a bounded map.

Proof. Lemma 17 holds with corresponding estimates for $\partial_2, \ldots, \partial_N$, and so we obtain the gradient estimate (17) for $u = W(v)$. ∎

COROLLARY 19. *If $u_0 \in H^1(G)$, then the map*

$$W : H^1((0,T); H_0^1(G)) \to C([0,T]; L^2(G))$$

is compact.

Proof. This follows from Theorem 18, Theorem 14, and the Aubin–Lions–Simon lemma [50]. ∎

4. Existence of a solution. We will demonstrate that the integral equation (9) has a solution by showing the existence of a fixed point for the map $u \mapsto W(\mathcal{Q}(u))$ in $L^2((0,T); L^2(G))$ if the ellipticity coefficient $\kappa(\cdot)$ is bounded as in assumption (A1). Then we establish estimates on the solution that show this coefficient satisfies those bounds in the case of (6) when $F(\cdot) = 0$.

4.1. The fixed point. We use the Schaefer fixed point theorem to establish the existence of a fixed point (see [32, p. 242]).

THEOREM 20 (Schaefer fixed point). *Suppose that X is a Banach space, and $T : X \to X$ is a continuous compact map. If the set*

$$\{x \in X : x = \lambda T(x) \text{ for some } \lambda \in [0,1]\}$$

is bounded, then T has a fixed point.

PROPOSITION 21. *Suppose that u_0 is in $H^1(G)$. Then $W \circ \mathcal{Q} : L^2((0,T); L^2(G)) \to L^2((0,T); L^2(G))$ is compact and continuous.*

Proof. By Corollary 9, $\mathcal{Q} : L^2((0,T); L^2(G)) \to H^1((0,T); H_0^1(G))$ is continuous and bounded. By Corollary 19, we have $W : H^1((0,T); H_0^1(G)) \to L^2((0,T); L^2(G))$ is continuous and compact. It follows that the composition is then continuous and compact. ∎

PROPOSITION 22. *For $u \in L^2((0,T); L^2(G))$, suppose $u = \lambda W(\mathcal{Q}(u))$ for $\lambda \geq 0$. Then u satisfies the estimate*

$$\|u(t)\|_{L^2} \leq h_\lambda(t) e^{\int_0^t a_\lambda(s) ds},$$

where

$$h_\lambda(t) = \lambda \|u_0\|_{L^2} + \tfrac{2\lambda K_\alpha}{\eta} m(G)^{1/2} t + \lambda \int_0^t \left(\tfrac{2}{\eta} \|P(s)\|_{L^2} + K m(G)^{1/2} C(s) \right) ds,$$

$$a_\lambda = \tfrac{(1+\lambda)K_\alpha}{\eta} + KC(s).$$

Consequently, the set $\{u : u = \lambda W(\mathcal{Q}(u))$ *for some* $0 \le \lambda \le 1\}$ *is a bounded subset of* $C([0,T]; L^2(G))$.

Proof. For $\lambda = 0$, there is nothing to prove. Otherwise, we have

$$\begin{aligned} &\tfrac{1}{\lambda} u(t) + \tfrac{1}{\eta} \int_0^t \left[\alpha(\tfrac{1}{\lambda} u(s)) + P(s) \right] ds \\ &\quad = u_0 + \int_0^t J\left(u(s), \tfrac{1}{\eta}(\alpha(u(s)) + P(s)) + \nabla \cdot (1 - u(s)) \nabla \Delta^{-1} F(s) \right) ds. \end{aligned}$$

Taking the norm of both sides gives

$$\begin{aligned} \|u(t)\|_{L^2} \le \lambda \|u_0\|_{L^2} &\\ &+ \tfrac{\lambda}{\eta} \int_0^t \left\| \alpha\left(\tfrac{1}{\lambda} u(s)\right) + P(s) \right\|_{L^2} ds \\ &+ \lambda \int_0^t \left\| J\left(u(s), \tfrac{1}{\eta} \left(\alpha(u(s)) + P(s)\right) + \boldsymbol{\nabla} \cdot (1 - u(s)) \boldsymbol{\nabla} \Delta^{-1} F(s) \right) \right\|_{L^2} ds. \end{aligned}$$

By assumption (A2) and Lemma 1, we get

$$\begin{aligned} \|u(t)\|_{L^2} \le \lambda \|u_0\|_{L^2} + \tfrac{\lambda}{\eta} \int_0^t K_\alpha \left(\tfrac{1}{\lambda} \|u(s)\|_{L^2} + m(G)^{1/2} \right) ds + \tfrac{\lambda}{\eta} \int_0^t \|P(s)\| \, ds \\ + \lambda \int_0^t (\tfrac{1}{\eta} \left(K_\alpha \left(\|u(s)\|_{L^2} + m(G)^{1/2} \right) + \|P(s)\|_{L^2} \right) \\ + K \left\| \boldsymbol{\nabla} \cdot (1 - u(s)) \boldsymbol{\nabla} \Delta^{-1} F(s) \right\|_{H^{-1}}) ds. \end{aligned}$$

Here, $K = \|J(u, \cdot)\|_{\mathcal{L}(H^{-1}, L^2)}$. After simplifying, this becomes

$$\begin{aligned} \|u(t)\|_{L^2} \le \lambda \|u_0\|_{L^2} + \tfrac{2\lambda K_\alpha}{\eta} m(G)^{1/2} t + \lambda \int_0^t \left(\tfrac{2}{\eta} \|P(s)\|_{L^2} + K m(G)^{1/2} C(s) \right) ds \\ + \int_0^t \left(\tfrac{(1+\lambda)K_\alpha}{\eta} + KC(s) \right) \|u(s)\|_{L^2} ds. \end{aligned}$$

Applying Grönwall's inequality finishes the proof. ∎

As seen in the proof, $u = W(\mathcal{Q}(u))$ is the form solutions must take.

THEOREM 23. *Suppose that* u_0 *is in* $L^2(G)$. *Then there exists a solution to* (9) *in* $H^1((0,T); L^2(G))$. *It is given by a fixed point* $u = W(\mathcal{Q}(u))$ *with* $\mathcal{Q}(u) \in H^1((0,T), H_0^1(G))$ *and* W *defined by* (12)*, and it satisfies*

$$\eta u' + \alpha(u) + P \in L^2((0,T); H_0^1(G)).$$

Proof. Assume that $u_0 \in H^1(G)$. By applying the Schaeffer fixed point theorem in $L^2((0,T); L^2(G))$, we see there exists a $u \in L^2((0,T); L^2(G))$ satisfying (9). But then $v = \mathcal{Q}(u)$ and $u = W(v)$ are necessarily in $H^1((0,T); L^2(G))$.

For $u_0 \in L^2(G)$, let $u_0^n \in H^1(G)$ be a sequence with $\|u_0^n - u_0\|_{L^2(G)} \to 0$ with u_n the corresponding solutions to (9) with initial data u_0^n. Then the previous proposition guarantees that $\{u_n\}$ is a bounded subset of $C([0,T];L^2(G))$. This implies that $\{\mathcal{Q}(u_n)\}$ is a bounded subset of $H^1((0,T);H_0^1(G))$ and is thus a compact subset of $C([0,T];L^2(G))$. Furthermore, we also have that $W(\mathcal{Q}(u_n))$ is bounded in $H^1((0,T);L^2(G))$. Together, this lets us take a subsequence $\left\{\mathcal{Q}(u_{n_j})\right\}$ and a function $v \in H^1((0,T);L^2(G))$ such that $\left\|v - \mathcal{Q}(u_{n_j})\right\|_{C([0,T];L^2(G))} \to 0$ and $\mathcal{Q}(u_{n_j}) \rightharpoonup v$ in $H^1((0,T);L^2(G))$. We now let $w = W(v) \in H^1((0,T);L^2(G))$ with initial condition $w(0) = u_0$. We will show that $u_{n_j}(t) \to w(t)$ in $L^2(G)$ and $v = \mathcal{Q}(w)$. To simplify notation, we rename our subsequence to u_j.

For the former, we multiply $\|w(t) - u_j(t)\|_{L^2}^2$ by the scaling factor $e^{-2kt/\eta}$ and differentiate. We then have, by assumption (A2) and the definition of w and u_j, that

$$\begin{aligned}
&\frac{d}{dt} e^{-2kt/\eta} \|w(t) - u_j(t)\|_{L^2}^2 \\
&\qquad = 2e^{-2kt/\eta} \left(w'(t) - u_j'(t) - \tfrac{k}{\eta}(w(t) - u_j(t)), w(t) - u_j(t)\right) \\
&\qquad \le 2e^{-2kt/\eta} \left(w'(t) - u_j'(t) + \tfrac{1}{\eta}(\alpha(w(t)) - \alpha(u_j(t))), w(t) - u_j(t)\right) \\
&\qquad = 2e^{-2kt/\eta} \left(v'(t) - Q(u_j)(t), w(t) - u_j(t)\right).
\end{aligned}$$

Integration then gives

$$\begin{aligned}
\|w(t) - u_j(t)\|_{L^2}^2 &\le \|u_0 - u_0^n\|_{L^2}^2 \\
&\quad + 2\int_0^t \left(v'(s) - Q(u_j)(s), e^{2k(t-s)/\eta}(w(s) - u_j(s))\right) ds.
\end{aligned}$$

If we apply integration by parts to this last term, we have

$$\begin{aligned}
&\int_0^t \left(v'(s) - Q(u_j)(s), e^{2k(t-s)/\eta}(w(s) - u_j(s))\right) ds \\
&\quad = \left[\left(v(s) - \mathcal{Q}(u_j)(s), e^{2k(t-s)/\eta}(w(s) - u_j(s)\right)\right]_{s=0}^{s=t} \\
&\quad\quad - \int_0^t \left(v(s) - \mathcal{Q}(u_j)(s), e^{2k(t-s)/\eta}((w'(s) - u_j'(s)) - \tfrac{2k}{\eta}(w(s) - u_j(s)))\right) ds.
\end{aligned}$$

We have that $\mathcal{Q}(u_j)$ converges to v uniformly and that the terms on the right of the inner product are bounded in $L^2((0,T);L^2(G))$. We can thus conclude the limit of the integral on the right is 0, and so $\lim_{j\to\infty} \|w(t) - u_j(t)\|_{L^2} = 0$ uniformly. It then follows that $\mathcal{Q}(u_j) \to \mathcal{Q}(w)$.

Uniqueness of limits then gives $\mathcal{Q}(w(t)) = v$, and so $w = W(v) = W(\mathcal{Q}(w))$, which is the desired result. □

4.2. The estimates. In this subsection, we show that when the overburden pressure P is independent of space and there are no fluid sources, the solution $u(t)$ will remain strictly within the unit interval, so assumption (A1) holds. In particular, we show that if the initial porosity u_0 is bounded between $\underline{k}$ and $\bar{k}$, then so is the solution $u(t)$.

THEOREM 24. *Assume that α is monotone (nondecreasing) with $\alpha(0) = 0$. Also assume that the overburden pressure, P, is independent of x, and that $F \equiv 0$. If both $u_0(x) \le \bar{k}$ for all x and $-P(t) \le \alpha(\bar{k})$, then $u(t,x) \le \bar{k}$ for all $x \in G$, $t \ge 0$. If $\underline{k} \le u_0(x)$ for all x and $\alpha(\underline{k}) \le -P(t)$, then we have $\underline{k} \le u(t,x)$ for all $x \in G$, $t \ge 0$.*

Proof. The previous section establishes that there exists a u in $H^1((0,T); H_0^1(G))$ such that

$$u'(t) + A_{u(t)}(\eta u'(t) + \alpha(u(t)) + P(t)) = 0.$$

Let $A(t) \equiv A_{u(t)}$ and $\sigma(t) = \eta u'(t) + \alpha(u(t))$. Rearranging this formula gives

$$u'(t) + \tfrac{1}{\eta}(\alpha(u(t)) - \sigma(t)) = 0,$$

which can be substituted into our solution to give

$$A(t)(\sigma(t) + P(t)) - \tfrac{1}{\eta}(\alpha(u(t)) - \sigma(t)) = 0.$$

Multiplying these latest two equations by $\text{sgn}^+(\alpha(u(t)) - \alpha(\bar{k})$ and $\text{sgn}^+(\sigma(t) - \alpha(\bar{k}))$, respectively, adding and integrating give

$$\begin{aligned}
&\frac{d}{dt}\int_G (u(t) - \bar{k})^+ dx \\
&\quad + \tfrac{1}{\eta}\int_G (\alpha(u(t)) - \sigma(t))(\text{sgn}^+(\alpha(u(t)) - \alpha(\bar{k})) - \text{sgn}^+(\sigma(t) - \alpha(\bar{k})))dx \\
&\quad + \tfrac{1}{\eta}\int_G A(t)(\sigma(t) + P(t))\,\text{sgn}^+(\sigma(t) - \alpha(\bar{k}))dx = 0.
\end{aligned}$$

The monotonicity of sgn^+ and $\alpha(\cdot)$ makes the integrand of the second integral nonnegative. For the third integral, we consider replacing sgn^+ by its Yosida approximation, sgn_ε^+. We then have that $\sigma(t,x) = -P(t) \le \alpha(\bar{k})$ on the boundary as $\sigma(t) + P(t)$ is in $H_0^1(G)$. This gives us that $\text{sgn}_\varepsilon^+(\sigma(t) - \alpha(\bar{k}))$ vanishes on the boundary. As $\sigma + P$ is in $H_0^1(G)$ and P is constant with respect to x, we have σ is in $H^1(G)$. Furthermore, as sgn_ε^+ is Lipschitz, we can conclude that our composition is sufficiently smooth to be in $H^1(G)$, and so vanishing on the boundary gives that it is in $H_0^1(G)$. We then have

$$\begin{aligned}
&\int_G A(t)(\sigma(t) + P(t))\,\text{sgn}_\varepsilon^+(\sigma(t) - \alpha(\bar{k}))dx \\
&\quad = \int_G \kappa(u(t))\boldsymbol{\nabla}(\sigma(t) + P(t)) \cdot \boldsymbol{\nabla}\,\text{sgn}_\varepsilon^+(\sigma(t) - \alpha(\bar{k}))dx \\
&\quad = \int_G \kappa(u(t))(\text{sgn}_\varepsilon^+)' \left|\boldsymbol{\nabla}\sigma(t)\right|^2 dx \ge 0.
\end{aligned}$$

By taking the limit as ε decreases to 0, we see that the third integral is nonnegative. The only term left is the first integral. When integrated with respect to t, we have

$$\int_G (u(t) - \bar{k})^+ dx \le \int_G (u_0(x) - \bar{k})^+ dx = 0.$$

It follows with $(u(t) - \bar{k})^+ \ge 0$ that it must vanish, so we can conclude that $u(t) \le \bar{k}$, completing the result for this inequality.

For the other inequality, replace sgn^+ in the previous with

$$\text{sgn}^-(\xi) = \begin{cases} 0, & \xi > 0, \\ -1, & \xi \le 0. \end{cases}$$

The change in direction of the inequality follows from the change the sign changes introduced by sgn^-. ∎

This tells us that if there exist constants $\underline{k}$, $\bar{k}$ such that $0 < \underline{k} \leq u_0(x) \leq \bar{k} < 1$, then we have $0 < \underline{k} \leq u(t,x) \leq \bar{k} < 1$ for all $x \in G$, $t \geq 0$. In particular, an initial condition that represents a porosity will lead to a solution that represents a porosity.

The importance of the assumptions on fluid source term and pressure are physically meaningful as well, as an unconstrained pressure or fluid could force our model to no longer be applicable.

5. Regularity. In this section, we will show that assumption (A2) gives us information on how discontinuities of the initial condition propogate.

5.1. Local regularity. Let $u_0 \in L^2(G)$ and $u = W(v) \in H^1((0,T); L^2(G))$, $v = \mathcal{Q}(u) \in H^1((0,T); H_0^1(G))$ be the corresponding solution obtained above as a fixed point. Theorem 18 implies that if $u_0 \in H^1(G)$ and $P \in L^2(0,T; H^1(G)$, then $u \in C([0,T]; H^1(G))$. Moreover, *local* versions of this result follow similarly from Theorem 13.

Let G_0 be an open subset of G. We define

$$H_j^1(G_0) = \{v \in L^2(G_0) : \partial_j v \in L^2(G_0)\}, \quad j = 1, 2, \ldots, N.$$

Let $u_0 \in L^2(G)$. For each $\delta > 0$ denote as before $G_{0,\delta} = \{x \in G_0 : \mathrm{dist}(x, \partial G_0) > \delta\}$, the subset of points bounded away from ∂G_0. If additionally $u_0 \in H_j^1(G_0)$, we follow the proof of Lemma 17 and apply (13) to the appropriate difference quotient on the set $G_{0,\delta}$ as before to get

$$\begin{aligned} (18) \quad \|\partial_j u(t)\|_{L^2(G_{0,\delta})} &\leq \|\partial_j u(0)\|_{L^2(G_0)} e^{\frac{k}{2}t} \\ &+ \int_0^t e^{\frac{k}{2}(t-s)} \left(\|\partial_j v'(s)\|_{L^2(G_0)} + \frac{1}{\eta} \|\partial_j P(s)\|_{L^2(G_0)} \right) ds, \quad 0 \leq t \leq T. \end{aligned}$$

Then we let $\delta \to 0$ to get the corresponding bound on $\|\partial_j u(t)\|_{L^2(G_0)}$ for $j = 1, 2, \ldots, N$. This is summarized in the following.

THEOREM 25. *Let G_0 be an open domain contained in G. Let $u_0 \in L^2(G)$ and $u = W(v) \in H^1((0,T); L^2(G))$ with $v = \mathcal{Q}(u) \in H^1((0,T); H_0^1(G))$ be the corresponding solution obtained in Theorem* 23*. For each $j = 1, 2, \ldots, N$, if $u_0 \in H_j^1(G_0)$ and $P \in L^2(0,T; H_j^1(G_0))$, then $u = W(v) \in C([0,T]; H_j^1(G_0))$. Likewise, if $u_0 \in H^1(G_0)$ and $P \in L^2(0,T; H^1(G_0))$, then $u \in C([0,T]; H^1(G_0))$.*

These results show that regularity in individual directions is preserved up to $H^1(G)$, locally as well as overall in G. In the next section we show that discontinuities along interfaces are preserved, so such interfaces are necessarily autonomous.

5.2. Perseverence of interfaces. Assume that G is the interior of the closure of two disjoint bounded domains G_1, G_2 in $\mathbb{R}^N$ with corresponding piecewise C^1 boundaries $\partial G_1, \partial G_2$ which intersect in a C^1 manifold of dimension $N-1$. By a change of variable, we may assume that $\partial G_1 \cap \partial G_2$ is *flat*, that is, $G_1 = \{x \in G : \ x_1 < 0\}$ and $G_2 = \{x \in G : \ x_1 > 0\}$. On the common boundary, $\partial G_1 \cap \partial G_2$, the respective unit outward normals are $\mathbf{n}_1 = -\mathbf{n}_2 = (1, 0, \ldots, 0)$.

Denote the corresponding continuous trace maps by $\gamma^i : H^1(G_i) \to L^2(\partial G_i)$, $i = 1, 2$. We can identify each $v \in H^1(G)$ as an element of $H^1(G_1) \oplus H^1(G_2)$ via restrictions, $v = [v_1, v_2] \in H^1(G)$, $v_i = v|_{G_i}$ with $\gamma^1(v_1) = \gamma^2(v_2)$ on $\partial G_1 \cap \partial G_2$. Conversely, $[v_1, v_2] \in H^1(G_1) \oplus H^1(G_2)$ belongs to $H^1(G)$ if $\gamma^1(v_1) = \gamma^2(v_2)$ on $\partial G_1 \cap \partial G_2$.

Let Γ be a relatively open domain in the flat interface $\partial G_1 \cap \partial G_2$ such that $\bar{\Gamma} \cap \partial G = \varnothing$, and define the space

$$H^1_\Gamma(G) \equiv \{v \in H^1(G_1) \oplus H^1(G_2) : \gamma^1(v_1) = \gamma^2(v_2) \text{ on } \partial G_1 \cap \partial G_2 \sim \Gamma\}.$$

Then we have the *saltus* or jump on Γ of $v \in H^1_\Gamma(G)$ given by the trace difference $s_\Gamma(v) = (\gamma^1(v_1) - \gamma^2(v_2))|_\Gamma$. By construction $s_\Gamma : H^1_\Gamma \to L^2(\Gamma)$ is a continuous linear map, and its kernel is $H^1(G)$. Finally, we note that each trace map γ^i agrees with the trace map $\gamma : H^1(G) \to L^2(\partial G)$ on $\partial G \cap \partial G_i$, i.e., $\gamma(v) = \gamma^i(v_i)$ on this intersection, for $i = 1, 2$ and $v \in H^1(G)$.

For $v \in H^1(G_i)$, we define the translate and difference on $L^2(G_{i,\delta})$ by (15), but only for $h < 0$ if $i = 1$ and only for $h > 0$ if $i = 2$. These correspond to the forward translate and backward difference operators on G_1. Likewise, they are the backward translate and forward difference operators on G_2. The proof of Lemma 15 yields the following.

LEMMA 26. *For $i = 1, 2$ if $v \in L^2(G_i)$ and $\partial_1 v \in L^2(G_i)$, then*

$$\|\nabla_h v\|_{L^2(G_{i,\delta})} \le \|\partial_1 v\|_{L^2(G_i)}$$

for those h of appropriate sign and $0 < |h| < \delta$. For each v in $H^1(G_i)$, the corresponding one-sided limit satisfies $\lim_{h\to 0}(\nabla_h v) = \partial_1 v$ in $L^2(G_{i,\delta})$ for each $\delta > 0$.

THEOREM 27. *Assume that G is constructed from the two disjoint bounded domains G_1, G_2 as above. Let $u_0 \in L^2(G)$ and $u = W(v) \in H^1((0,T); L^2(G))$ with $v = \mathcal{Q}(u) \in H^1((0,T); H^1_0(G))$ be the corresponding solution obtained in Theorem* 23. *If $u_0 \in H^1_\Gamma(G)$ and $P \in L^2(0,T; H^1(G))$, then $u = W(v) \in C([0,T]; H^1_\Gamma(G))$. The boundary trace satisfies the initial-value problem*

$$\eta(\gamma u)'(t) + \alpha((\gamma u)(t)) + \gamma P(t) = 0,\ (\gamma u)(0) = \gamma u_0 \text{ in } L^2(\partial G), \tag{19a}$$

and the saltus satisfies

$$(s_\Gamma u)'(t) + \tfrac{1}{\eta} s_\Gamma(\alpha(u(t))) = 0 \text{ in } L^2(\Gamma) \tag{19b}$$

for $0 \le t \le T$.

Proof. Let $u = W(\mathcal{Q}(u))$ with u_0 in $H^1_\Gamma(G)$. Then as previously argued, we have u is in $C([0,T]; H^1_\Gamma(G))$. The continuity of α gives $\gamma^i \alpha(u(t)) = \alpha(\gamma^i u(t))$, and so

$$(\gamma^i u)'(t) + \tfrac{1}{\eta}(\alpha(\gamma^i u(t)) + \gamma^i P(t)) = \gamma^i \mathcal{Q}(u)(t), \quad i = 1, 2.$$

These give (19a) on ∂G, and by taking the difference of the equations for $i = 1$ and $i = 2$ on Γ we obtain (19b) since $\mathcal{Q}(u)(t) \in H^1(G)$. ∎

The solutions of the pointwise equations (19) can be easily estimated.

COROLLARY 28. *The saltus satisfies the estimate*

$$|s_\Gamma u(t)| \le e^{\frac{k}{\eta} t} |s_\Gamma u_0| .$$

If additionally for fixed $K \in \mathbb{R}$, α satisfies

$$(\alpha(\xi_1) - \alpha(\xi_2))(\xi_1 - \xi_2) \le K |\xi_1 - \xi_2|^2, \quad \xi_1, \xi_2 \in \mathbb{R}, \tag{20}$$

then

$$|s_\Gamma u(t)| \ge e^{-\frac{K}{\eta} t} |s_\Gamma u_0| .$$

Proof. By our assumption (10b) on α, we have

$$(s_\Gamma \alpha(u(t)))(s_\Gamma u(t)) \geq -k\, s_\Gamma u(t)^2,$$

so multiplying (19b) by $(s_\Gamma u(t))$ gives

$$\frac{1}{2}\frac{d}{dt}(s_\Gamma u(t))^2 = -\frac{1}{\eta}(s_\Gamma \alpha(u(t)))(s_\gamma u(t)) \leq \frac{k}{\eta} s_\Gamma u(t)^2.$$

This gives the first estimate. The second estimate follows similarly from (20). □

From (19a) similar estimates hold as well for the trace, $\gamma u(t)$. Note that if α is Lipschitz continuous, then (10b) and (20) follow and both of the estimates hold. In this case, a discontinuity in the initial porosity is maintained and may grow or decay at most exponentially. If α is strongly monotone ($k < 0$), then any discontinuity introduced in the initial porosity decays exponentially. Finally, we note that if α is monotone, then (20) is equivalent to α being Lipschitz.

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