# A NONLINEAR PSEUDOPARABOLIC DIFFUSION EQUATION* 

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#### Abstract

Diffusion in a fissured medium with absorption or partial saturation effects leads to a pseudoparabolic equation nonlinear in both the enthalpy and the permeability. The corresponding initialboundary value problem is shown to have a solution in various Sobolev-Besov spaces, and sufficient conditions are given for the problem to be well-posed.


Introduction. This is the second of two papers dealing with a certain pseudoparabolic diffusion equation. It was shown in [6] (see also [2]) that diffusion processes in fissured media lead to the following problem:

Let $G \subset \mathbb{R}^{N}$ be a bounded domain (the place where the diffusion process takes place), and denote by $S:=[0, T]$ a finite time interval. We are looking for functions $u=u(x, t)$ (concentration) and $v=v(x, t)$ (a flow potential), such that

$$
\begin{align*}
& u^{\prime}+\frac{1}{\varepsilon}(\alpha(u)-v)=f_{1}, \\
& -\operatorname{div}(k(u) \nabla v)+\frac{1}{\varepsilon}(v-\alpha(u))=f_{2},  \tag{0.1}\\
& u(x, 0)=u_{0}(x), \quad v_{\partial G}=0 .
\end{align*}
$$

Here $u$ ':= $\partial u / \partial t$, "div" denotes the usual divergence operator, " $\nabla$ " stands for the gradient with respect to $x=\left(x_{1}, \cdots, x_{N}\right) \in \mathbb{R}^{N}$, and $t \in S$. The functions $f_{i}=f_{i}(x, t)$, $i=1,2$, and $u_{0}$ are given and $k=k(u), \alpha=\alpha(u)$ are specified by properties of the field or medium. For each $u \in L^{1}(G)$ define $A_{u}:=-\operatorname{div}(k(u) \nabla)$ and consider this elliptic operator subject to Dirichlet boundary conditions. By eliminating $v$ in (0.1) we obtain the following equivalent ordinary differential equation involving only the single variable $u$

$$
\begin{align*}
& u^{\prime}(t)+\frac{1}{\varepsilon}\left(I-\left(I+\varepsilon A_{u}\right)^{-1}\right) \alpha(u)=f_{1}+\left(I+\varepsilon A_{u}\right)^{-1} f_{2},  \tag{0.2}\\
& u(0)=u_{0} .
\end{align*}
$$

Applying $I+A_{u(t)}$ to both sides of the equation in (0.2), one formally obtains the pseudo-parabolic problem

$$
\begin{align*}
& u^{\prime}+\varepsilon A_{u}\left(u^{\prime}\right)+A_{u}(\alpha(u))=\left(I+\varepsilon A_{u}\right)\left(f_{1}\right)+f_{2}, \\
& u(0)=u_{0},  \tag{0.3}\\
& \left.\alpha(u)\right|_{\partial G}=0 .
\end{align*}
$$

[^0]We note that problems of the type (0.3) also arise in the quite different contexts of heat conduction modelled by two-temperature systems [8], certain weak formulations of two-phase Stefan problems [6], [15] and in the description of some non-Newtonian fluids [9], [17]. Further references can be found in [7].

In [6] we considered the case $k=k(x, t), \alpha=\alpha(u)$ monotone, Lipschitz. There we showed existence and uniqueness of solutions under fairly weak assumptions on the data. Furthermore, for this case comparison and maximum principles were shown. Here we are concerned with the additional nonlinearity $k=k(u)$ which arises in the diffusion model [6] due to saturation or absorption effects on the permeability. Specific properties of $k$ and $\alpha$, as they are used later are listed below under (H1)-(H6).

We prove the existence of solutions to (0.2) or (0.3) in various spaces. Theorems 2.1, 2.3 and Corollary 2.2 contain existence results for solutions $u=u(t)$ taking their values in $B V(G), W_{0}^{s, p}(G)$ and $H_{0}^{1}(G)$, respectively. As the formulation of Theorem 2.1 shows, there remains a "gap". If $p \in[1, N / 2), s \in(0,1]$, then there are solutions in $W_{0}^{s, p}(G)$ (provided, $u_{0}, f(t) \in W_{0}^{s, p}(G)$ ). If $p>N / 2$, we only have some (sufficiently small) $s>0$, such that $u(t) \in W_{0}^{s, p}(G)$. Theorem 2.6 deals with $W^{2, p}$-existence of the solutions in the two-dimensional case. By interpolation methods we obtain results for $W_{0}^{1+\tau, p_{-} \text {existence for } \tau \in(0,1) \text { (Corollary 2.4). As a consequence we get some sufficient }}$ conditions on the data which imply that $u(t) \in W_{0}^{1, p}(G)$ for $N=2$ and for certain $p>2$ (Corollary 2.5). These seem not to be optimal since the assumed regularity of the data is higher than that of the solutions obtained. We continue by proving a uniqueness- and continuous-dependence result, the assumptions of which can be met at least in the oneand two-dimensional cases. The final theorem states some useful pointwise estimates, which in particular imply a weak maximum principle for ( 0.2 ).

The paper is organized as follows. Section 1 contains notations and lists some function spaces which we use. Section 2 contains the precise formulation of the results. Section 3 is concerned with the proofs. We conclude with a short appendix which presents some facts on interpolation.

1. Notation and spaces. Let $G \subset \mathbb{R}^{N}$ be a smooth and bounded domain, $\Gamma:=\partial G$ the boundary,
$S:=[0, T]$-a finite (time) interval, $S_{t}:=[0, t]$ for $t \in S$,
$Q_{t}:=(0, t) \times G, Q:=(0, T) \times G$
$D^{\alpha}:=D^{\alpha_{1}} D^{\alpha_{2}} \cdots D^{\alpha_{N}}$ for a multiindex $\alpha=\left(\alpha_{1}, \cdots, \alpha_{N}\right), D^{\alpha_{i}}:=\partial / \partial x_{i}$,
$x=\left(x_{1}, \cdots, x_{N}\right) \in G$,
$s \in[0,2], p \in[1, \infty], r \in[1, \infty], \sigma:=N s+p$ if $s \in(0,1)$,
$\lambda \in[0,1], k \in \mathbb{N}$,

$$
W^{s, p}(G):=\left\{\begin{array}{l}
W^{s, p}(G)-\text { the usual Sobolev space, if } s \text { is an integer, }{ }^{1} \\
B_{p, p}^{s}(G) \text {-the usual Besov space, if } s \text { is not an integer }
\end{array}\right.
$$

the norm in $W^{s, p}(G)$ is denoted by $\|\cdot\|_{s, p}$,
$L^{p}(G):=$ the usual space of $p$-integrable real-valued functions, $L^{p}(G)$ normed by the usual $L^{p}$-norm $|\cdot|_{p}$. Special case. If $p=2$ and $s=1$, then $|v|:=|v|_{2},\|v\|:=\|v\|_{1,2}$.
" $(\cdot, \cdot)$ " denotes the usual scalar product in $L^{2}(G)$ and the dual pairing between $H^{-1}(G)$ and $H_{0}^{1}(G), "((\cdot, \cdot))$ " stands for the scalar product in $H_{0}^{1}(G)$.

By $W_{0}^{s, p}(G)$ we denote the Sobolev-Besov space of those functions in $W^{s, p}(G)$ having zero-trace on $\partial G . W_{0}^{1, p}(G)$ is assumed to be normed by $\|v\|_{1, p}:=\sum_{|\alpha|=1}\left|D^{\alpha} v\right|_{p}$.

[^1]This norm is on $W_{0}^{1, p}(G)$ equivalent to the usual $W^{1, p}(G)$-norm, so that we do not introduce extra notation.

Furthermore:
$C^{k}(\bar{G}):=\{v: \bar{G} \rightarrow \mathbb{R}, k$ times continuously differentiable $\}$, equipped with the usual max-norm $\|\cdot\|_{C^{k}(\bar{G})}$,
$C^{0, \mu}(\bar{G})$ denotes the space of all Hölder continuous functions with Hölder exponent $\mu \in(0,1],\| \| \|_{C^{0, \mu}(\bar{G})}$-norm of this space.
$B V(G):=\left\{v \in L^{1}(G):\|v\|_{B V(G)}<\infty\right\}$-the set of all $L^{1}$-functions with finite total variation,

$$
\begin{aligned}
\|v\|_{B V(G)} & :=|v|_{L^{1}(G)}+[v]_{B V(G)}, \\
{[v]_{B V(G)} } & :=\sup \left\{\int_{G} v \operatorname{div} \vec{w} d x, \vec{w} \in C_{0}^{1}(\bar{G})^{N},|\vec{w}(x)| \leqq 1 \text { for all } x \in \bar{G}\right\} . \\
C_{0}^{1}(G) & :=\left\{v \in C^{1}(G): \operatorname{supp} v \subset \subset G\right\} .
\end{aligned}
$$

Generally, we do not distinguish in our notation between the norms in the space $\{V,|\cdot|\}$ and the norm in $V \times V \times \cdots \times V$, e.g., " $|\nabla g|_{p}$ " means $|\nabla g|_{L^{p}(G)^{N}}$. The same applies for scalar products.

If $\{V,|\cdot|\}$ is a normed space, then:
$L^{r}(S, V):=L^{r}(0, T ; V)$-the usual space of $V$-valued, to the power $r$ Bochner-integrable functions on $S$ and equipped with the norm $\left\|\|_{L^{r}(S, V)}\right.$ (sometimes we also write $\|_{L^{r}(S, V)}$ for the same standard norm).
$W^{1, r}(S, V):=\left\{v \in L^{r}(S, V): \quad v^{\prime} \in L^{r}(S, V)\right\}$.The norm in $W^{1, r}(S, V)$ is $\|v\|_{W^{1, r}(S, V)}:=|v|_{L^{r}(S, V)}+\left|v^{\prime}\right|_{L^{r}(S, V)}$.
$C(S, V):=C(0, T ; V)$-the space of all continuous functions mapping $S$ in $V$. $\|v\|_{C(S, V)}:=\max \{|v(s)|: s \in S\}$,
$C^{1}(S, V):=\left\{v \in C(S, V): v^{\prime} \in C(S, V)\right\},\|v\|_{C^{1}(S, V)}:=\|v\|_{C(S, V)}+\left\|v^{\prime}\right\|_{C(S, V)}$.
$C^{0, \mu}(S, V):=\left\{v \in C(S, V):\|v\|_{C^{0, \mu}(S, V)}<\infty\right\}$.

$$
\|v\|_{C^{0, \mu}(S, V)}:=\|v\|_{C(S, V)}+\sup \left\{\frac{|v(s)-v(t)|}{|t-s|^{\mu}}, t \neq s, t, s \in S\right\} .
$$

By " $\subset$ " we denote (beside set theoretic inclusion) continuous imbeddings, and " $\subset \subset$ " denotes compact imbeddings.
" $c$ " always stands for a nonnegative constant. Sometimes, we indicate on what quantities $c$ might depend.
$" \rightarrow "$ denotes strong convergence, " $\rightarrow$ "-weak convergence, " $\stackrel{*}{ }$ "-weak-star convergence.
2. Results. We will use the following hypotheses on the coefficients $k$ and $\alpha$, respectively.
(H1) $\quad k: \mathbb{R} \rightarrow \mathbb{R}$-continuous, and there are constants $k_{0}, k_{1}$ such that $0<k_{0} \leqq k(u) \leqq k_{1}$ for $u \in \mathbb{R}$,
(H2) $\quad \alpha: \mathbb{R} \rightarrow \mathbb{R}$-Lipschitz continuous with Lipschitz constant $L_{\alpha}$,
(H3) $\alpha$ is monotone and $\alpha(0)=0$,
(H4) $\quad k \in W^{1, \infty}(\mathbb{R}),\left|k^{\prime}\right|_{\infty}=: L_{k}$,
(H5) $\quad \alpha \in W^{2, \infty}(\mathbb{R}),\|\alpha\|_{2, \infty}=: \alpha_{1}$,
(H6) $\quad \gamma:=k \cdot \alpha^{\prime}$-Lipschitz continuous with Lipschitz constant $L_{\gamma}$.

Now we are going to formulate what we understand by a solution of (0.2) and (0.3), respectively. Assume $k$ and $\alpha$ are sufficiently regular so that all the appearing terms make sense. For the sake of (purely technical) simplicity we set $f_{1}=f$ and $f_{2}=0$ hereafter.

Formulation of (0.2) as an ordinary differential equation. Let us assume for a moment, we are given a sufficiently regular solution $u, v$ of $(0.1)$. If $u(t) \in L^{1}(G)$, then $\hat{k}(t):=k(u(t)) \in L^{\infty}(G)(k$ as in (H1)). Set $A(t)=-\operatorname{div}(\hat{k}(t) \nabla(\cdot))$. By the existence theory for elliptic operators which are subject to Dirichlet boundary conditions, we can define

$$
B(t):=(I+\varepsilon A(t))^{-1} \quad(I=\text { identity })
$$

and we have a continuous linear operator

$$
B(t): L^{p}(G) \rightarrow L^{p}(G) \quad \text { for all } p \in[1, \infty]
$$

(see Lemma 3.1 below). Now, set $A_{v}:=-\operatorname{div}(k(v) \nabla(\cdot))$ for $v \in L^{1}(G)$. By the preceding remarks,

$$
B_{v}:=\left(I+\varepsilon A_{v}\right)^{-1}: L^{p}(G) \rightarrow L^{p}(G) \quad \text { for all } p \in[1, \infty]
$$

Set for abbreviation

$$
A_{v}^{\varepsilon}:=\frac{1}{\varepsilon}\left(I-B_{v}\right)
$$

Thus, (0.2) is equivalent to

$$
\begin{equation*}
u^{\prime}(t)+A_{u(t)}^{\varepsilon}(\alpha(u(t)))=f(t) \quad \text { for } t \in S \tag{2.1}
\end{equation*}
$$

This leads to the formulation of (0.2) as an ordinary differential equation in $L^{p}(G)$ : Let

$$
r \in[1, \infty], \quad p \in[1, \infty], \quad f \in L^{r}\left(S, L^{p}(G)\right), \quad u_{0} \in L^{p}(G)
$$

We call $u \in W^{1, r}\left(S, L^{p}(G)\right)$ a solution, if (2.1) holds for a.a. $t \in S$ as an equation in $L^{p}(G)$ and

$$
\begin{equation*}
u(0)=u_{0} \quad \text { in } L^{p}(G) \tag{2.2}
\end{equation*}
$$

Formulation in variational form. Formally applying $I+\varepsilon A_{u(t)}$ on both sides of (2.1), we obtain (0.3). Given $r \in[1, \infty], p \in[1, \infty], 1 / p+1 / p^{\prime}=1, u_{0} \in W_{0}^{1, p}(G), f \in$ $L^{r}\left(S, W_{0}^{1, p}(G)\right)$, we call $u \in W^{1, r}\left(S, W_{0}^{1, p}(G)\right)$ a solution of $(0.3)$ if

$$
\begin{align*}
\left(u^{\prime}(t), v\right)+\varepsilon & \left(k(u(t)) \nabla u^{\prime}(t), \nabla v\right)+(k(u(t)) \nabla \alpha(u(t)), \nabla v)  \tag{2.3}\\
& =(f(t), v)+\varepsilon(k(u) \nabla f, \nabla v) \quad \text { for all } v \in W_{0}^{1, p^{\prime}}(G), \quad \text { a.a. } t \in S, \tag{2.4}
\end{align*}
$$

Notice that under appropriate regularity assumptions on $u$, the fact that $u$ satisfies (2.1), (2.2) implies that it also satisfies (2.3), (2.4) and vice versa.

Our results are as follows.
Theorem 2.1. Assume (H1), (H2),

$$
u_{0} \in W_{0}^{s, p}(G), \quad f \in C\left(S, W_{0}^{s, p}(G)\right)
$$

and one of the following conditions is satisfied:
(a) $N \geqq 2, p \in[1, \min \{2 N /(N+2), N / 2\}], s \in(0,1]$,
(b) $2 \leqq N \leqq 6, p \in[2 N /(N+2), N / 2), s \in(0,(2-N) / 2+N / p)$,
(c) $p>\max \{1, N / 2\}, \mathrm{s} \in(0,1]$ sufficiently small,
(d) $N=1, p=1, s \in(0,1]$.

Then (2.1), (2.2) has a solution $u \in C^{1}\left(S, W_{0}^{s, p}(G)\right)$.
Remark 2.1. Let $k$ and $\alpha$ be as in Theorem 2.1. If $u_{0} \in B V(G), f \in C(S, B V(G))$, then (2.1), (2.2) is solvable by a $u \in C^{1}(S, B V(G))$. For merely integrable right-hand sides $f$ we have

Corollary 2.2. Let $k, \alpha, p$ and $s$ be as in Theorem 2.1, $r \in[1, \infty]$. If $f \in$ $L^{r}\left(S, W_{0}^{s, p}(G)\right), u_{0} \in W_{0}^{s, p}(G)$, then there is a solution $u \in W^{1, r}\left(S, W_{0}^{s, p}(G)\right)$ satisfying (2.1), (2.2). The corresponding remark holds for $f \in L^{r}(S, B V(G)), u_{0} \in B V(G)$.

Theorem 2.3. Let $N \geqq 1, r \in[1, \infty]$.
(a) If $f \in L^{r}\left(S, H_{0}^{1}(G)\right), \quad u_{0} \in H_{0}^{1}(G)$, then (2.3), (2.4) has a solution $u \in$ $W^{1, r}\left(S, H_{0}^{1}(G)\right)$.
(b) If $f \in C\left(S, H_{0}^{1}(G)\right)$, then $u \in C^{1}\left(S, H_{0}^{1}(G)\right)$.

We remark that the proofs also yield several estimates for norms of $u$ in terms of the data.

Looking at (2.1), (2.2), one should expect $u$ to be exactly as regular as $u_{0}$ and $f$, since $B_{u(t)}$ is for many function spaces at least a regularity-preserving operator. But the nonlinearity of the problem causes some problems. With respect to a higher than square integrability of the first derivatives we get only a partial result which will be a consequence of Theorem 2.6 formulated below and the following corollary, the formulation of which seems to be rather technical.

Corollary 2.4. Let $N=2$, and assume (H1), (H2), (H4), (H5) and (H6). Furthermore, let $\theta, \tau \in(0,1), p^{*} \geqq 2, a, a^{\prime}>1, p>2$ be such that

$$
\frac{1}{a}+\frac{1}{a^{\prime}}=1, \quad \theta+\tau \leqq 1, \quad \theta=\frac{a^{\prime} p-2}{2 a^{\prime}(p-1)}, \quad \frac{1}{p^{*}}=\frac{(1-\tau)}{2}+\frac{\tau}{p}, \quad \tau \geqq \frac{2}{p^{*}}-\frac{2}{a p} .
$$

If $u_{0} \in W_{0}^{1+\tau, p^{*}}(G), f \in C\left(S, W_{0}^{1+\tau, p^{*}}(G)\right)$, then (2.3) (2.4) has a solution $u \in$ $C^{1}\left(S, W_{0}^{1+\tau, p^{*}}(G)\right)$. Furthermore, $u \in C^{1}\left(S, W_{0}^{1, a p}(G)\right)$.

To illustrate the assumptions under which this corollary is valid, we formulate Corollary 2.5.
(a) Let $a>1, p>2$ and set

$$
\tau:=\frac{a p-2}{2 a(p-1)}, \quad p^{*}:=\frac{2 a p(p-1)}{p(a+1)-2}, \quad \theta:=\frac{a(p-2)+2}{2 a(p-2)} .
$$

These numbers satisfy the conditions imposed in Corollary 2.4.
(b) Vice versa-let $N=2, k$ and $\alpha$ as in Corollary 2.4, $q>2$ a given number. We have for a solution $u \in C^{1}\left(S, W_{0}^{1, q}(G)\right)$, provided the data satisfy

$$
u_{0} \in W_{0}^{1+\tau, p^{*}}(G), \quad f \in C\left(S, W_{0}^{1+\tau, p^{*}}(G)\right)
$$

where for a $p \in[2, q)$

$$
p^{*}=\frac{2 q(p-1)}{q+p-2}, \quad \theta=\frac{q(1-1 / p)+2}{2 q(1-2 / p)}, \quad \tau=\frac{q-2}{2 q(1-1 / p)} .
$$

Concerning $W^{2, p}$-regularity, we have
Theorem 2.6. Let $N=2, k$ and $\alpha$ as in Corollary 2.4, $\tau \in[1, \infty]$, and $p \in[2, \infty]$.
(a) If $f \in L^{r}\left(S, W^{2, p}(G) \cap W_{0}^{1, p}(G)\right), u_{0} \in W^{2, p}(G) \cap W_{0}^{1, p}(G)$, then we have for a solution of (2.1), (2.2),

$$
u \in W^{1, r}\left(S, W^{2, p}(G) \cap W_{0}^{1, p}(G)\right)
$$

(b) If $f \in C\left(S, W^{2, p}(G) \cap W_{0}^{1, p}(G)\right)$, then $u \in C^{1}\left(S, W^{2, p}(G) \cap W_{0}^{1, p}(G)\right)$.

The next theorem reflects some sufficient conditions, which ensure unique solvability and continuous dependence of the solution on the data for problem (2.3), (2.4). In particular, these conditions are automatically fulfilled if $N=1$. For $N=2$, Corollary 2.5 provides information about such properties of $u_{0}$ and $f$ giving at least one sufficiently regular solution which meets these conditions.

Theorem 2.7. Let $N \geqq 1$, and $k$ satisfying (H1), (H4), (H6). Set $p>2$ if $N=2$, otherwise, $p:=N$.
(a) If $u_{0} \in H_{0}^{1}(G), f \in L^{1}\left(S, W_{0}^{1, p}(G)\right)$ and if there is at least one solution of (2.3), (2.4) with $u \in W^{1,1}\left(S, W_{0}^{1, p}(G)\right)$, then (2.1), (2.2) is uniquely solvable.
(b) The map

$$
\left\{u_{0}, f\right\} \in H_{0}^{1}(G) \times L^{r}\left(S, W^{1, p}(G)\right) \mapsto u \in W^{1, r}\left(S, H_{0}^{1}(G)\right)
$$

is locally Lipschitz for all $r \in[1, \infty]$.
Finally, we obtain some pointwise estimates on solutions and briefly indicate their usefulness.

Theorem 2.8. Let $k$ and $\alpha$ satisfy (H1)-(H3) and let $u \in W^{1,1}\left(S, L^{1}(G)\right)$ be a solution of (2.1), (2.2). Then, for a.a. $t \in S$ we have

$$
\begin{aligned}
& \left|u^{+}(t)\right|_{\infty} \leqq\left|u_{0}^{+}\right|_{\infty}+\int_{0}^{t}\left|f^{+}(s)\right|_{\infty} d s, \\
& \left|u^{-}(t)\right|_{\infty} \leqq\left|u_{0}^{-}\right|_{\infty}+\int_{0}^{t}\left|f^{-}(s)\right|_{\infty} d s .
\end{aligned}
$$

In particular, if $u_{0}(x) \geqq 0$ a.e. $f(s, x) \geqq 0$ a.e., then $u(t, x) \geqq 0$ a.e. If, in addition, there is a number $c_{0}>0$ with $\alpha\left(c_{0}\right)=0$ and $u_{0}(x) \geqq c_{0}$ a.e. in $G$, then $u(t, x) \geqq c_{0}$ a.e. in 0 .

The preceding is particularly relevant in the diffusion model of [6] where there is some interval $[0, L]$ on which $\alpha$ is identically zero. (This occurs because of partial saturation or absorption in the model.) To illustrate the usefulness of Theorem 2.8, suppose in this situation we know only that $k(u)$ is defined and continuous at each $u>0$. With $u_{0}$ and $f$ as given in Theorem 2.8, we choose $k_{0}$ to be the minimum and $k_{1}$ the maximum of $k(\cdot)$ on the interval $\left[c_{0},\left\|u_{0}\right\|_{L^{\infty}}+\int_{0}^{1}\|f(s)\|_{l^{\infty}} d s\right]$. Then extend $k$ outside this interval so as to satisfy ( H 1$)$. By Theorem 2.8 it follows the solution is independent of the extension, so we may assume without loss of generality that the original $k$ satisfies (H1). These remarks are useful in the diffusion model [6] where possibly $k(u) \rightarrow+\infty$ as $u \rightarrow 0^{+}$or $k(u) \rightarrow 0$ as $u \rightarrow+\infty$.
3. Proofs. The formulation of (0.2) as an operator equation (2.1) involves the resolvent $B_{v}:=\left(I+\varepsilon A_{v}\right)^{-1}$. We have to justify that $B_{v}$ exists. The point which has to be observed is that for fixed $v$ the operator $A_{v}$ has a coefficient $k=k(v)$, which is due to the lack of regularity of $k$ not too smooth. The following lemma lists some properties of the resolvent $B$ of a related operator $A$.

Lemma 3.1. Let $\hat{k}: \mathbb{R}^{n} \rightarrow \mathbb{R}$ be measureable, $0<k_{0} \leqq \hat{k}(x) \leqq k_{1}$ for a.a. $x \in \mathbb{R}^{n}$ $\left(k_{0}, k_{1}\right.$ as in the definition of $\left.k(\cdot)\right)$. Set $A:=-\operatorname{div}(\hat{k}(x) \nabla(\cdot)), B:=(I+\varepsilon A)^{-1}$, and consider the elliptic operator $A$ as subject to Dirichlet boundary conditions. In each of the following cases $B$ is defined and we have
(i) If $1<p \leqq 2 N /(N+2), 1 / p^{*}=1 / p-1 / N$, then $B: L^{p}(G) \rightarrow W_{0}^{1, p^{*}}(G)$.
(ii) If $p>N / 2$, then there is $a \lambda \in(0,1)$ such that $B: L^{p}(G) \rightarrow H_{0}^{1}(G) \cap C^{0, \lambda}(\bar{G})$.
(iii) If $p=1, q \in[1, N(N-1))$, then $B: L^{1}(G) \rightarrow W_{0}^{1, q}(G)$.
(iv) If $2 N /(N+2)<p \leqq \min \{N / 2,2\}$, then $B: L^{p}(G) \rightarrow W_{0}^{1,2}(G)$.
(v) If $2 \leqq p<N, 1 / p^{*}=1 / p-1 / N$, then $B: L^{p}(G) \rightarrow W_{0}^{1,2}(G) \cap L^{p^{*}}(G)$.

In each of these cases, $B$ is a linear and continuous map and its norm depends at most on $G, p^{*}, p, N, \lambda, \varepsilon k_{0}, k_{1} . \lambda$ depends at most on $G, p, N, \varepsilon k_{0}, k_{1}$.

Proof. (i), (iii) are part of [16, Thm. 4.5], for (ii) see [12, Chap. III], (iv) and (v) follows from (i).

The next lemma list some known imbedding properties.
Lemma 3.2. Consider the situation in Lemma 3.1 and let $p, p^{*}, \lambda, N$ be as in (i)-(v). Take in (i)-(iv) $s \in(0,1)$, in (v) $s \in(0(2-N) / 2+N / p)$. Then
(i) ${ }^{2} W^{1, p^{*}}(G) \subset \subset W^{s, p^{*}}(G) \subset W^{s, p} \subset \subset L^{p}(G)$
(ii) $W^{s, p}(G) \subset \subset L^{p}(G), C^{0, \lambda}(\bar{G}) \subset L^{\infty}(G)$
(iii) $W^{1, p^{*}}(G) \subset \subset W^{s, p^{*}}(G) \subset W^{s, 1}(G) \subset \subset L^{1}(G)$
(iv) $W^{1, p^{*}}(G) \subset \subset W^{s, p}(G) \subset \subset L^{p}(G)$
(v) $W^{1,2}(G) \subset \subset W^{s, p}(G) \subset \subset L^{p}(G)$, if $N \leqq 6$.

Proof. See [1], [3]. The compactness results from $W^{1, p} \subset \subset L^{p}$ for any $p \geqq 1$, and the fact, that ${ }^{3} W^{s, p}=\left[L^{p}, W^{1, p}\right]_{s}$ and general interpolation theory.

Corollary 3.3. Let $v \in L^{1}(G)$, set, as before, $A_{v}:=-\operatorname{div}(k(v(x)) \nabla(\cdot)), B_{v}:=$ $\left(I+\varepsilon A_{v}\right)^{-1} . B_{v}$ has exactly the same properties as $B$ in Lemma 3.1. Furthermore, $A_{v}^{\varepsilon}:=(1 / \varepsilon)\left(I-B_{v}\right): L^{p}(G) \rightarrow L^{p}(G)$ is Lipschitz. Thus, $A_{v}^{\varepsilon} \circ \alpha: L^{p}(G) \rightarrow L^{p}(G)$ is Lipschitz.

It follows that (2.1) can be considered as an ordinary differential equation in $L^{p}(G)$. Moreover, we have the following.

Lemma 3.4. Let $v \in L^{1}\left(S, L^{1}(G)\right)$ be given. Then
(i) For each $w \in L^{p}(G)$

$$
\begin{equation*}
t \in S \rightarrow A_{v(t)}^{\varepsilon}(w) \in L^{p}(G) \tag{3.1}
\end{equation*}
$$

is measureable.
(ii) If $v \in C\left(S, L^{1}(G)\right)$, then the map in (3.1) is continuous.

Proof. (i) see [14].
(ii) Let $t_{n} \rightarrow t$ in $S$, set $g_{n}:=\left(I+\varepsilon A_{v\left(t_{n}\right)}\right)^{-1}(w)$. By definition $g_{n}$ satisfies

$$
\begin{equation*}
\left(I+\varepsilon A_{v\left(t_{n}\right)}\right)\left(g_{n}\right)=w . \tag{3.2}
\end{equation*}
$$

Consider case (i) in Lemma 3.1. We have $\left\|g_{n}\right\|_{1, p^{*}} \leqq c|w|_{p}$. Therefore, for a subsequence $g_{n_{j}} \rightarrow \tilde{g}$ in $L^{p^{*}}(G)$ and $g_{n_{j}} \rightarrow \tilde{g}$ in $W_{0}^{1, p^{*}}(G)$. By the continuity of $k, \tilde{g}$ satisfies

$$
\left(I+\varepsilon A_{v(t)}\right) \tilde{g}=w .
$$

[^2]But, (3.2') is uniquely solvable, so that $g_{n} \rightarrow \tilde{g}$ in $L^{p}(G)$, which proves the assertion. The other cases in Lemma 3.1 can be dealt with in a similar manner.

Proof of Theorem 2.1. Fix $\bar{u} \in C\left(S, L^{p}(G)\right)$. By Corollary 3.3 and Lemma 3.4, there is exactly one solution

$$
\begin{equation*}
u \in C^{1}\left(S, L^{p}(G)\right) \tag{3.3}
\end{equation*}
$$

satisfying

$$
\begin{equation*}
u^{\prime}(t)+A \frac{\varepsilon}{\bar{u}(t)} \alpha(u(t))=f(t), \quad u(0)=u_{0} \tag{3.4}
\end{equation*}
$$

We employ Schauder's theorem to show that the map

$$
\mathscr{T}: C\left(S, L^{p}(G)\right) \rightarrow C\left(S, L^{p}(G)\right)
$$

defined by (3.3), (3.4) has a fixed point. Obviously, a fixed point of $\mathscr{T}$ solves (2.3), (2.4). The next lemma summarizes some properties of $\mathscr{T}$. In particular, the proof yields several estimates of the solutions of (2.1), (2.2).

Lemma 3.5. (i) $\mathscr{T}$ maps $C\left(S, L^{1}(G)\right)$ into a bounded subset of $C^{1}\left(S, L^{p}(G)\right)$.
(ii) Let $s, N$ be as in Lemma 3.2, $t \in S, \bar{u} \in C\left(S, L^{1}(G)\right)$. Then $(\mathscr{T} \bar{u})(t)$ is in a bounded subset of $W^{s, p}(G)$.
(iii) $\mathscr{T}: C\left(S, L^{1}(G)\right) \rightarrow C\left(S, L^{p}(G)\right)$ is continuous.

Proof. (i). Integrate (3.4) over ( $0, t$ ), and take the $L^{p}(G)$-norm on both sides. Then, by Lemmas 3.1, 3.2 and $\alpha$ 's Lipschitz continuity

$$
\begin{aligned}
&|u(t)|_{p} \leqq\left|u_{0}\right|_{p}+\int_{0}^{t}\left|A_{\tilde{u}(s)}^{\varepsilon} \alpha(u(s))\right|_{p} d s+\int_{0}^{t}|f(s)|_{p} d s \\
& \leqq\left|u_{0}\right|_{p}+c \int_{0}^{t}|\alpha(u(s))|_{p} d s+\int_{0}^{t}|f(s)|_{p} d s \\
& \leqq\left|u_{0}\right|_{p}+c \int_{0}^{t}\left\{|\alpha(0)| \cdot|G|+L_{\alpha}|u(s)|_{p}\right\} d s+\int_{0}^{t}|f(s)|_{p} d s
\end{aligned}
$$

hence, by Gronwall's inequality

$$
\begin{equation*}
|u|_{C\left(0, t ; L^{p}(G)\right)} \leqq c\left\{\left|u_{0}\right|_{p}+1+|f|_{L^{1}\left(S, L^{p}(G)\right)}\right\} \tag{3.5}
\end{equation*}
$$

where $c=c\left(\varepsilon k_{0}, k_{1},|G|, N, p, L_{\alpha}, \lambda, s, T\right)$. By (3.4) and (3.5)

$$
\begin{equation*}
|u|_{C^{1}\left(0, t ; L^{p}(G)\right)} \leqq c\left\{\left|u_{0}\right|_{p}+1+|f|_{C\left(S, L^{p}(G)\right)}\right\} . \tag{3.6}
\end{equation*}
$$

To obtain further estimates, we notice that $u$ as a solution of an ordinary differential equation is the $C\left(S, L^{p}(G)\right)$-limit of the sequence $\left\{u_{n}\right\}$ defined by

$$
\begin{align*}
& u_{1}:=u_{0}  \tag{3.7}\\
& u_{n+1}(t)=u_{0}+\int_{0}^{t} f(s) d s-\int_{0}^{t} A_{\bar{u}(s)}^{\varepsilon} \alpha\left(u_{n}(s)\right) d s, t \in S . \tag{3.8}
\end{align*}
$$

We have
Lemma 3.6. Let $u_{n} \in C\left(S, W^{s, p}(G)\right)$, where $s, p$ are taken as in Lemma 3.1, 3.2, $s \in(0, \lambda)$ if $p>N / 2(\lambda$ arises in Lemma 3.1, (ii)). Then
a) $u_{n+1} \in C^{1}\left(S, W^{s, p}(G)\right)$ and
b)

$$
\begin{equation*}
\left\|u_{n}\right\|_{C\left(S, W^{s, p}(G)\right)} \leqq c\left\{\left\|u_{0}\right\|_{s, p}+1+\|f\|_{L^{1}\left(S, W^{s, p}(G)\right)}\right\} \tag{3.9}
\end{equation*}
$$

with $c$ as in (3.5).

Proof. By assumption $u_{n}(t) \in W^{s, p}(G)$; hence $\alpha\left(u_{n}\right) \in W^{s, p}(G) \subset L^{p}(G)$. By Lemma 3.1, 3.2 $A_{\bar{u}(t)}^{\varepsilon}\left(\alpha\left(u_{n}(t)\right)\right) \in W^{s, p}(G)$ if $p \leqq N / 2$, and

$$
\begin{align*}
\left\|A_{\bar{u}(t)}^{\varepsilon}\left(\alpha\left(u_{n}(t)\right)\right)\right\|_{s, p} & \leqq \frac{1}{\varepsilon}\left\|\alpha\left(u_{n}(t)\right)\right\|_{s, p}+\frac{c}{\varepsilon}\left|\alpha\left(u_{n}(t)\right)\right|_{p}  \tag{3.10}\\
& \leqq \frac{c}{\varepsilon}\left\|u_{n}(t)\right\|_{s, p} \quad(c \text { as in }(3.5)) .
\end{align*}
$$

Therefore, by (3.8), $u_{n+1}(t) \in W^{s, p}(G)$. The $t$-continuity properties follow as in Lemma 3.4. Equations (3.8) and (3.10) yield (3.9) by an iteration argument (cf. [4]). To show (3.9) for $p>N / 2$, take the $W^{s, p}$-norm on both sides of (3.8). Thus

$$
\begin{align*}
\left\|u_{n+1}(t)\right\|_{s, p} \leqq & \left\|u_{0}\right\|_{s, p}+\int_{0}^{t}\|f(s)\|_{s, p} d s+\frac{1}{\varepsilon} \int_{0}^{t}\left\|\alpha\left(u_{n}(s)\right)\right\|_{s, p} d s  \tag{3.11}\\
& +\frac{1}{\varepsilon} \int_{0}^{t}\left\|\left(I+\varepsilon A_{\bar{u}(s)}\right)^{-1} \alpha\left(u_{n}(s)\right)\right\|_{s, p} d s .
\end{align*}
$$

We have $\left\|\alpha\left(u_{n}\right)\right\|_{s, p} \leqq L_{\alpha}\left\|u_{n}\right\|_{s, p}$ and for

$$
g(t, x):=\left(I+\varepsilon A_{\bar{u}(t)}\right)^{-1}\left(\alpha\left(u_{n}(t)\right)\right)
$$

by Lemma 3.1, 3.2.
(ii)

$$
\begin{aligned}
\|g(t)\|_{s, p}^{p} & =\iint_{G G} \frac{|g(t, x)-g(t, y)|^{p}}{|x-y|^{\mu}} d x d y, \quad \mu=N+s p \\
& \leqq \iint_{G G} c \cdot|x-y|^{\lambda p-\mu} d x d y \leqq \text { const. }
\end{aligned}
$$

Therefore (3.11) and Gronwall's inequality imply (3.9).
To complete the proof of Lemma 3.5(ii), we note that (3.9) and (3.6) imply

$$
\begin{equation*}
\left\|u_{n}\right\|_{C^{1}\left(S, W^{s, p}(G)\right)} \leqq c\left\{\left\|u_{0}\right\|_{s, p}+\|f\|_{C\left(S, W^{s, p}(G)\right)}+1\right\} \tag{3.12}
\end{equation*}
$$

where $c$ is as in (3.5). By weak-star compactness, $u \in W^{1, \infty}\left(S, W^{s, p}(G)\right)$ and by (3.4), $u \in C^{1}\left(S, W^{s, p}(G)\right)$, where (3.12) implies

$$
\begin{equation*}
\|u\|_{C^{1}\left(S, W^{s, p}(G)\right)} \leqq c\left\{\left\|u_{0}\right\|_{s, p}+\|f\|_{C\left(S, W^{s, p}(G)\right)}+1\right\} . \tag{3.13}
\end{equation*}
$$

This proves (ii) of Lemma 3.5. To see the continuity of $\mathscr{T}$ let $\bar{u}_{k} \rightarrow \bar{u}$ in $C\left(S, L^{1}(G)\right)$, set $g_{k}:=B_{\bar{u}_{k}}\left(\alpha\left(u_{k}\right)\right), u_{k}:=\mathscr{T}\left(\bar{u}_{k}\right)$, so that $g_{k}$ and $u_{k}$, resp. satisfy

$$
\begin{align*}
& \varepsilon A_{\bar{u}_{k}}\left(g_{k}\right)=-g_{k}+\alpha\left(u_{k}\right)  \tag{3.14}\\
& u_{k}^{\prime}(t)+\frac{1}{\varepsilon} \alpha\left(u_{k}(t)\right)=\frac{1}{\varepsilon} g_{k}(t)+f(t), \quad u_{k}(0)=u_{0} \tag{3.15}
\end{align*}
$$

By estimates (3.6), (3.13) and $\alpha$ 's Lipschitz continuity there is a subsequence

$$
\begin{align*}
& u_{k_{j}} \stackrel{*}{u} \text { in } W^{1, \infty}\left(S, W^{s, p}(G)\right),  \tag{3.16}\\
& \alpha\left(u_{k_{j}}\right) \xrightarrow{*} \alpha(u) \quad \text { in } L^{\infty}\left(S, W^{s, p}(G)\right) .
\end{align*}
$$

Using the estimates for $u_{k}$ from (3.15) (cf. estimates (3.6), (3.13)) to estimate $g_{k}$ in (3.14) ((3.14) is an elliptic problem for $g_{k}$, various norms of $g_{k}$ can be estimated in terms of $\alpha\left(u_{k}\right)$ by Lemma 3.1, and $\alpha\left(u_{k}\right)$ can be estimated by (3.13), (3.6)), we arrive for $p \leqq(N / 2)$ at

$$
\begin{aligned}
\left|g_{k}\right|_{L^{\infty}\left(S, W^{s, p}(G)\right)} & \leqq c\left|\alpha\left(u_{k}\right)\right|_{L^{\infty}\left(S, L^{p}(G)\right)} \leqq c\left\{1+\left|u_{k}\right|_{L^{\infty}\left(S, L^{p}(G)\right)}\right\} \\
& \leqq c\left\{1+\left|u_{0}\right|_{p}+|f|_{L^{1}\left(S, L^{p}(G)\right)}\right\},
\end{aligned}
$$

and for $p>N / 2$, at

$$
\left|g_{k}\right|_{L^{\infty}\left(S, H_{0}^{1}(G)\right)} \leqq c\left\{1+\left|u_{0}\right|_{p}+|f|_{L^{1}\left(S, L^{p}(G)\right)}\right\} .
$$

Therefore, we have for a subsequence

$$
\begin{equation*}
g_{k_{j}} \stackrel{*}{-} g \text { in } L^{\infty}\left(S, W_{0}^{1, p}(G)\right) . \tag{3.17}
\end{equation*}
$$

Equations (3.16), (3.17) and relation (3.14) imply that $g$ satisfies

$$
\begin{equation*}
\varepsilon A_{\bar{u}}(g)=-g+\alpha(u) . \tag{3.18}
\end{equation*}
$$

(To pass to the limit $j \rightarrow \infty$ in (3.14), use the strong convergence properties of $k\left(\bar{u}_{k_{j}}\right)$-remember, $\bar{u}_{k} \rightarrow \bar{u}$ in $C\left(S, L^{1}(G)\right)$ and $k$ is continuous.) Since (3.18) is uniquely solvable, we have $u_{k} \stackrel{*}{u} u$ in $W^{1, \infty}\left(S, W^{s, p}(G)\right)$, i.e., the whole sequence converges, in particular $u_{k}=\mathscr{T}\left(\bar{u}_{k}\right) \rightarrow u=\mathscr{T}(\bar{u})$ in $C\left(S, L^{p}(G)\right)$, i.e., $\mathscr{T}$ is continuous, which finishes the proof of Lemma 3.5.

Corollary 3.7. Lemma 3.5 implies that $\mathscr{T}$ has a fixed point $u$. u solves problem (2.3), (2.4) and satisfies the estimates given by (3.6), (3.9). Moreover, $u$ is-according to (3.7), (3.8)—the limit of the sequence $\left\{u_{n}\right\}$ defined by

$$
\begin{align*}
& u_{1}:=u_{0} \\
& u_{n+1}(t)=u_{0}+\int_{0}^{t} f(s) d s-\int_{0}^{t} A_{u_{n}(s)}^{e}\left(\alpha\left(u_{n}(s)\right)\right) d s, \tag{3.19}
\end{align*}
$$

and

$$
\begin{equation*}
u_{n} \rightarrow u \quad \text { in } C\left(S, L^{p}(G)\right), \quad u_{n} \stackrel{*}{\rightarrow} u \quad \text { in } W^{1, \infty}\left(S, L^{p}(G)\right) . \tag{3.20}
\end{equation*}
$$

This finishes the proof of Theorem 2.1.
Proof of Remark 2.1. Construct as in (3.3), (3.4) an operator

$$
\mathscr{T}: C\left(S, L^{1}(G)\right) \rightarrow C^{1}\left(S, L^{1}(G)\right)
$$

One has already estimate (3.6), so that Lemma 3.5(i) follows. The third statement of this lemma has already been proved and the second has to be changed to

Lemma 3.5'. (ii') If $\bar{u} \in C\left(S, L^{1}(G)\right)$, then $(\mathscr{T} u)(t)$ is in a bounded subset of $B V(G)$, which does not depend either on $t$ or on $\bar{u}$.

To see this, look at the iteration procedure (3.7), (3.8) which yields $u$ as the $C\left(S, L^{1}(G)\right)$-limit of $\left\{u_{n}\right\}$. By Lemma 3.1(iii) we have $\left(I+\varepsilon A_{\bar{u}(s)}\right)^{-1} \alpha\left(u_{n}(s)\right) \in$ $W^{1, p^{*}}(G) \subset B V(G)$ so that by the same lemma

$$
\begin{aligned}
\left\|A_{\bar{u}(s)}^{\varepsilon} \alpha\left(u_{n}(s)\right)\right\|_{B V(G)} & \leqq c \cdot \frac{1}{\varepsilon}\left\{\left\|\alpha\left(u_{n}\right)\right\|_{B V(G)}+\left\|\alpha\left(u_{n}\right)\right\|_{L^{1}(G)}\right\} \\
& \leqq c \cdot \frac{1}{\varepsilon}\left\{1+\left\|u_{n}\right\|_{B V(G)}\right\} .
\end{aligned}
$$

Taking the $B V(G)$-norm on both sides of (3.8) yields

$$
\left\|u_{n}\right\|_{C^{1}(S, B V(G))} \leqq c\left\{\left\|u_{0}\right\|_{B V(G)}+\|f\|_{C(S, B V(G))}+1\right\}
$$

Since $B V(G) \subset \subset L^{p^{*}}(G) \subset L^{1}(G)$ for $p^{*} \in(1, N /(N-1))$, Arzela-Ascoli's theorem yields for a subsequence

$$
u_{n_{j}} \rightarrow u \quad \text { in } C\left(S, L^{p^{*}}(G)\right) .
$$

Because of the reflexivity of $B V(G)$ we have by weak-star compactness

$$
\begin{aligned}
& u_{n_{j}} \stackrel{*}{\rightharpoonup} u \quad \text { in } W^{1, \infty}(S, B V(g)) \quad \text { and } \\
& \|u\|_{W^{1, \infty}(S, B V(G))} \leqq c\left\{\left\|u_{0}\right\|_{B V(G)}+1+\|f\|_{C(S, B V(G))}\right\} .
\end{aligned}
$$

Since $u$ satisfies (3.4), we obtain after a short calculation

$$
\begin{align*}
& u^{\prime} \in C(S, B V(G)) \quad \text { and } \\
& \|u\|_{C^{1}(S, B V(G))} \leqq c\left\{\left\|u_{0}\right\|_{B V(G)}+1+\|f\|_{C(S, B v(G))}\right\} . \tag{3.21}
\end{align*}
$$

Therefore $\mathscr{T}$ has a fixed point $u \in C^{1}(S, B V(G))$ which solves (2.3), (2.4).
Proof of Corollary 2.2. We modify estimates (3.6), (3.13). One has

$$
\begin{aligned}
\left|u^{\prime}(t)\right|_{p} & \leqq\left. A_{u(t)}^{\varepsilon}(\alpha(u(t)))\right|_{p}+|f(t)|_{p} \\
& \leqq c\left\{1+|u(t)|_{p}\right\}+|f(t)|_{p}
\end{aligned}
$$

from arguments which led to estimate (3.5). Taking (3.5) into account, one gets for $r \in[1, \infty]$

$$
\begin{equation*}
\left|u^{\prime}\right|_{L^{r}\left(S, L^{p}(G)\right)} \leqq c\left\{1+\left|u_{0}\right|_{p}+|f|_{L^{2}\left(S, L^{p}(G)\right)}\right\} . \tag{3.22}
\end{equation*}
$$

Similarly, (3.11) implies

$$
\|u\|_{C\left(S, W^{s, p}(G)\right)} \leqq c\left\{1+\left\|u_{0}\right\|_{s, p}+|f|_{L^{1}\left(S, W^{s, p}(G)\right)}\right\}
$$

and (2.3) yields

$$
\left\|u^{\prime}(t)\right\|_{s, p} \leqq\left\|A_{u(t)}^{\varepsilon}(\alpha(u(t)))\right\|_{s, p}+\|f(t)\|_{s, p}
$$

Therefore,

$$
\begin{equation*}
\left|u^{\prime}\right|_{L^{r}\left(S, W^{s, p}(G)\right)} \leqq c\left\{1+\left\|u_{0}\right\|_{s, p}+|f|_{L^{r}\left(S, W^{s, p}(G)\right)}\right\} . \tag{3.23}
\end{equation*}
$$

Proof of Theorem 2.3. We employ the Galerkin method. Let $\left\{w_{i}\right\} \subset H_{0}^{1}(G) \cap H^{2}(G)$ be an orthonormal eigenvalue basis of the Laplacian subject to Dirichlet boundary conditions, i.e,.

$$
\begin{equation*}
-\Delta w_{i}=\lambda_{i} w_{i},\left(w_{i}, w_{j}\right)=\delta_{i j}, w_{i} \perp w_{j} \quad \text { in } H_{0}^{1}(G) \tag{3.24}
\end{equation*}
$$

Set $V_{n}:=\operatorname{span}\left\{w_{1}, \cdots, w_{n}\right\}$, denote by $P_{n}$ the orthogonal projection in $H_{0}^{1}(G)$ onto $V_{n}$. We are looking for an absolutely continuous

$$
u_{n}(t):=\sum_{j=1}^{n} h_{n j}(t) w_{j}
$$

which satisfies

$$
\begin{align*}
\left(u_{n}^{\prime}(t), v\right)+ & \varepsilon\left(k\left(u_{n}\right) \nabla u_{n}^{\prime}, \nabla v\right)+\left(k\left(u_{n}\right) \nabla \alpha\left(u_{n}\right), \nabla v\right)  \tag{3.25}\\
& =\varepsilon\left(k\left(u_{n}\right) \nabla f, \nabla v\right)+(f, v) \quad \forall v \in V_{n}, \text { a.a. } t \in S, \\
u_{n}(0)=u_{0 n} & :=P_{n} u_{0} . \tag{3.26}
\end{align*}
$$

By using a fixed point argument and applying the results of [6] or a reduction of (3.25), (3.26) to an ordinary differential equation (cf. [5, Lemma 1]) to obtain the standard form for an application of Caratheodory's theorem, one shows, that (3.25')-(3.26) is (at least) locally solvable. The following lemma implies that these solutions are globally defined. One has

Lemma 3.8. There is a constant $c=c\left(\varepsilon k_{0}, k_{1}, L_{\alpha}, T,|G|, \varepsilon\right)$ such that for $r \in[1, \infty]$

$$
\begin{equation*}
\left\|u_{n}\right\|_{W^{1, r}\left(S, H_{0}^{1}(G)\right)} \leqq c\left\{\left\|u_{0}\right\|+\|f\|_{L^{1}\left(S, H_{0}^{1}(G)\right)}\right\} . \tag{3.27}
\end{equation*}
$$

Proof of Lemma 3.8. Choose in (3.25) $v:=u_{n}^{\prime}(t)$, use the boundedness properties of $k(\cdot)$ and $\alpha(\cdot)$ and apply Hölder's and Young's inequalities

$$
\begin{align*}
& \left|u_{n}^{\prime}(t)\right|^{2}+\varepsilon k_{0}\left|\nabla u_{n}^{\prime}(t)\right|^{2}  \tag{3.28}\\
& \quad k_{1} L_{\alpha}\left|\nabla u_{n}\right|\left|\nabla u_{n}^{\prime}\right|+\varepsilon k_{1}|\nabla f|\left|\nabla u_{n}^{\prime}\right|+|f| \cdot\left|u_{n}^{\prime}\right| \\
& \quad \leqq \frac{\varepsilon k_{0}}{2}\left|\nabla u_{n}^{\prime}\right|^{2}+c\left\{\left|\nabla u_{n}\right|^{2}+|\nabla f|^{2}+|f|\right\}+\frac{1}{2}\left|u_{n}^{\prime}\right|^{2}, \\
& \left|\nabla u_{n}^{\prime}(t)\right|^{\hat{r}} \leqq c\left\{|f|^{\hat{r}}+|\nabla f|^{\hat{r}}+\left|\nabla u_{0 n}\right|^{\hat{r}}+\int_{0}^{t}\left|\nabla u_{n}^{\prime}(s)\right|^{\hat{r}} d s\right\} . \tag{3.29}
\end{align*}
$$

If $r<\infty$, then $\hat{r}:=r$; if $r=\infty$, then $\hat{r}:=1$.
Gronwall's inequality in connection with (3.26) yields

$$
\left\|u_{n}^{\prime}\right\|_{L^{r}\left(S, H_{0}^{1}(G)\right)} \leqq c\left\{\|f\|_{L^{\prime}\left(S, H_{0}^{1}(G)\right)}+\left\|u_{0}\right\|\right\}
$$

and therefore (3.27). The usual compactness argument completes the proof and shows that (2.1), (2.2) has a solution, which satisfies

$$
\begin{equation*}
\|u\|_{W^{1, r}\left(S, H_{0}^{1}(G)\right)} \leqq c\left\{\left\|u_{0}\right\|+\|f\|_{L^{r}\left(S, H_{0}^{1}(G)\right)}\right\} \tag{3.30}
\end{equation*}
$$

Before proving Corollaries 2.4, 2.5, we prove Theorem 2.6.
Proof of Theorem 2.6(a). We use the Galerkin method and continue at (3.25), (3.26). We have already estimate (3.27) and will show that

$$
\begin{equation*}
\left\|u_{n}\right\|_{W^{1, r}\left(S, W^{2, p}(G) \cap W_{0}^{1, p}(G)\right)} \leqq \text { const. } \tag{3.31}
\end{equation*}
$$

To see this, note that due to (3.24)

$$
v:=-\Delta u_{n}^{\prime}(T) \in V_{n}
$$

is an admissible function in (3.25). Integration by parts and reordering yield

$$
\begin{aligned}
\left|\nabla u_{n}^{\prime}(t)\right|^{2}+ & \varepsilon\left(k\left(u_{n}\right) \Delta u_{n}^{\prime}, \Delta u_{n}^{\prime}\right) \\
= & \varepsilon\left(k^{\prime}\left(u_{n}\right) \nabla u_{n}^{\prime}, \nabla u_{n} v\right)+(f, v)-\varepsilon\left(k^{\prime}\left(u_{n}\right) \nabla u_{n}, \nabla f v\right) \\
& \quad-\varepsilon\left(k\left(u_{n}\right) \Delta, f, v\right)+\left(k^{\prime}\left(u_{n}\right) \Delta u_{n} \cdot \nabla u_{n}, \alpha^{\prime}\left(u_{n}\right) v\right) \\
& +\left(k\left(u_{n}\right) \alpha^{\prime \prime}\left(u_{n}\right)\left|\nabla u_{n}\right|^{2}, v\right)+\left(k\left(u_{n}\right) \alpha^{\prime}\left(u_{n}\right) \Delta u_{n}, v\right) .
\end{aligned}
$$

By (H1), (H2), (H4), (H5), Hölder's, Young's and one of Sobolev's inequalities we obtain

$$
\begin{aligned}
& \left|\nabla u_{n}^{\prime}(t)\right|^{2}+\varepsilon k_{0}\left|\Delta u_{n}^{\prime}(t)\right|^{2} \\
& \leqq \varepsilon L_{k}\left|\nabla u_{n}^{\prime}\right|_{4}\left|\nabla u_{n}\right|_{4}|v|_{2}+|f|_{2}|v|_{2}+\varepsilon L_{k}\left|\nabla u_{n}\right|_{4}|\nabla f|_{4}|v|_{2} \\
& +\varepsilon k_{1}|\Delta f|_{2}|v|_{2}+L_{k} \alpha_{1}\left|\nabla u_{n}\right|_{4}^{2}|v|_{2}+k_{1} \alpha_{1}\left|\nabla u_{n}\right|_{4}^{2}|v|_{2}+k_{1} \alpha_{1}\left|\Delta u_{n}\right|_{2}|v|_{2} \\
& \leqq \frac{\varepsilon k_{0}}{4}|v|_{2}^{2}+c\left\{\left|\nabla u_{n}^{\prime}\right|_{2}\left|\Delta u_{n}^{\prime}\right|_{2}\left|\nabla u_{n}\right|_{2}+|f|_{2}^{2}\right. \\
& \left.+\left|\nabla u_{n}\right|\left|\Delta u_{n}\right||\nabla f|_{4}^{2}+|\Delta f|_{2}^{2}+\left|\nabla u_{n}\right|^{2}\left|\Delta u_{n}\right|^{2}+\left|\Delta u_{n}\right|^{2}\right\} .
\end{aligned}
$$

(This is the only place where the restriction $N=2$ is essential. The case $N=3$ allows only estimates like $\left|\nabla u_{n}^{\prime}\right|_{4} \leqq c\left|\nabla u_{n}^{\prime}\right|_{2}^{1 / 3}\left|\Delta u_{n}^{\prime}\right|_{2}^{2 / 3}$, which finally would require some restrictions on $\mid \nabla f_{4}$ and $\left\|u_{0}\right\|_{2,2}$, to obtain (3.31).)

From now on we use again our convention concerning the notation of $L^{2}$-norms. Using again Young's inequality, we arrive at

$$
\begin{aligned}
\left|\Delta u_{n}^{\prime}(t)\right|^{2} \leqq c & \left\{\left|\nabla u_{n}^{\prime}\right|^{2}\left|\nabla u_{n}\right|^{2}\left|\Delta u_{n}\right|^{2}+|f|^{2}+|\Delta f|^{2}\right. \\
& \left.+\left|\nabla u_{n}\right|\left|\Delta_{n}\right||\nabla f|_{4}^{2}+\left|\nabla u_{n}\right|^{2}\left|\Delta u_{n}\right|^{2}+\left|\Delta u_{n}\right|^{2}\right\} \\
\left|\Delta u_{n}^{\prime}(t)\right| \leqq c & \left\{\left|\nabla u_{n}^{\prime}\right|\left|\nabla u_{n}\right|\left|\Delta u_{n}\right|+|f|+\left|\nabla u_{n}\right|+\left|\Delta u_{n}\right||\nabla f|_{4}^{2}\right. \\
& \left.\quad+|\Delta f|+\left|\nabla u_{n}\right|\left|\Delta u_{n}\right|+\left|\nabla u_{n}\right||\nabla f|_{4}^{2}+\left|\Delta u_{n}\right|\right\} .
\end{aligned}
$$

By (3.27),

$$
\left|\nabla u_{n}\right|_{C\left(S, L^{2}(G)^{N}\right)} \leqq c\left\{\left\|u_{0}\right\|+\|f\|_{L^{1}\left(S, H_{0}^{1}(G)\right)}\right\}=: \bar{c},
$$

so that

$$
\begin{align*}
\left|\Delta u_{n}^{\prime}(t)\right| \leqq c\left\{\bar{c}\left|\nabla u_{n}^{\prime}\right|\right. & \left(\left|\Delta u_{0 n}\right|+\int_{0}^{t}\left|\Delta u_{n}^{\prime}(s)\right| d s\right)+\|f\|_{2,2}+\bar{c}  \tag{3.32}\\
& \left.+\left(\left|\Delta u_{0 n}\right|+\int_{0}^{t}\left|\Delta u_{n}^{\prime}(s)\right| d s\right)|\nabla f|_{4}^{2}+\cdots\right\} .
\end{align*}
$$

If $r=\infty$, then estimate (3.30) and Gronwall's inequality yield $\left|\Delta u_{n}^{\prime}\right|_{C\left(0, t ; L^{2}(G)\right)} \leqq$ const., where the const. depends on

$$
\begin{equation*}
\varepsilon k_{0}, k_{1}, L_{k}, T, N=2,|G|,|f|_{L^{r}\left(S, W^{1,4}(G)\right)},\left|\Delta u_{0}\right|_{2} . \tag{3.33}
\end{equation*}
$$

If $r \in[1, \infty)$, take on both sides of (3.32) the $r$ th power and integrate over $(0, t) \subset[0, T]$, which yields by Gronwall's inequality

$$
\left|\Delta u_{n}^{\prime}\right|_{L^{r}\left(0, t ; L^{2}(G)\right)} \leqq \text { const. }
$$

Because of $u_{n} / \partial G=\Delta u_{n}^{\prime} / \partial G=0$, this implies

$$
\left\|u_{n}\right\|_{W^{1, r}\left(S, W^{2,2}(G) \cap H_{0}^{1}(G)\right)} \leqq \text { const }=\hat{c},
$$

$\hat{c}$ as in (3.33). The usual compactness argument finishes the proof and yields an estimate

$$
\begin{equation*}
\|u\|_{W^{1, r}\left(S, W^{2,2}(G) \cap H_{0}^{1}(G)\right)} \leqq \hat{c} \tag{3.34}
\end{equation*}
$$

for the solution $u$ of (2.1), (2.2).
Proof of part (b). (Sketch, a similar argument is used in the proof of Corollary 2.4.) Under the conditions of part (b) we set that (2.1), (2.2) holds in $L^{1}(G)$ a.e. Therefore, we can multiply equation (2.2) on both sides by $v:=-\left|\Delta u^{\prime}(t)\right|^{p-2} \Delta u^{\prime}(t)$ and arrive, using similar arguments as in (a) and estimate (3.34), at

$$
\begin{equation*}
\|u\|_{W^{1, r}\left(S, W^{2, p}(G)\right)} \leqq c\left\{1+|\Delta f|_{L^{\prime}\left(S, L^{p}(G)\right)}+\left|\Delta u_{0}\right|_{p}\right\} \tag{3.35}
\end{equation*}
$$

which is sufficient to complete the proof. $c$ depends via (3.34) on the data.
Proof of Corollary 2.4. We approximate $f$ and $u_{0}$ respectively by regular $f_{\delta}, u_{0 \delta}$, and obtain by Theorem 2.6 regular solutions $u_{\delta}$ for which we show the estimate (3.39). The basic tool to obtain (3.39) are the estimates (3.42), (3.43) for a related linear problem (3.40). Let

$$
\begin{equation*}
f_{\delta} \in C\left(S, W^{2, \infty}(G) \cap W_{0}^{1, \infty}(G)\right), u_{0 \delta} \in W^{2, \infty}(G) \cap W_{0}^{1, \infty}(G) \tag{3.36}
\end{equation*}
$$

such that

$$
\begin{equation*}
f_{\delta} \rightarrow f \text { in } C\left(S, W_{0}^{1+\tau, p^{*}}(G)\right), \quad u_{0 \delta} \rightarrow u_{0} \quad \mathrm{n} W_{0}^{1+\tau, p^{*}}(G) . \tag{3.37}
\end{equation*}
$$

By Theorem 2.6, there are solutions

$$
\begin{equation*}
u_{\delta} \in C^{1}\left(S, W^{2, \infty}(G) \cap W_{0}^{1, \infty}(G)\right) \tag{3.38}
\end{equation*}
$$

satisfying (2.1), (2.2) with $u_{0 \delta}, f_{\delta}$ as data. Set $\hat{k}(t, x):=k\left(u_{\delta}(t, x)\right), \hat{\gamma}(t, x):=\hat{k}(t, x)$. $\alpha^{\prime}\left(u_{\delta}(t, x)\right)$. By our hypotheses, $0<k_{0} \leqq \hat{k}(t, x) \leqq k_{1} \forall t \in S, x \in G$ and $|\hat{\gamma}|_{\infty}<\infty$. It is our goal to show the following estimate. There is a constant $c$ merely depending on the
bounds of the coefficients $k, \alpha$ and their derivatives as appearing in (H1)-(H6) and on

$$
\left\|u_{0}\right\|_{1+\tau, p^{*}},\|f\|_{C\left(S, W^{1+\tau, p^{*}}(G)\right)}
$$

such that

$$
\begin{equation*}
\left\|u_{\delta}\right\|_{C^{1}\left(S, W_{0}^{1+\tau, p^{*}}(G)\right)} \leqq c . \tag{3.39}
\end{equation*}
$$

To this end consider for given $g, v_{0}$ the linear problem

$$
\begin{align*}
& v^{\prime}-\varepsilon \operatorname{div}\left(\hat{k} \nabla v^{\prime}\right)-\operatorname{div}(\hat{\gamma} \nabla v)=g-\varepsilon \operatorname{div}(\hat{k} \nabla), \\
& v(0)=v_{0} \tag{3.40}
\end{align*}
$$

We note that choosing $g:=f_{\delta}, v_{0}:=u_{0 \delta}, v:=u_{\delta}$ satisfies this equation.
Denote by $S_{t}:=[0, t], t \in S$, a subinterval of $S$. As in Theorems 2.3 and 2.5 one shows the existence of solution operators $P_{i, t}, i=0,1$,

$$
\begin{aligned}
& P_{0, t}: H_{0}^{1}(G) \times C\left(S_{t}, H_{0}^{1}(G)\right) \rightarrow C\left(S_{t}, H_{0}^{1}(G)\right), \\
& P_{1, t}:\left(W^{2, p}(G) \cap W_{0}^{1, p}(G)\right) \times C\left(S_{t}, W^{2, p}(G)\right) \rightarrow C\left(S_{t}, W^{2, p}(G) \cap W_{0}^{1, p}(G)\right), \\
& P_{i, t}:\left\{v_{0}, g\right\} \mapsto v^{\prime}, v_{0}, g, v, v^{\prime} \quad \text { as in }(3.40) .
\end{aligned}
$$

(The numbers $p$ and $p^{*}$ are related by the hypotheses of this corollary.) These operators are linear and bounded in the given spaces and we denote by $M_{i, t}$ their respective norms. By interpolation it follows that $P_{1, t}$ can be restricted to $W_{0}^{1+\tau, p^{*}}(G) \times$ $C\left(S_{t}, W^{1+\tau, p^{*}}(G)\right.$ ) (see Lemma A5, appendix) and, denoting the restriction by $P_{\tau, t}$,

$$
P_{\tau, t}: W_{0}^{1+\tau, p^{*}}(G) \times(G) \times C\left(S_{t}, W_{0}^{1+\tau, p^{*}}(G)\right) \rightarrow C\left(S_{t}, W_{0}^{1+\tau, p^{*}}(G)\right), \quad \tau \in(0,1) .
$$

$P_{\tau, t}$ is linear and bounded and its norm $M_{\varepsilon, t}$ can be estimated by

$$
\begin{equation*}
M_{\tau, t} \leqq c M_{0, t}^{1-\tau} M_{1, t}^{\tau} \tag{3.41}
\end{equation*}
$$

with some numerical constant depending on $\tau$, but independent of $t$. We prove the following

Lemma 3.9. There exists a constant $c$, depending at most on $\tau, G, T, p^{*}$ the several bounds of the coefficients as appearing in the hypotheses of this corollary, such that
(a) $M_{0, t} \leqq c$,
(b) $\quad M_{1, t} \leqq c\left\{1+\left\|u_{\delta}\right\|_{C\left(S_{t}, W^{1+\tau, p^{*}}(G)\right)}^{1 /(1-\theta)}\right\}$.

Proof of Lemma 3.9. First we notice that due to the rather technical assumptions of this corollary we have

$$
\begin{align*}
& W_{0}^{1+\tau, p^{*}}(G) \subset W_{0}^{1, a p}(G),  \tag{3.44}\\
& {\left[W_{0}^{1,2}(G), W^{2, p}(G) \cap W_{0}^{1, p}(G)\right]_{\tau}=W_{0}^{1+\tau, p^{*}}(G)} \tag{3.45}
\end{align*}
$$

(see the appendix, Lemmas A4, A6). The proof of (a) is at the beginning almost identical to that of estimate (3.27) in Lemma 3.8. To show (3.43) we begin with the case $p=2$. Computing the div-terms in (3.40), multiplying by $w:=-\Delta v^{\prime}(t)$ and integrating
over $G$ by parts, one obtains for $s \in S_{t}$

$$
\begin{aligned}
& \left|\nabla v^{\prime}(s)\right|^{2}+\left(\varepsilon \hat{k} \Delta v^{\prime}, \Delta v^{\prime}\right) \\
& \quad=\left(\varepsilon \hat{k} \Delta g+\hat{\gamma} \Delta v, \Delta v^{\prime}\right)+\left(g,-\Delta v^{\prime}\right)+\left(\varepsilon \nabla \hat{k} \cdot \nabla v^{\prime}+\nabla \hat{\gamma} \cdot \nabla v,-\Delta v^{\prime}\right)+\left(\varepsilon \nabla \hat{k} \cdot \nabla g, \Delta v^{\prime}\right) .
\end{aligned}
$$

To obtain (3.43), it is sufficient to assume that

$$
\begin{equation*}
\left\|v_{0}\right\|_{2, p}+\|g\|_{C\left(S_{t}, W^{2, p}(G)\right)} \leqq 1 . \tag{3.46}
\end{equation*}
$$

Using the boundedness properties of $\hat{k}$ and $\hat{\alpha}$ and Hölder's inequality, we obtain with $c=c\left(\varepsilon k_{0}, k_{1}, L_{\alpha}\right)$

$$
\begin{aligned}
& \left|\nabla v^{\prime}(s)\right|^{2}+\varepsilon k_{0}\left|\Delta v^{\prime}(s)\right|^{2} \\
& \quad \leqq c\left\{|\Delta g|+|\Delta v|+|g|+|\nabla \hat{k}|_{4}\left|\nabla v^{\prime}\right|_{4}+|\nabla \hat{\gamma}|_{4}|\nabla v|_{4}+|\nabla \hat{k}|_{4}|\nabla g|_{4}\right\}\left|\Delta v^{\prime}\right| .
\end{aligned}
$$

By $v^{\prime} / \partial G=0$, there is a constant $c=c(G)$,

$$
\begin{equation*}
\left|\Delta v^{\prime}(s)\right|_{p}+\left|v^{\prime}(s)\right|_{p} \geqq c\left\|v^{\prime}(s)\right\|_{2, p} \tag{3.47}
\end{equation*}
$$

by interpolation, $\left|\nabla v^{\prime}\right|_{4}^{2} \leqq c(G)\left|\Delta v^{\prime}\right|_{2}\left\|v^{\prime}\right\|_{2,2}$. This and (3.46) imply by using Young's inequality and (3.42),

$$
\begin{aligned}
\left\|v^{\prime}(s)\right\|_{2,2} \leqq & c\left\{\|g\|_{2,2}+\left\|v_{0}\right\|_{2,2}+\int_{0}^{s}\left\|v^{\prime}(r)\right\|_{2,2} d r\right. \\
& \left.\quad+\left.|\nabla \hat{k}|_{4}^{2}\right|^{2}\left|\nabla v^{\prime}\right|_{2}+|\nabla \hat{\gamma}|_{4}^{2}|\nabla v|_{2}+\|v\|_{2,2}+|\nabla \hat{\gamma}|_{4}\|g\|_{2,2}\right\} \\
\leqq & c\left\{1+|\nabla \hat{k}|_{4}^{2}+|\nabla \hat{\gamma}|_{4}^{2}+\int_{0}^{s}\left|v^{\prime}(r)\right|_{2,2} d r\right\} .
\end{aligned}
$$

Because of

$$
\left.\begin{array}{rl}
|\nabla \hat{k}|_{4}+|\nabla \hat{\gamma}|_{4} & \leqq c\left\|u_{\delta}\right\|_{1,4} \\
& \leqq\left\|u_{\delta}\right\|_{3 / 2,2}
\end{array} \quad \text { (cf. def. of } \hat{k}, \hat{\gamma}\right)
$$

this and Gronwall's inequality imply (3.43) for $p=2, \tau=\frac{1}{2}$. To deal with the general case $p \geqq 2, \tau \in(0,1)$ as in the assumptions of this corollary, we compute in (3.40) the div-terms, multiply by $w:=-\left|\Delta v^{\prime}(s)\right|^{p-2} \Delta v^{\prime}(s)$, integrate over $G$ parts, use the boundedness of $\hat{k}$ and $\hat{\gamma}$ as implied by (H1), (H2) and apply Hölder's inequality to arrive at

$$
\begin{align*}
&\left|\Delta v^{\prime}(s)\right|_{p} \leqq c\left\{\left|v^{\prime}(s)\right|_{p}+|g|_{p}+|\nabla \hat{k} \cdot \nabla g|_{p}+\left|\hat{k}_{\Delta g}\right|_{p}\right. \\
&\left.+|\hat{\gamma} \Delta v|_{p}+|\nabla \hat{\gamma} \cdot \nabla v|_{p}+\left|\nabla \hat{k} \cdot \nabla v^{\prime}\right|_{p}\right\}\left|\Delta v^{\prime}(s)\right|_{p} .
\end{align*}
$$

By (3.42),

$$
\left|v^{\prime}(s)\right|_{p} \leqq c(G)\left|v^{\prime}\right|_{C\left(S_{t}, H_{0}^{1}(G)\right)} \leqq c\left\{\left\|v_{0}\right\|+\|g\|_{C\left(S_{t}, H_{0}^{1}(G)\right)}\right\},
$$

so that by (3.46) and the embedding theorems

$$
\left|v^{\prime}(s)\right|_{p} \leqq c .
$$

Furthermore, $|\nabla g|_{a^{\prime} p} \leqq c\|g\|_{2, p}$ (we have $N=2, p \geqq 2$ !), so that by (3.46) $|\nabla g|_{a^{\prime} p} \leqq c$ (indep. of $g$ ). By Sobolev's inequalities $\left|\nabla v^{\prime}\right|_{a^{\prime} p} \leqq c\left(G, a^{\prime}, p, p\right)\left|\nabla v^{\prime}\right|_{2}^{1-\theta}\left\|v^{\prime}\right\|_{2, p}^{\theta}(\theta$ defined in the assumptions) $|v|_{a^{\prime} p} \leqq c\left(G, a^{\prime}, p, p\right)\left|\nabla v^{\prime}\right|_{2}^{1{ }^{1}-\boldsymbol{\theta}}\left\|v^{\prime}\right\|_{2, p}^{\theta}$,

$$
\begin{aligned}
\left|\nabla v^{\prime}(s)\right|_{2}+|\nabla v(s)|_{2} & \leqq c\left\{\left\|v_{0}\right\|+\|g\|_{C\left(S_{t}, H_{0}^{1}(G)\right)}\right\} \quad \text { by }(3.42) \\
& \leqq c \quad \text { by }(3.46) .
\end{aligned}
$$

Finally, by (3.44), $\left|\nabla \hat{k}\left\|_{a_{p}}+|\nabla \hat{\gamma}|_{a_{p}} \leqq c\right\| u_{\delta}\left\|_{1, a_{p}} \leqq c\right\| u_{\delta} \|_{1+\tau, p^{*}}\right.$. Therefore, (3.47), (3.48') and Young's inequality imply

$$
\begin{aligned}
\left\|v^{\prime}(s)\right\|_{2, p} & \leqq c\left\{1+\left\|u_{\delta}\right\|_{1+\tau, p^{*}}^{1 /(1-\theta)}+\|v(s)\|_{2, p}\right\} \\
& \leqq c\left\{1+\left\|u_{\delta}\right\|_{1+\tau, p^{*}}^{1 /(1-\theta)}+\left\|v_{0}\right\|_{2, p}+\int_{0}^{s}\left|v^{\prime}(r)\right|_{2, p} d r\right\} .
\end{aligned}
$$

Set for abbreviation $V:=W_{0}^{1+\tau, \tau^{*}}(G)$. Gronwall's inequality and (3.46) show that

$$
\begin{gather*}
\left|v^{\prime}\right|_{C\left(S_{t}, W^{2, p}(G)\right)} \leqq c\left\{1+\left\|u_{\delta}\right\|_{C\left(S_{t}, V\right)}^{1 /(1-\theta)}\right\} \cdot \exp (c T)  \tag{3.48}\\
\text { where } c=c\left(\varepsilon k_{0}, k_{1}, L_{\alpha},|G|, T, p, \theta, \tau, p^{*},\left|k^{\prime}\right|_{\infty},\left|\alpha^{\prime \prime}\right|_{\infty}\right)
\end{gather*}
$$

which implies (3.43). Lemma 3.9 is proved.
To show (3.9), we first notice, that by (3.37), (3.41)-(3.43) and by $\theta+\tau \leqq 1,{ }^{4}$

$$
\begin{aligned}
\left\|u_{\delta}^{\prime}\right\|_{C\left(S_{t}, V\right)} & \leqq c\left\{1+\left\|u_{\delta}\right\|_{C\left(S_{t}, V\right)}^{1 /(1-\theta)}\right\}^{\tau}\left\{\left\|u_{0 \delta}\right\|_{V}+\left\|f_{\delta}\right\|_{C\left(S_{t}, V\right)}\right\} \\
& \leqq c\left\{1+\left\|u_{\delta}\right\|_{C\left(S_{t}, V\right)}\right\}\left\{\left\|u_{0}\right\|_{V}+\|f\|_{C(S, V)}\right\} \\
& \leqq c\left\{1+\left\|u_{\delta}\right\|_{C\left(S_{t}, V\right)}\right\} \\
& \leqq c\left\{1+\left\|u_{0 \delta}\right\|_{V}+\int_{0}^{t}\left\|u_{\delta}^{\prime}(s)\right\|_{V} d s\right\} .
\end{aligned}
$$

Gronwall's inequality and (3.37) yield (3.39). Therefore, problem (3.40) with $u_{0 \delta}, f_{\delta}$ as data has $u_{\delta}$ as a solution, which by (3.39) satisfies

$$
\left\|u_{\delta}\right\|_{W^{1, \infty}(S, V)} \leqq \text { const. }
$$

The usual compactness argument yields a (sub-) sequence

$$
u_{\delta} \stackrel{*}{\xrightarrow{*} u} \text { in } W^{1, \infty}(S, V) .
$$

By means of the approximate equation (3.40) one easily shows that $u$ satisfies (2.1), (2.2). This completes the proof of Corollary 2.4.

Proof of Corollary 2.5. (a) follows directly from Corollary 2.4.
(b) set $a:=q / p$ and apply (a). Then, $u \in C^{1}\left(S, W_{0}^{1+\tau, p^{*}}(G)\right)$ and by (3.44), $u \in$ $C^{1}\left(S, W_{0}^{1, q}(G)\right)$.

Proof of Theorem 2.7. Let $u_{i}$ be a solution of (2.1), (2.2) with respect to the initial value $u_{0 i}$ and the right-hand side $f_{i}, i=1,2$. Set for abbreviation $f:=f_{1}-f_{2}, w:=u_{1}-u_{2}$,

$$
{ }^{4} V:=W_{0}^{1+\tau, p^{*}}(G),\|\cdot\|_{V}:=\|\cdot\|_{1+\tau, p^{*}} .
$$

$w_{0}:=u_{01}-u_{02}$ and assume

$$
\begin{array}{ll}
u_{2} \in W^{1,1}\left(S, W_{0}^{1, p}(G)\right), & u_{1} \in W^{1,1}\left(S, H_{0}^{1}(G)\right) \\
f_{2} \in L^{1}\left(S, W^{1, p}(G)\right), & f_{1} \in L^{1}\left(S, H_{0}^{1}(G)\right)
\end{array}
$$

Subtract (2.2) for $u_{2}$ from that for $u_{1}$, choose $w^{\prime}$ as a test-function in the variational formulation, integrate over $G$ and make some rearrangements. Thus, for $t \in S$

$$
\begin{aligned}
\left|w^{\prime}(t)\right|^{2}+ & \varepsilon k_{0}\left|\nabla w^{\prime}(t)\right|^{2} \\
\leqq & \left|w^{\prime}(t)\right|^{2}+\varepsilon\left(k\left(u_{1}\right) \nabla w^{\prime}, \nabla w^{\prime}\right) \\
= & \left(f, w^{\prime}\right)+\varepsilon\left(k\left(u_{1}\right) \nabla f, \nabla w^{\prime}\right)+\varepsilon\left(\left(k\left(u_{1}\right)-k\left(u_{2}\right)\right) \nabla f_{2}, \nabla w^{\prime}\right) \\
& +\varepsilon\left(\left(k\left(u_{1}\right)-k\left(u_{2}\right)\right) \nabla u_{2}^{\prime}, \nabla w^{\prime}\right)+\left(\gamma\left(u_{1}\right) \nabla w, \nabla w^{\prime}\right) \\
& +\left(\left(\gamma\left(u_{1}\right)-\gamma\left(u_{2}\right)\right) \nabla u_{2}, \nabla w^{\prime}\right) .
\end{aligned}
$$

By using the Lipschitz continuity of $k$ and $\gamma$ and applying Hölder's inequality, we obtain

$$
\begin{aligned}
& \left|w^{\prime}(t)\right|^{2}+\varepsilon k_{0}\left|\nabla w^{\prime}(t)\right|^{2} \\
& \leqq|f|\left|w^{\prime}\right|+\left\{\varepsilon k_{1}\|f\|+\varepsilon L_{k}|w|_{p^{\prime}}\left\|f_{2}\right\|_{1, p}+\varepsilon L_{k}|w| p^{\prime}\left\|u_{2}^{\prime}\right\|_{1, p}+|\gamma|_{\infty}\|w\|+L_{\gamma}\left\|u_{2}\right\|_{1, p}|w|_{p^{\prime}}\right\}\left\|w^{\prime}(t)\right\| .
\end{aligned}
$$

By the imbedding theorems, $|w|_{p^{\prime}} \leqq c(G)\|w\|$. Also, $\|w(t)\|^{r} \leqq\left\|w_{0}\right\|^{r}+c \int_{0}^{t}\|w(r)\|^{r} d r$, if $r \in[1, \infty)$, with $c=c(r, T)$. This and Young's inequality imply for $\bar{r}:=r$ if $r<\infty, \bar{r}=1$ if $r=\infty$,

$$
\begin{aligned}
& \psi:=\left\|f_{2}\right\|_{1, p}^{\bar{r}}+\left\|u_{r}^{\prime}\right\|_{1, p}^{\bar{r}}+\left\|u_{2}\right\|_{1, p}^{\bar{r}} \\
& \left\|w^{\prime}(t)\right\|^{r} \leqq \varepsilon k_{1}\|f(t)\|^{r}=c \psi(t)\left\{\left\|w_{0}\right\|^{r}+\int_{0}^{t}\left\|w^{\prime}(s)\right\|^{r} d s\right\} .
\end{aligned}
$$

Integration and Gronwall's inequality yield

$$
\left\|w^{\prime}\right\|_{L^{r}\left(S, H_{0}^{1}(G)\right)} \leqq c\left\{\|f\|_{L^{r}\left(S, H_{0}^{1}(G)\right.}+\left\|w_{0}\right\| \cdot|\psi|_{L^{1}(S)}\right\} \cdot \exp \left\{c|\psi|_{L^{1}(S)}\right\} .
$$

For $r:=\infty$ we obtain

$$
\left\|w^{\prime}\right\|_{L^{\infty}\left(S, H_{0}^{1}(G)\right)} \leqq c\left\{\|f\|_{L^{\infty}\left(S, H_{0}^{1}(G)\right)}+\left\|w_{0}\right\| \cdot|\psi|_{L^{\infty}(S)}\right\} \cdot \exp \left\{c|\psi|_{L^{\infty}(S)}\right\} .
$$

This proves (b) of the theorem, and (a) follows by setting $u_{01}=u_{02}, f_{1}=f_{2}$.
Proof of Theorem 2.8. First of all, notice that $\hat{k}(t):=(u(t, \cdot)) \in L^{\infty}(G), \hat{k}(t, x) \geqq k_{0}$ $>0$ for a.a. $t \in S, x \in G$. Thus, $A(t):=-\operatorname{div}(\hat{k}(t) \nabla(\cdot))$ is an $m$-accretive operator on $L^{1}(G)$. Now, [6, Thm. 1] yields the desired estimates. If $u_{0} \geqq 0, f(T) \geqq 0$, then obviously, $|u(t)|_{L^{\infty}(G)}=0$, i.e., $u(t, s) \geqq 0$ a.e.

Appendix. We list some facts on complex interpolation of $B$-spaces and, in particular, Sobolev spaces. Basic references are Bergy-Löfstrom [3], Lions-Magenes [13], and Triebel [18].

Let $\tau \in[0,1]$ be a parameter, $\bar{A}:=\left\{A_{0}, A_{1}\right\}, \bar{B}:=\left\{B_{0}, B_{1}\right\}$-two couples of compatible $B$-spaces " $[\cdot, \cdot]_{\tau}$ " denotes the interpolation functor for the complex interpolation method (cf. [3], [13], [18]), $\overline{A_{\tau}}:=\left[A_{0}, A_{1}\right]_{\tau}$. We have

Lemma A1. $\left[A_{0} \times B_{0}, A_{1} \times B_{1}\right]_{\tau}=\left[A_{0}, A_{1}\right]_{\tau} \times\left[B_{0}, B_{1}\right]_{\tau}$ (algebraically and topologically).

Lemma A2. Let $\mathscr{P} \in \mathscr{L}\left(A_{i}, B_{i}\right)$ with norm $M_{i}, i=0,1$. Then $\mathscr{P} \in \mathscr{L}\left(\bar{A}_{\tau}, \bar{B}_{\tau}\right)$ and $\mid \mathscr{P}_{\mathscr{L}\left(\bar{A}_{r}, \bar{B}_{\tau}\right)} \leqq M_{0}^{1-\tau} M_{1}^{\tau}$.

LEMMA A3. [ $\left.C\left(S, A_{0}\right), C\left(S, A_{1}\right)\right]_{\tau}=C\left(S, \overline{A_{\tau}}\right)$ (algebraically and topologically).
Lemma A4. (a) Let $s_{i} \in \mathbb{R}, p_{i} \in(1, \infty), \tau \in(0,1), s^{*}:=(1-\tau) s_{0}+\tau s_{2}$,

$$
\frac{1}{p^{*}}=\frac{1-\tau}{p_{0}}+\frac{\tau}{p_{1}}
$$

$s_{0} \neq s_{1}, G \subset \mathbb{R}^{N} a($ bounded $)$ domain. Then $\left[W^{s_{0}, p_{0}}(G), W^{s_{1}, p_{1}}(G)\right]_{\tau}=W^{s^{*}, p^{*}}(G)$.
(b) Let

$$
p \in(1, \infty), \quad \tau \in(0,1), \quad \frac{1}{p}=\frac{1-\tau}{2}+\frac{\tau}{p} .
$$

Then $\left[W_{0}^{1,2}(G), W^{2, p}(G) \cap W_{0}^{1, p}(G)\right]_{\tau}=W_{0}^{1+\tau, \bar{p}}(G)$.
Lemma A5. Let $\tau, p$ and $\bar{p}$ be as in Lemma A4(b), $S:=[0, T]$ an interval of any finite length,
$\mathscr{P}_{1}:\left(W^{2, p}(G) \cap W_{0}^{1, p}(G)\right) \times C\left(S, W^{2, p}(G)\right) \rightarrow C\left(S, W^{2, p}(G) \cap W_{0}^{1, p}(G)\right)$
a linear and bounded operator with norm $M_{1}$,
$\mathscr{P}_{0}$ a linear and bounded extension of $\mathscr{P}_{1}$, such that

$$
\mathscr{P}_{0}: W_{0}^{1,2}(G) \times C\left(S, W^{1,2}(G)\right) \rightarrow C\left(S, W_{0}^{1,2}(G)\right) .
$$

Then
$\mathscr{P}_{\tau}:=$ restriction of $\mathscr{P}_{0}$ to $W_{0}^{1+\tau, \bar{p}}(G) \times C\left(S, W^{1+\tau, \bar{p}}(G)\right)$ maps continuously into $C\left(S, W_{0}^{1+\tau, \bar{p}}(G)\right)$.
Let $M_{0}$ denote the norm of $\mathscr{P}_{0}, M_{\tau}$ that of $\mathscr{P}_{\tau}$. Then

$$
M_{\tau} \leqq M_{0}^{1-\tau} M_{1}^{\tau} .
$$

Note that this estimate does not depend on $S$.
References/proofs. Lemma A1 follows from [3, Thm. 4.1.2]. Lemma A2 is a special case of a theorem in [11]. Lemma A3 can be found in [13], if $A_{1}, A_{0}$ are Hilbert spaces, otherwise cf. [10]. Lemma A4(a)—cf. [3, Thm. 6.4.5], (b)—cf. [18, Thm. 4.3.3]. Lemma A5 follows from Lemmas A1-A4.

Lemma A6. [3, Thm. 6.5.1]. Let $s_{i} \in \mathbb{R}, p_{i}>1, i=1,2$. If $s_{1}-N / p_{1} \geqq s_{2}-N / p_{2}$, then $W^{s_{1}, p_{1}}(G) \subset W^{s_{2}, p_{2}}(G)$.

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[^1]:    ${ }^{1}$ A complete reference for these spaces may be found in Adams [1].

[^2]:    ${ }^{2}$ " $\subset$ " denotes algebraic and topological imbedding, " $\subset \subset$ " the compact imbedding.
    3 " $[\cdot, \cdot]$ " is the complex interpolation space generator (see appendix).

