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DIFFUSION IN FISSURED MEDIA*

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Abstract. The nonlinear initial-boundary value problem

$$\frac{\partial u}{\partial t}+\frac{1}{\varepsilon}(\alpha(u)-v)=f_1, \qquad -\operatorname{div}(k\operatorname{grad} v)+\frac{1}{\varepsilon}(v-\alpha(u))=f_2 \quad \text{in } G\times(0,T),$$
$$u(x,0)=u_0(x) \quad \text{in } G, \qquad v(s,t)=0 \quad \text{on } \partial G\times(0,T)$$

is a well-posed model of diffusion in a fissured porous medium. Special features of the solution include the perseverance of local spatial continuity or singularities in the concentration u, the instantaneous propagation of the partially-saturated region throughout G, the delayed and limited advance of the fully-saturated region into G, and the concentration discontinuity on the boundary of the fully-saturated region. Weak maximum and order-comparison principles are obtained as L^∞ and L^1 estimates on a solution and a difference of solutions, respectively.

1. Introduction. Our objectives here are to derive a system of partial differential equations as a model for nonstationary flow of a fluid through a fissured porous medium, to demonstrate that the appropriate initial-boundary-value problem is mathematically well-posed, and to describe special features of such a flow model which distinguish it from the classical porous medium equation. The system obtained is actually equivalent to a single evolution equation, the *fissured medium equation*, which can be regarded as a regularization of the porous medium equation. Also, the porous medium equation is known to be the homogeneous limit of the fissured medium equation with increasing degree of fissuring [7].

Section 2 contains the derivation of the system of differential equations for flow in fissured media. Initially we follow [1], where only a special linear case was considered, but we include in our model the nonlinearities arising from fluid type (liquid or gas), concentration (porosity, absorption or saturation), and the exchange rate [6], [11]. The essential requirement is that the fluid be compressible. The considerably more difficult case wherein permeability is concentration-dependent will be discussed in [2]. We briefly describe an analogous heat conduction model. In §3 we show the Cauchy problem for the fissured medium equation has a unique generalized solution and we give weak maximum and order-comparison principles in the form of L^∞ and L^1 estimates. In contrast to the case of (possibly degenerate) parabolic equations, we find that for the fissured medium equation the local spatial regularity or a singularity in the solution is stationary and may persevere for all time. We consider in §4 the evolution of a system originating with a uniform positive pressure in a portion G' of G and with null concentration in the complement of G' in G. It is shown that the partially-saturated region expands instantly to all of G, the positive-pressure set is nondecreasing, propagates only after a delay, and an upper bound is given for its measure.

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Our notation is standard. G is a *bounded domain* in Euclidean space $\mathbf{R}^N$, $Q=G\times(0,T)$ is the indicated space-time cylinder, and ∂G denotes the boundary of G. $L^p(G)$ and $W^{m,p}(G)$ are the usual Lebesgue and Sobolev spaces, and $C^{m,\lambda}(G)$ is the Schauder space of functions whose derivatives of order m are Hölder-continuous with exponent λ, $0<\lambda<1$. For a Banach space B, we let $L^q(0,T;\ B)$ and $C(0,T;\ B)$ denote the spaces of q-summable or uniformly-continuous B-valued functions on $[0,T]$, respectively, and $W^{1,q}(0,T;\ B)$ denotes those strongly absolutely continuous functions whose derivatives belong to $L^q(0,T;\ B)$. The positive and negative parts of $u\in\mathbb{R}$ are given by $u^+=\max(u,0)$ and $u^-=\min(u,0)$, respectively, so $u=u^++u^-$. The Heaviside function is $H_0(u)=1$ for $u>0$ and $H_0(u)=0$ for $u\leqq 0$; its maximal monotone extension [3] is denoted by $H(u)=\{H_0(u)\}$ for $u\neq 0$ and $H(0)=[0,1]$. Likewise the sign function is $\operatorname{sgn}_0(u)=u/|u|$ for $u\neq 0$, $\operatorname{sgn}_0(0)=0$, and sgn denotes the maximal monotone extension. The gradient in $\mathbb{R}^N$ will be denoted by $\vec{\nabla}$ and similarly $\vec{\nabla}\cdot$ denotes the corresponding divergence operator.

2. Fissured medium equation. We consider the flow of a liquid or gas through a fissured porous medium, a structure consisting of porous permeable blocks separated by a system of fissures. The distribution of fissures prevents direct diffusion between adjacent blocks, and the system of fissures occupies a region of negligible relative volume. Thus the blocks provide for the local storage of fluid mass, and the fissures are the essential flow-paths for all the diffusion. The essential point in the construction of the fissured medium model is to introduce at each point in space two fluid pressures, the pressure p_1 in the blocks and the pressure p_2 in the fissures, where each is an average over a neighborhood which contains a substantial number of blocks.

The fluid under consideration may be any compressible liquid or gas whose density ρ and pressure p are related by an equation-of-state $\rho=s(p)$ for which the compressibility satisfies $s'(p)>0$ for $p\geqq 0$ and $s(0)\geqq 0$. The total concentration of fluid is given by $u=P(s(p_1)-s(0)+\xi(L+s(0)))$ where $P>0$ is porosity of the blocks, $L\geqq 0$ is that density of fluid which is immobilized due to absorption or chemical reaction with the medium, and the saturation level $0\leqq\xi\leqq 1$ is that fraction of $L+s(0)$ already immobilized or absorbed. Note $p_1(1-\xi)=0$ so $\xi\in H(p_1)$, the Heaviside graph. Thus u is a monotone graph of p_1 whose inverse $p_1\equiv\alpha(u)$ is a monotone function Lipschitz-continuous with constant K. The medium is *completely saturated* when $u\geqq P(L+s(0))$, hence, $\xi=1$, and *partially saturated* (*strictly partially saturated*) when $u>0$ (respectively, $0<u<P(L+s(0))$.)

The exchange of fluid between blocks and fissures occurs with a volume rate per volume of medium given by $(p_2-p_1)/\mu\varepsilon$ where μ is the viscosity of the fluid and $1/\varepsilon$ is a characteristic of the medium related to the degree of fissuring or the surface area common to the blocks and fissures. Thus the mass of fluid which flows from blocks to fissures per unit time is given by $\rho(p_1-p_2)/\mu\varepsilon$ where ρ is the average density on the pressure-interval $[p_1,p_2]$. Denoting by $S(p)\equiv\int_0^p s(r)\,dr$ the antiderivative of $s(p)$, or "flow potential" [6, p. 60], we have $\rho=(p_2-p_1)^{-1}\int_{p_1}^{p_2}s(p)\,dp=(p_2-p_1)^{-1}(S(p_2)-S(p_1))$. The fluid mass exchanged per unit time is $(S(p_1)-S(p_2))/\mu\varepsilon$. Thus the continuity equation for conservation of fluid mass in the blocks gives

$$\frac{\partial u}{\partial t}+\frac{1}{\varepsilon\mu}(S(p_1)-S(p_2))=f_1(x,t), \tag{2.1}$$

where f_1 is the volume-distributed source rate in the blocks.

We shall assume the velocity of the fluid in the fissures is given by Darcy's law. Thus, $V=-(k/\mu)\vec{\nabla}p_2$ where k is the permeability of the system of fissures. The flux in the fissures is computed by the chain rule as $\rho_2 V = -(k/\mu)\vec{\nabla} S(p_2)$. Since the relative volume of the fissure system is null, the enclosed concentration is negligible and the conservation of fluid mass in the fissure system gives

$$-\frac{1}{\mu}\vec{\nabla}\cdot k\vec{\nabla}S(p_2)+\frac{1}{\varepsilon\mu}(S(p_2)-S(p_1))=f_2(x,t). \tag{2.2}$$

Here f_2 denotes a volume-distributed source rate in the system of fissures.

The porosity and permeability may depend on the pressures. Given the small volume of the fissures the pressure p_2 will not appreciably affect the block porosity, so we may expect a mild dependence $P=P(p_1)>0$ of block porosity on block pressure. This does not alter the assumptions above on the relation $p_1=\alpha(u)$. Due to the relative volumes of blocks and fissures, any variation of the fissure permeability is essentially due to the block pressure p_1. This is equivalent to the assumption that the fluid in fissures does not participate in the support of the structure. In contrast to the slight variations of $k(p_1)$ for $p_1>0$, any swelling of the blocks due to saturation or absorption of fluid can result in a dramatic decrease of fissure permeability owing to their relative volumes. This sensitivity of permeability to saturation due to swelling is typical of consolidated sandstones containing clay or silt [6, p. 13]. We shall account for such phenomena by setting $k=k(u)$ in (2.2). The function $k(\cdot)$ is continuous, positive and nonincreasing on $0\leqq u$; furthermore, the model suggests $k(u)$ is essentially constant for $u\geqq P(L+s(0))$, the saturated zone.

In summary, the process of diffusion in a fissured medium is prescribed by the system of partial differential equations (2.1), (2.2) with $p_1=\alpha(u)$ and $k=k(u)$. The initial concentration $u(x,0)=u_0(x)$ is given over the region G of interest; this is equivalent to specifying initial block-pressure $p_1(x)$ and initial saturation $\xi_0(x)$ with $\xi_0(x)\in H(p_1(x))$. The description is completed by setting fissure-pressure $p_2=0$ on the boundary of the region G. Note that no boundary conditions are given for p_1, since all fluid flow in the blocks is accounted for in (2.1), even in a neighborhood of the boundary. Similarly, the initial pressure distribution in the fissures is determined by (2.2).

We shall write the above system as a single nonlinear evolution equation. Thus, for a function u on G of a type made precise below, let $A_u(v)=-(1/\mu)\vec{\nabla}\cdot k(u)\vec{\nabla}v$ be the indicated linear elliptic partial differential operator in divergence form subject to null Dirichlet boundary conditions. By adding (2.1) and (2.2), then substituting (2.2) we obtain

$$(I+\varepsilon\mu A_{u(t)})\left(\frac{\partial u}{\partial t}\right)+A_{u(t)}(S(\alpha(u)))=(I+\varepsilon\mu A_{u(t)})f_1(t)+f_2(t). \tag{2.3}$$

Alternatively, we may resolve (2.2) for $S(p_2)$ and substitute in (2.1) to obtain

$$\frac{\partial u}{\partial t}+\frac{1}{\varepsilon\mu}\left[I-(I+\varepsilon\mu A_{u(t)})^{-1}\right]S(\alpha(u))=f_1(t)+(I+\varepsilon\mu A_{u(t)})^{-1}f_2(t). \tag{2.4}$$

The equation (2.4) we shall call the *fissured medium equation*. It is actually equivalent to the above system. When $S(\alpha(u))$ is smooth and satisfies the Dirichlet boundary condition, i.e., belongs to the domain of $A_{u(t)}$, then (2.4) implies the stronger form (2.3). Note that formally taking $\varepsilon\to 0^+$ in either one leads to the classical *porous medium equation*

$$\frac{\partial u}{\partial t}-\frac{1}{\mu}\vec{\nabla}\cdot k(u)\vec{\nabla}S(\alpha(u))=f_1+f_2 \tag{2.5}$$

when $L=0$ and to the Stefan free-boundary problem in weak form when $L>0$. This corresponds to increasing the degree of fissuring, $1/\varepsilon$, and thereby approximating the homogeneous limiting case (2.5) [3], [7].

We shall briefly describe an analogous model for heat conduction in a heterogeneous medium consisting of two components. This thermal conduction model is formally equivalent to the fissured medium equation. Thus, assume the first component occurs in small blocks isolated by the second component which is distributed throughout the medium with negligible measure. We permit the first component material to undergo a phase change as in a Stefan free-boundary problem. A model for the situation is water (the first component) contained in a metal (second component) structure of thin walls forming a structure much like an ice-cube tray. Letting T_1 and T_2 denote temperatures (averaged) in the water and metal, respectively, we obtain the system

$$\begin{aligned} &\frac{\partial u}{\partial t}+\frac{1}{\varepsilon}(T_1-T_2)=f_1, \\ (2.6)\qquad &-k\Delta T_2+\frac{1}{\varepsilon}(T_2-T_1)=f_2, \\ &u\in C(T_1)+LH(T_1). \end{aligned}$$

Here the heat content u is given by the specific heat $C(T_1)$ in water and the latent heat L in the melted region ($T_1>0$), k is the conductivity of the second component material, and the heat exchange between water chambers and metal dividers is assumed proportional to the difference of their temperatures. The local description and derivation of the equations follows exactly as in [13]. This system is formally equivalent to a special case of (2.1) and (2.2). Unlike the diffusion model, we are interested in temperatures which are not necessarily nonnegative; these are permitted in our discussion below. A completely-saturated region in the diffusion model corresponds to a completely melted or water region ($u\geqq L$) in the conduction model, and a strictly-partially-saturated region corresponds to a region of *mush*, a mixture of ice and water in equilibrium at the freezing temperature. As we shall see below, the solution to such a conduction model is dramatically different from the classical Stefan problem solution. Specifically, (2.6) is *not* the Stefan problem for the pseudo-parabolic equation of heat conduction [5] as given in [8].

3. The Lipschitz case. We begin our discussion of (2.4) by considering the special case in which $k(u)$ is independent of u but is a function of $(x,t)\in Q$. In the diffusion problem this corresponds to the case of a rigid structure in which the permeability is not affected by the total concentration (density and saturation). We shall denote by

$$A(t)v\equiv -\overline{\nabla}\cdot\big(k(x,t)\overline{\nabla}v\big)$$

the indicated elliptic differential operator whose coefficient $k\in L^\infty(Q)$ is assumed to satisfy $0<k_0\leqq k(x,t)$, a.e. $(x,t)\in Q$. In the Banach space $L^1(G)$ the domain of $A(t)$ is $\operatorname{dom}(A(t))=\{v\in W_0^{1,1}(G)\colon A(t)v\in L^1(G)\}$, where $A(t)v$ is understood in the sense of distributions. This L^1-realization of $A(t)$ can be obtained as the L^1-closure of its restriction to $L^p(G)$, $1<p<+\infty$. Each such restriction, including $A(t)$ itself, is a linear m-accretive operator on the corresponding Banach sapce, $L^p(G)$. See [4], [10] for these and additional properties of these elliptic operators in L^p. Here we shall consider the realization of the fissured medium equation (2.4) in $L^1(G)$ in the form

$$(3.1)\qquad u'(t)+\frac{1}{\varepsilon}\big(I-(I+\varepsilon A(t))^{-1}\big)\alpha(u(t))=f(t),\qquad 0\leqq t\leqq T.$$

Without loss of generality we have set $\mu=1$, $S=I$, and $f(t)=f_1(t)+(I+\varepsilon A(t))^{-1}f_2(t)$ in $L^1(G)$. We assume hereafter that α is a nondecreasing Lipschitz continuous function on $\mathbb{R}$ with $\alpha(0)=0$. Thus the substitution operator $v\mapsto\alpha(v)$ is Lipschitz on each $L^p(G)$ and we easily obtain the following L^p-existence-uniqueness result.

LEMMA 1. *If $u_0\in L^p(G)$ and $f\in L^q(0,T;\ L^p(G))$, $1\leqq p,\ q\leqq+\infty$, then there is a unique $u\in W^{1,q}(0,T;\ L^p(G))$ which satisfies* (3.1) *and* $u(0)=u_0$.

Proof. Since each $(I+\varepsilon A(t))^{-1}$ is a contraction and $\alpha(\cdot)$ is Lipschitz on each $L^p(G)$, it follows that the u-dependence in (3.1) is Lipschitz, uniformly in t. From [12] it follows that the operator-valued map $t\mapsto(I+\varepsilon A(t))^{-1}$ is strongly-measurable into $\mathscr{L}(H^{-1}(G),H_0^1(G))$ and an elementary closure argument shows it is strongly-measurable into $\mathscr{L}(L^p(G))$. The classical successive-approximations finishes the proof.

In order to obtain "pointwise estimates" on solutions of (3.1) we write it in the form

$$u'(t)+\frac{1}{\varepsilon}\alpha(u(t))=\frac{1}{\varepsilon}(I+\varepsilon A(t))^{-1}\alpha(u(t))+f(t),\qquad 0\leqq t\leqq T. \tag{3.2}$$

This splitting of (3.1) displays explicitly its structure as an ordinary differential equation (in t) and an elliptic partial differential equation (in x). Moreover, it suggests we consider the ordinary initial-value problem

$$w'(t)+\frac{1}{\varepsilon}\alpha(w(t))=g(t),\qquad 0\leqq t\leqq T,\quad w(0)=w_0. \tag{3.3}$$

For each $g\in L^1(0,T)$ and $w_0\in\mathbb{R}$ there is a unique solution $w\in W^{1,1}(0,T)$. If $w_j\,(j=1,2)$ are solutions corresponding to data g_j, w_0^j, we subtract the equations, multiply by $H_0(w_1(t)-w_2(t))$ and integrate to obtain (since $(\alpha(w_1)-\alpha(w_2))H_0(w_1-w_2)\geqq0$)

$$[w_1(t)-w_2(t)]^+\leqq[w_0^1-w_0^2]^++\int_0^t[g_1(s)-g_2(s)]^+\,ds.$$

Moreover, if each $g_j\in L^1(Q)$ and $w_0^j\in L^1(G)$, the above holds for a.e. $x\in G$ and a further integration over G yields

$$\|[w_1(t)-w_2(t)]^+\|_{L^1(G)}\leqq\|[w_0^1-w_0^2]^+\|_{L^1(G)}+\int_0^t\|[g_1(s)-g_2(s)]^+\|_{L^1(G)}ds,\quad 0\leqq t\leqq T.$$

Thus, the operator W: $L^1(G)\times L^1(Q)\to C(0,T;\ L^1(G))$ defined by (3.3) with $w=W(w_0,g)$ is an order-preserving contraction. The elliptic operator $A(t)$ satisfies a similar estimate [4, Lemma 3*]

$$\left\|\left[(I+\varepsilon A(t))^{-1}g\right]^+\right\|_{L^1(G)}\leqq\|g^+\|_{L^1(G)},\qquad g\in L^1(G),$$

and trivially so also does α: $L^1(G)\to L^1(G)$.

The relevance of the preceding remarks is that a solution of (3.1) is characterized by

$$u=W\big(u_0,(1/\varepsilon)(I+\varepsilon A)^{-1}\alpha(u)+f\big). \tag{3.4}$$

The right side of (3.4) is Lipschitz with an integral bound implying it has a unique fixed point. This provides an alternate proof of Lemma 1 with $p=q=1$ but, more important, it yields the following comparison principle.

LEMMA 2. *Let u_1 and u_2 be the respective solutions of the initial-value problem for* (3.1) *with data $u_0^1,u_0^2\in L^1(G)$ and $f_1,f_2\in L^1(Q)$. If $u_0^1\geqq u_0^2$ a.e. in G and if $f_1\geqq f_2$ a.e. in Q, then $u_1\geqq u_2$ in Q.*

Proof. For $j=1,2$, we have $u_j=\lim_{n\to\infty}\mathscr{W}_j^{(n)}(u_0^j)$ in $C(0,T;\ L^1(G))$ where $\mathscr{W}_j(v)\equiv W(u_0^j,(1/\varepsilon)(I+\varepsilon A)^{-1}\alpha(v)+f_j)$. The preceding remarks show $\mathscr{W}_1(v_1)\geqq\mathscr{W}_2(v_2)$ whenever $v_1\geqq v_2$ and $u_0^1\geqq u_0^2$, so the desired follows.

In a similar manner, we can deduce an L^∞ estimate on the solution. However, this procedure is inefficient and does not yield the optimal estimates in either case; these will be obtained below. Although (3.2) has so far served only to motivate the comparison and maximum principles and to provide elementary proofs, it will be used below to directly obtain very distinctive and surprising results on local regularity of solutions. All of these we state as follows.

THEOREM 1. *Let $\{A(t):0\leqq t\leqq T\}$ be the uniformly elliptic family of elliptic operators on $L^1(G)$ as given above. Suppose $\varepsilon>0$ and α is monotone with $\alpha(0)=0$ and Lipschitz constant K.*

(a) *For each $u_0\in L^p(G)$ and $f\in L^q(0,T;\ L^p(G))$, $1\leqq p,\ q\leqq+\infty$, there is a unique solution $u\in W^{1,q}(0,T;\ L^p(G))$ of (3.1) with $u(0)=u_0$.*

(b) *This solution satisfies*

$$\|u(t)^+\|_{L^\infty(G)}\leqq\|u_0^+\|_{L^\infty(G)}+\int_0^t\|f(s)^+\|_{L^\infty(G)}ds,\qquad 0\leqq t\leqq T,$$

and similar estimates for $\|u(t)^-\|_{L^\infty(G)}$, $\|u(t)\|_{L^\infty(G)}$.

(c) *For $j=1,2$ let u_j be the solution with corresponding data $u_0^j\in L^1(G)$, $f_j\in L^1(Q)$. Then*

$$\|[u_1(t)-u_2(t)]^+\|_{L^1}\leqq\|[u_0^1-u_0^2]^+\|_{L^1}+\int_0^t\|[f_1(s)-f_2(s)]^+\|_{L^1}ds,\qquad 0\leqq t\leqq T,$$

and similarly for $\|[u_1(t)-u_2(t)]^-\|_{L^1}$ and $\|[u_1(t)-u_2(t)]\|_{L^1}$.

(d) *Assume $p>N/2$ and G' is a subdomain whose closure is contained in G. There are constants $C>0$, $\lambda>0$ such that*

$$\begin{aligned}[u(x_1,t)-u(x_2,t)]^+\leqq{}&[u_0(x_1)-u_0(x_2)]^+\\&+\int_0^t[f(x_1,s)-f(x_2,s)]^+ds+C|x_1-x_2|^\lambda,\\&\qquad\qquad x_1,x_2\in G',\quad 0\leqq t\leqq T,\end{aligned}$$

and similarly for $[u(x_1,t)-u(x_2,t)]^-$ and $|u(x_1,t)-u(x_2,t)|$.

(e) *Assume $p>N/2$, and let $x\in G$, $v\in\mathbb{R}^N$ be a unit vector, and denote the saltus or jump of a function w at x by $\sigma(w(x))\equiv\lim_{h\to0^+}(w(x+hv)-w(x))$. Assume there is a $g\in L^1(0,T)$ for which*

$$|f(x+hv,t)|\leqq g(t),\quad 0<h<h_0,\quad 0\leqq t\leqq T$$

and each of $\sigma(u_0(x))$, $\sigma(f(x,t))$ exists. Then $\sigma(u(x,t))$ exists for each $t\in[0,T]$ and

$$\sigma(u(x,t))^+\leqq\sigma(u_0(x))^++\int_0^t\sigma(f(x,s))^+ds,$$

$$\sigma(u(x,t))^+\geqq e^{-Kt/\varepsilon}\left\{\sigma(u_0(x))^++\int_0^t e^{Ks/\varepsilon}\sigma(f(x,s))^-ds\right\}$$

with similar estimates for $\sigma(u(x,t))^-$ and $\sigma(u(s,t))$.

Proof. Part (a) is just Lemma 1. To prove (b) note first that the Yoshida approximation

$$A_\varepsilon(t)\equiv\frac{1}{\varepsilon}\left(I-(I+\varepsilon A(t))^{-1}\right),$$

satisfies the resolvent identity

$$(I+\lambda A_\varepsilon(t))^{-1}=(\varepsilon/(\varepsilon+\lambda))I+(\lambda/(\varepsilon+\lambda))(I+(\lambda+\varepsilon)A(t))^{-1}, \qquad \lambda>0,$$

which implies that $A_\varepsilon(t)$ satisfies the conditions in [4, Theorem 1]. From (3.1) in $L^\infty(G)$ subtract $\|f(t)^+\|_{L^\infty}$ and multiply by $H_0(u(x,t)-k-\int_0^t\|f^+(s)\|_{L^\infty}ds)$ where $k\geqq 0$ will be chosen below. Integrating the product gives

$$\int_G(u'(t)-\|f^+(t)\|_{L^\infty})H_0\left(u-k-\int_0^t\|f^+\|_{L^\infty}ds\right)ds$$
$$+\int_G A_\varepsilon(t)(\alpha(u(t)))H_0\left(u-k-\int_0^t\|f^+\|_{L^\infty}ds\right)dx\leqq 0.$$

The second term is nonnegative by the fundamental [4, Lemma 2] and the first term is equal to $\frac{d}{dt}\int_G[u(x,t)-k-\int_0^t\|f(s)^+\|ds]^+\,dx$. Thus we obtain

$$\int_G\left[u(x,t)-k-\int_0^t\|f(s)^+\|_{L^\infty}ds\right]^+dx\leqq\int_G[u_0(x)-k]^+dx,$$

and choosing $k=\|u_0^+\|_{L^\infty}$ proves (b). Part (c) is proved similarly: subtracting (3.1) for $j=1,2$ and multiplying by $H_0(u_1-u_2)$ yields

$$\frac{d}{dt}\int_G[u_1(x,t)-u_2(x,t)]^+dx+\int_G A_\varepsilon(t)(\alpha(u_1)-\alpha(u_2))H_0(u_1-u_2)\,dx$$
$$=\int_G(f_1-f_2)H_0(u_1-u_2)\,dx\leqq\|(f_1-f_2)^+\|_{L^1}.$$

The second term is nonnegative as before and this leads to the end of proof of (c).

Consider the situation of part (d). Since the solution u is bounded in $L^p(G)$, so also is $\alpha(u)$ and it follows that $v(t)\equiv(I+\varepsilon A(t))^{-1}\alpha(u(t))$ is bounded in a Schauder space $C^{0,\lambda}(G')$ [10, p. 192]. Thus, there is a constant C_1 for which

$$|v(x_1,t)-v(x_2,t)|\leqq C_1|x_1-x_2|^\lambda, \qquad x_1,x_2\in G', \quad 0\leqq t\leqq T.$$

The splitting (3.2) and the estimates following (3.3) with $w_j(t)=u(x_j,t)$ lead directly to the proof of (d).

For (e) we difference (3.2) at $x_1=x+hv$ and $x_2=x$ and use the preceding estimates and the Lebesgue theorem to obtain

$$\frac{d}{dt}\sigma(u(x,t))+\frac{1}{\varepsilon}\sigma(\alpha(u(x,t)))=\sigma(f(x,t)).$$

Since $\sigma(\alpha(u))=\alpha(\sigma(u))$, the desired estimates follow as above or by Gronwall's inequality. This finishes the proof.

Estimates of the forms in (b) and (c) are known as weak maximum principles and as comparison principles, respectively. Those given are optimal as can be seen by taking $\alpha\equiv 0$. They imply that nonnegative data yield nonnegative solutions.

If the coefficients $k(\cdot,t)$ and the boundary of G are smooth, then in the situation of (d) we get $v(t)$ bounded in $W^{2,p}(G)$, hence, in $C^{1,\lambda/2}(G)$. This leads to pointwise estimates on smoothness of first-order spatial derivatives of the solution. Such estimates on higher (than first) order derivatives appear to require assumptions on the global regularity of the data.

From (d) it follows the solution is exactly as smooth in x as $u_0(\cdot)$ and $\int_0^T f(\cdot,s)\,ds$, up to Hölder continuity with constant λ in each neighborhood in G. Likewise, (e) shows any jump discontinuity in data persists at the same point for a positive time interval, and for all time if $\sigma(f^-)\sigma(u_0)^+=0$ at that point. This striking persistence of local regularity is a consequence of the form (3.2) of the fissured medium equation.

We consider the meaning of a jump discontinuity in the solution of (3.1) when the equation is used as a model for diffusion. First, recall that the variables introduced in the diffusion model were defined pointwise as averages over a neighborhood of an idealized variable, e.g., pressure. It follows for an integrable ideal variable that such averaged variables are necessarily absolutely continuous in their spatial dependence. Thus within the medium the data and hence the solution are continuous. Second, we note that a discontinuity in data can be induced by fitting together two regions with independently prescribed concentration distributions. This discontinuity along the common interface will then persist on that stationary interface. This is consistent with the fissured medium diffusion model, because the two regions are coupled only by way of the fissure system, a relatively weak coupling.

4. Propagation and saturation. We consider now the fissured medium equation (3.1) and assume for definiteness that $\alpha(u)=0$ for $0\leqq u\leqq L$ and that $\alpha(u)>0$ for $u>L$. The medium is called *partially saturated* (or *strictly partially saturated*) at $(x,t)\in Q$ if $u(x,t)>0$ (respectively, $0<u(x,t)<L$). From Theorem 1 it follows that each strictly-partially-saturated point remains so over some time interval. In order to follow the advance of the fluid through the medium we consider for each $t\in[0,T]$ the set $P(t)\equiv\{x\in G: u(x,t)>L\}=\{x\in G:\alpha(u(x,t))>0\}$ wherein the block-pressure is strictly positive and, hence, the medium is *completely saturated.*

THEOREM 2. *In the situation of Theorem* 1 *assume further that* $\alpha^{-1}(0)=[0,L]$, $p>N/2$ *and both* u_0 *and* f *are nonnegative. Thus* $u\geqq 0$ *and we also have the following*:

(a) *The set* $P(t)$ *is nondecreasing in* $t\in[0,T]$. *If* $P(t_0)$ *is nonempty then the medium is partially saturated at every* $(x,t)\in Q$ *with* $t\geqq t_0$.

(b) *Assume* $f\equiv 0$, *let* G_1 *be a measurable subset of* G, ρ_0 *and* L *be strictly positive, and set* $u_0(x)=\rho_0+L$ *for* $x\in G_1$, *and* $u_0(x)=0$ *for* $x\in G\sim G_1$. *Denoting Lebesgue measure by* $m(\cdot)$ *we have*

$$m(G_1)\leqq m(P(t))\leqq(1+\rho_0/L)m(G_1),\qquad 0\leqq t\leqq T.$$

(c) *Assume further that* $k=k(x)$ *is autonomous, there is a* $\delta>0$ *with* $\alpha(s)\geqq\delta(s-L)^+$, *and* $m(G_1)>0$. *Then for each* $x\in G$ *there is a* $C(\varepsilon,x)>0$ *such that* $\rho_0/L>C(\varepsilon,x)$ *implies that* $x\in P(t)$ *for all* t *sufficiently large.*

Proof. (a) Since K is the Lipschitz constant for α and $\alpha(L)=0$ we have

$$\varepsilon u_t+K(u-L)\geqq\varepsilon u_t+\alpha(u)-\alpha(L)=(I+\varepsilon A)^{-1}\alpha(u)+\varepsilon f(t)\geqq 0,$$

so there follows

$$u(x,t)-L\geqq e^{-(K/\varepsilon)(t-t_0)}(u(x,t_0)-L),\qquad t\geqq t_0,\quad x\in P(t_0).$$

This shows $P(t)\supset P(t_0)$. Similarly, we have

$$\varepsilon u_t+Ku\geqq(I+\varepsilon A)^{-1}\alpha(u)+\varepsilon f.$$

If for some $(x_0,t_0)\in\Omega$ we have $\alpha(u(x_0,t_0))>0$, then by the strong maximum principle [10, pp. 188–189] $((I+\varepsilon A)^{-1}\alpha(u))(x,t_0)>0$ for *all* $x\in G$ and there follows $u(x,t)>0$ for all $t\geqq t_0$.

(b) The first inequality follows from (a) since $P(0)=G_1$. The second is obtained from the L^1-estimate

$$Lm(P(t)) \leqq \int_{P(t)} u(x,t)\,dx \leqq \|u(t)\|_{L^1} \leqq \|u_0\|_{L^1} = (\rho_0+L)m(G_1).$$

(c) For $x\in G\sim G_1$, $u_0(x)=0$ so by continuity the number $T(x)\equiv \sup\{\tau \geqq 0 : u(x,t)\leqq L$ for all $0\leqq t\leqq \tau\}$ is strictly positive. We shall show $T(x)<\infty$. From the proof of (a) follows

$$\alpha(u(x,t)) \geqq \delta(u(x_1,t)-L) \geqq \delta\rho_0 e^{-Kt/\varepsilon}, \qquad x_1\in G_1, \quad t\geqq 0.$$

Define χ_1 as the characteristic function of G_1 and $\varphi_1\equiv (I+\varepsilon A)^{-1}\chi_1$. By the strong maximum principle $\varphi_1(x)>0$ for every $x\in G$. Since $\alpha(u(x,t))\geqq \delta\rho_0 e^{-Kt/\varepsilon}\chi_1(x)$ we obtain from the comparison principle

$$(I+\varepsilon A)^{-1}\alpha(u(x,t)) \geqq \delta\rho_0 e^{-Kt/\varepsilon}\varphi_1(x), \qquad x\in G, \quad t\geqq 0.$$

Thus, for $x\in G\sim G_1$ and $0\leqq t\leqq T(x)$ we have $\alpha(u(x,t))=0$ and from (3.1)

$$u_t(x,t) \geqq (\delta\rho_0/\varepsilon)e^{-Kt/\varepsilon}\varphi_1(x)$$

and therefore

$$u(x,t) \geqq (\delta\rho_0/K)(1-e^{-Kt/\varepsilon})\varphi_1(x), \qquad x\in G\sim G_1, \quad 0\leqq t\leqq T(x).$$

Thus, if $\rho_0/L\geqq K/\delta\varphi_1(x)$, then there is a $t^*=T(x)$ for which $u(x,t^*)=0$ and $u_t(x,t^*)>0$. This finishes the proof of the theorem.

The property expressed in (a), that every point in the medium is partially saturated as soon as any point has positive pressure, is a consequence of the instantaneous diffusion through the system of fissures. Although such infinite propagation speeds are standard for linear parabolic equations, the porous medium equation (2.5) is known to have finite propagation speeds for certain nonlinearities.

In the diffusion model leading to the situation described in (b), L is the amount of fluid required per volume to fill the voids or to overcome an absorption characteristic of the medium, and ρ_0 is the density of excess fluid available in the region G_1. The estimate in (b) is an explicit upper bound on the advance of the pressure set $P(t)$ in terms of the ratio ρ_0/L.

Similarly, (c) shows that for each point $x\in G$ there is a value of the ratio ρ_0/L which drives $P(t)$ to enclose x. The qualitative dependence of $C(\varepsilon,x)$ is clear from the proof and interesting. Specifically, for $x\in G\sim G_1$ we note $C(\varepsilon,x)$ increases as x approaches ∂G or as ε decreases to 0, $C(\varepsilon,x)$ decreases as x approaches G_1, and $C(\varepsilon,x)$ approaches $2k/\delta$ for x near G_1 and ε near 0.

Our knowledge of the regularity of the solution permits a description of its behavior along the "free-boundary" or interface Γ bounding the pressure set. Thus, let $Q_+\equiv\{(x,t)\in Q : u(x,t)>L\}$ and $Q_0\equiv\{(x,t)\in Q : 0\leqq u\leqq L\}$ in the situation of Theorem 2, and set $\Gamma=\partial Q_+$. At each point of Γ denote the unit normal by $(n_1,n_2,\cdots,n_N,n_t)$ and let n be the unit vector in $\mathbb{R}^N$ with direction of $(n_1,\cdots,n_N)$. Let σ_Γ denote the jump or saltus along Γ. The standard computation of (2.1) over Q_+, Q_0 and the divergence theorem lead to the interface condition

$$\sigma_\Gamma(u)n_t=0 \quad \text{on } \Gamma.$$

Thus, at each point of Γ, either the concentration is continuous or the interface is stationary. A similar computation on (2.2) shows

$$\sigma_\Gamma\left(k\frac{\partial}{\partial n}(S(p_2))\right)=0 \quad \text{on } \Gamma.$$

Thus flux is continuous across Γ. Note that in the classical Stefan problem it is only the *sum* of the preceding values which vanishes, thereby giving a constraint on the velocity $n_t/\|(n_1,\cdots,n_N)\|$ of Γ. The regularity of a generalized solution of (3.1) will not permit a nonstationary singularity.

Finally, we note that the above remarks have physically meaningful consequences for the thermal conduction model (2.6). In contrast to the completely contrary property of the classical Stefan problem, the solution of (2.6) will permit the appearance of a mush zone even if one were not present initially and no outside sources are present. Moreover, Theorem 3.2(a) implies that such mush regions always form over large regions from initial conditions containing both pure ice ($u=0$) and water at positive temperature.

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