Closed Range Theorem

Let V be a Hilbert space and denote by $\mathcal{R}_V : V \to V'$ the Riesz isomorphism, $\mathcal{R}_V v(w) = (v, w)_V$, $\forall v, w \in V$. For any subset $W \subset V$ we have $\mathcal{R}_V(W^\perp) = W^a \subset V'$. That is, *orthogonal complement* corresponds to *annihilator*. We identify $V = V''$.

Let S be another Hilbert space and $\mathcal{B} \in \mathcal{L}(V, S')$, that is, it is continuous and linear. Its dual $\mathcal{B}' \in \mathcal{L}(S, V')$, and a direct computation gives

$$\begin{aligned} \operatorname{Ker}(\mathcal{B}) &= \{v \in V : \mathcal{B}v(s) = 0 \ \forall s \in S\} \\ &= \{v \in V : \mathcal{B}'s(v) = 0 \ \forall s \in S\} = (\operatorname{Rg}\mathcal{B}')^a. \end{aligned}$$

Lemma 0.1. *If $\mathcal{B} \in \mathcal{L}(V, S')$, the following are equivalent:*

1. $\operatorname{Rg}\mathcal{B}$ *is closed in* S'.
2. $\operatorname{Rg}\mathcal{B} = (\operatorname{Ker}\mathcal{B}')^a$.
3. *There is a* $\mathcal{B}^R \in \mathcal{L}(\operatorname{Rg}\mathcal{B}, (\operatorname{Ker}\mathcal{B})^\perp)$ *and a constant* $c_b > 0$ *such that* $\mathcal{B}\mathcal{B}^R = I$ *on* $\operatorname{Rg}\mathcal{B}$ *and* $c_b\|\mathcal{B}^R(g)\|_V \le \|g\|_{S'}$ *for all* $g \in \operatorname{Rg}\mathcal{B}$.
4. *For some constant* $c_b > 0$,

$$\inf_{v\in V} \sup_{s\in S} \frac{\mathcal{B}v(s)}{\|v\|_{V/\operatorname{Ker}\mathcal{B}}\|s\|_S} \ge c_b.$$

Proof. From above we have $\operatorname{Ker}(\mathcal{B}') = (\operatorname{Rg}\mathcal{B})^a$ and so $\operatorname{Ker}(\mathcal{B}')^a = (\operatorname{Rg}\mathcal{B})^{aa} = \overline{\operatorname{Rg}\mathcal{B}}$. This shows (1) is equivalent to (2).

From (1) we have $\mathcal{B} : (\operatorname{Ker}\mathcal{B})^\perp \to \overline{\operatorname{Rg}\mathcal{B}}$ is continuous and injective, so it is necessarily an isomorphism, and this implies (3); (1) follows directly from (3) since $\mathcal{B}$ is continuous.

Finally, we note the equivalence of (3) and (4) follows from that of

$$c_b \inf_{w\in\operatorname{Ker}\mathcal{B}} \|v + w\|_V \le \|\mathcal{B}v\|_{S'} \text{ and } c_b\|v\|_{V/\operatorname{Ker}\mathcal{B}} \le \sup_{s\in S} \frac{\mathcal{B}v(s)}{\|s\|_S}.$$

□

Corollary 0.2. $\mathrm{Rg}\,\mathcal{B}'$ *is closed in* V'.

Proof. If $s \in (\mathrm{Ker}\,\mathcal{B}')^\perp$ then $\mathcal{R}_S(s) \in (\mathrm{Ker}\,\mathcal{B}')^a = \mathrm{Rg}\,\mathcal{B}$ and so $\mathcal{R}_S(s) = \mathcal{B}v$ where we define $v \equiv \mathcal{B}^R(g)$. Thus we have $c_b\|v\|_V \leq \|\mathcal{R}_S(s)\|_{S'} = \|s\|_S$ from which there follows

$$\|s\|_S^2 = \mathcal{R}_S s(s) = \mathcal{B}v(s) = \mathcal{B}'s(v) \leq \|\mathcal{B}'s\|_{V'}\|v\|_V \leq \|\mathcal{B}'s\|_{V'}\tfrac{1}{c_b}\|s\|_S.$$

This implies $c_b\|s\|_S \leq \|\mathcal{B}'s\|_{V'}$ (with the same constant c_b) and hence that $\mathrm{Rg}\,\mathcal{B}'$ is closed. □

The Corollary shows $\mathrm{Rg}\,\mathcal{B}$ is closed if and only if $\mathrm{Rg}\,\mathcal{B}'$ is closed, and we have obtained the **Closed Range Theorem**.

Theorem 0.3. *If* $\mathcal{B} \in \mathcal{L}(V, S')$, *the following are equivalent:*

1. $\mathrm{Rg}\,\mathcal{B}$ *is closed in* S'.
2. $\mathrm{Rg}\,\mathcal{B} = (\mathrm{Ker}\,\mathcal{B}')^a$.
3. *There is a right-inverse* $\mathcal{B}^R \in \mathcal{L}(\mathrm{Rg}\,\mathcal{B}, (\mathrm{Ker}\,\mathcal{B})^\perp)$ *and a constant* $c_b > 0$ *such that* $\mathcal{B}\mathcal{B}^R = I$ *on* $\mathrm{Rg}\,\mathcal{B}$ *and* $c_b\|\mathcal{B}^R(g)\|_V \leq \|g\|_{S'}$ *for all* $g \in \mathrm{Rg}\,\mathcal{B}$.
4. *For some constant* $c_b > 0$,

$$\inf_{v\in V}\sup_{s\in S}\frac{\mathcal{B}v(s)}{\|v\|_{V/\mathrm{Ker}\,\mathcal{B}}\|s\|_S} \geq c_b.$$

5. $\mathrm{Rg}\,\mathcal{B}'$ *is closed in* V'.
6. $\mathrm{Rg}\,\mathcal{B}' = (\mathrm{Ker}\,\mathcal{B})^a$.
7. *There is a right-inverse* $\mathcal{B}'^R \in \mathcal{L}(\mathrm{Rg}\,\mathcal{B}', (\mathrm{Ker}\,\mathcal{B}')^\perp)$ *and a constant* $c_b > 0$ *such that* $\mathcal{B}'\mathcal{B}'^R = I$ *on* $\mathrm{Rg}\,\mathcal{B}'$ *and* $c_b\|\mathcal{B}'^R(f)\|_S \leq \|f\|_{V'}$ *for all* $f \in \mathrm{Rg}\,\mathcal{B}'$.
8. *For some constant* $c_b > 0$,

$$\inf_{s\in S}\sup_{v\in V}\frac{\mathcal{B}'s(v)}{\|s\|_{S/\mathrm{Ker}\,\mathcal{B}'}\|v\|_V} \geq c_b.$$