John Zaheer
MTH 581
12/5/16

Homework 6

Problem 1

Consider the following system

$$\begin{cases} \dot{x} = a_{11}x + a_{12}y \\ \dot{y} = a_{21}x + a_{22}y \end{cases}$$

Which can be written in the form $\dot{X} = AX$ where $X = [x, y]^T$ and $A = \{a_{ij}\}$. Note that by the definition of A we have that by letting $p = a_{11} + a_{22}$ and $q = a_{11}a_{22} - a_{12}a_{21}$, $\det(A) = q$ and $\text{tr}(A) = p$. Finally let $\triangle = p^2 - 4q = \text{tr}(A)^2 - 4\det(A)$.

(a) Show that $x_0 = [0, 0]^T$ is a critical point of the system.

(b) Show that x_0 is a node if $q > 0$ and $\triangle \geq 0$.

(c) Show that x_0 is a saddle if $q < 0$.

(d) Show that x_0 is a spiral if $p \neq 0$ and $\triangle < 0$.

(e) Show that x_0 is a center if $p = 0$ and $q > 0$.

(f) Show that x_0 is asymptotically stable if $q > 0$ and $p < 0$.

(g) Show that x_0 is stable if $q > 0$ and $p = 0$.

(h) Show that x_0 is unstable if $q < 0$ or $p > 0$.

Solution: Assume the problem as presented.

(a) To check that x_0 is a critical points, we simply plug in the values into the system and check that it equals $[0, 0]^T$.

$$Ax_0 = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

For the following portion of the problem we will be doing analysis of the critical points in which we need to consider $\det(A - \lambda\mathbb{I}) = 0$. So we compute that here and it will be used for the remaining portion of the problem.

$$\begin{aligned} \det(A - \lambda\mathbb{I}) = \begin{vmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{vmatrix} &= (a_{11} - \lambda)(a_{22} - \lambda) - a_{12}a_{21} \\ &= a_{11}a_{22} - (a_{11} + a_{22})\lambda + \lambda^2 - a_{12}a_{21} \\ &= \lambda^2 - p\lambda + q \end{aligned}$$

Finding zeroes of the characteristic equation we have

$$\lambda_{\pm} = \frac{p \pm \sqrt{p^2 - 4q}}{2} = \frac{p \pm \sqrt{\triangle}}{2}.$$

(b) Assume that $q > 0$ and $\triangle \geq 0$. Thus we have that $\sqrt{\triangle} \in \mathbb{R}$ which implies that $\lambda_{\pm} \in \mathbb{R}$. This in turn implies that x_0 is a node.

(c) Assume that $q < 0$. Let $q = -g$ where $g > 0$. With $q < 0$ we have that $\triangle \geq 0$ so $\sqrt{\triangle} \in \mathbb{R}$, which implies that we have a node. Note that

$$\lambda_{\pm} = \frac{1}{2}\left(p \pm \sqrt{p^2 - 4q}\right) = \frac{1}{2}\left(p \pm \sqrt{p^2 + 4g}\right).$$

Furthermore, since $\sqrt{p^2 + 4g} > p$ we have that $p - \sqrt{p^2 + 4g} > 0$. From these two inequalities we have that for any value of p, $\lambda_+ > 0$ and $\lambda_- < 0$. Since $\lambda_{\pm}$ are real and have opposite signs we have that x_0 is a saddle.

(d) Suppose that $p \neq 0$ and $\triangle < 0$. Since $\triangle < 0$ we have that $\sqrt{\triangle} \in \mathbb{C}$, which implies that we have a spiral or a center. But since $p \neq 0$ we cannot have a center but in fact we we have that $\lambda_{\pm} = \frac{p \pm \sqrt{\triangle}}{2}$ in which, if $p < 0$ we have that x_0 is a stable spiral and if $p > 0$ we have an unstable spiral.

(e) Assume that $p = 0$ and $q > 0$. Thus we have

$$\lambda_{\pm} = \frac{\sqrt{-4q}}{2} = \frac{i\sqrt{4q}}{2} \in \mathbb{C}.$$

Since the real part of $\lambda_{\pm}$ is zero we in fact have a center.

(f) Assume that $q > 0$ and $p < 0$. By the formula for $\lambda_p m$ we can note that the only term that we need to be worried about is $\triangle$ since if $\triangle < 0$ we have that x_0 is a spiral and if $\triangle \geq 0$ we have that x_0 is a node. So if we have that $\triangle \geq 0$ we have the following implications

$$p \geq 4q \implies p^2 > p^2 - 4q \implies |p| > \sqrt{p^2 - 4q}$$

The last inequality tells us that $\lambda_{\pm} < 0$ which implies x_0 is a stable node which in turn implies asymptotically stable. Now suppose that $\triangle < 0$ we have the following implications

$$p^2 < 4q \implies \lambda_{\pm} = \frac{1}{2}\left(p \pm i\sqrt{4q - p^2}\right) \in \mathbb{C}.$$

Furthermore since $p < 0$ we have that x_0 is a stable spiral which in turn implies asymptotically stable.

(g) Assume that $q > 0$ and $p = 0$ then we have

$$\lambda_{\pm} = \pm\frac{1}{2}\sqrt{-4q} = \pm i\sqrt{q} \in \mathbb{C}.$$

Since $p = 0$ we have that x_0 is a center which in turn implies it is stable.

(h) Assume that $q < 0$ first. Let $q = -g$ with $g > 0$. Then we have $\triangle > 0$ so $\lambda_{\pm} \in \mathbb{R}$ which implies we are dealing with a node. Since we have that $\lambda_{\pm}$ is real all we need to worry about is the signs of λ. Since we have $p^2 - 4q > 0$ and $q < 0$ we have $\sqrt{\triangle} > |p|$. With this information we have 3 cases

   (i) If $p < 0$ then we have that $\lambda_+ > 0$ and $\lambda_- < 0$.

   (ii) If $p > 0$ then we have that $\lambda_+ > 0$ and $\lambda_- < 0$.

   (iii) If $p = 0$ then we have that $\lambda_+ > 0$ and $\lambda_- < 0$.

   In all the cases above we have that x_0 is a saddle which implies unstable.

   Now assume that $p > 0$. Then we have the following 3 cases.

   (i) If $q < 0$ then we have $\triangle > 0$ and $\sqrt{\triangle} > p$ which implies that $\lambda_+ > 0$ and $\lambda_- < 0$. From this we conclude that x_0 is a saddle which implies unstable.

   (ii) If $q = 0$ we have that $\lambda_+ = p$ and $\lambda_- = 0$ which implies we have an unstable system.

   (iii) If $q > 0$ then we have the following two cases.

      (1.) If $\triangle > 0$, then we consider $p < \sqrt{\triangle}$ in which we have $\lambda_+ > 0$ and $\lambda_- < 0$ which corresponds to a saddle which is unstable. We also need to consider if $p \geq \sqrt{\triangle}$ in which we have $\lambda_{\pm} > 0$ so it is unstable.

      (2.) If $\triangle < 0$. Then we have that $\lambda_{\pm} \in \mathbb{C}$ and since $p > 0$ we have that x_0 is an unstable spiral.

   In all the cases shown we have x_0 being unstable.

Problem 2

Consider the following system

$$\begin{cases} \dot{x} = -x + y - 2xy \\ \dot{y} = -4x - y + x^2 - y^2 \end{cases}$$

(a) Find all the critical points.

(b) Determine the local linear stability of the critical points.

(c) Draw a slope field for the system.

Solution: Consider the system as given.

(a) To solve for the critical points we solve for when the system is $\dot{x} = 0$ and $\dot{y} = 0$ at the same time. Looking at it as system of equations, from $\dot{x} = 0$ we have

$$-x + y - 2xy = 0 \implies y = \frac{x}{1 - 2x}.$$

Substituting this into $\dot{y} = 0$ we have

$$-4x - \frac{x}{1-2x} + x^2 - \frac{x^2}{(1-2x)^2} = 0 \implies x(4x^3 - 20x^2 + 18x - 5) = 0.$$

Clearly $x = 0$ is a root, then to solve for the others we use MATLAB's 'roots' function and attain $x = \{3.9379, 0.5311 + .1881i, 0.5311 - 1.881i\}$. Since we only care about the real roots we have the critical points to be considering being the set $x = \{0, 3.9379\}$ with corresponding $y = \{0, -0.5727\}$ respectively.

(b) To determine the stability we first compute the Jacobian of the system. That is we consider

$$J(X) = \begin{pmatrix} \frac{d\dot{x}}{dx} & \frac{d\dot{x}}{dy} \\ \frac{d\dot{y}}{dx} & \frac{d\dot{y}}{dy} \end{pmatrix} = \begin{pmatrix} -1-2y & 1-2x \\ -4+2x & -1-2y \end{pmatrix}.$$

Using the Jacobian we are also going to use this particular formula for the characteristic formula roots

$$\lambda_\pm = \frac{1}{2}\left(\text{tr}(A) \pm \sqrt{\text{tr}(A)^2 - 4\det(A)}\right)$$

such that $A = J(X_0)$ and X_0 are the roots found in (a).

Consider $X_0 = (0,0)$. Thus we have

$$J(X_0) = \begin{pmatrix} -1 & 1 \\ -4 & -1 \end{pmatrix}.$$

From this we can calculate $\text{tr}(J) = -2$ and $\det(J) = 5$. Furthermore we have

$$\lambda_\pm = -1 \pm 2i.$$

Here we see that $\lambda_\pm \in \mathbb{C}$ and since the real part of $\lambda_\pm$ is negative we have $X_0 = (0,0)$ being a stable spiral.

Consider $X_0 = (3.9379, -0.5727)$. From this we have

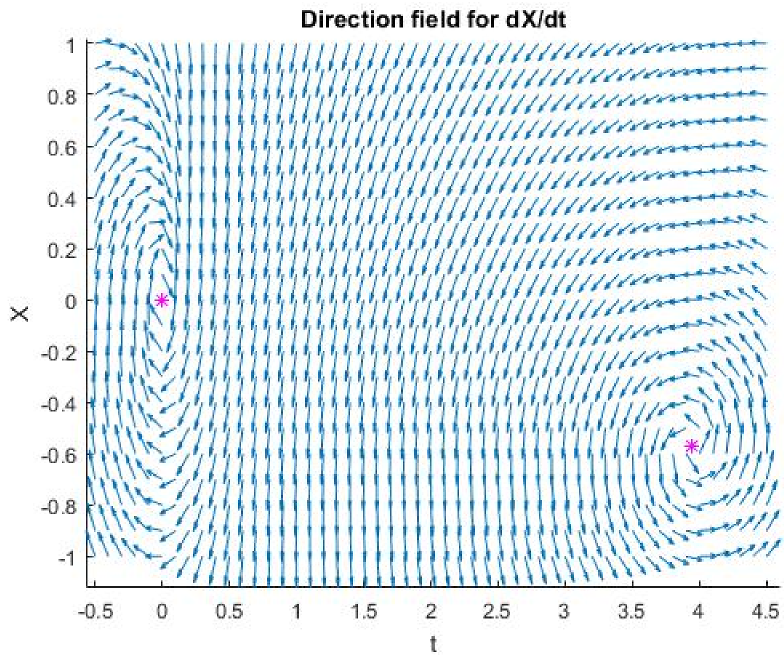
$$J(X_0) = \begin{pmatrix} 2.1454 & -6.8758 \\ 3.8758 & .1454 \end{pmatrix}$$

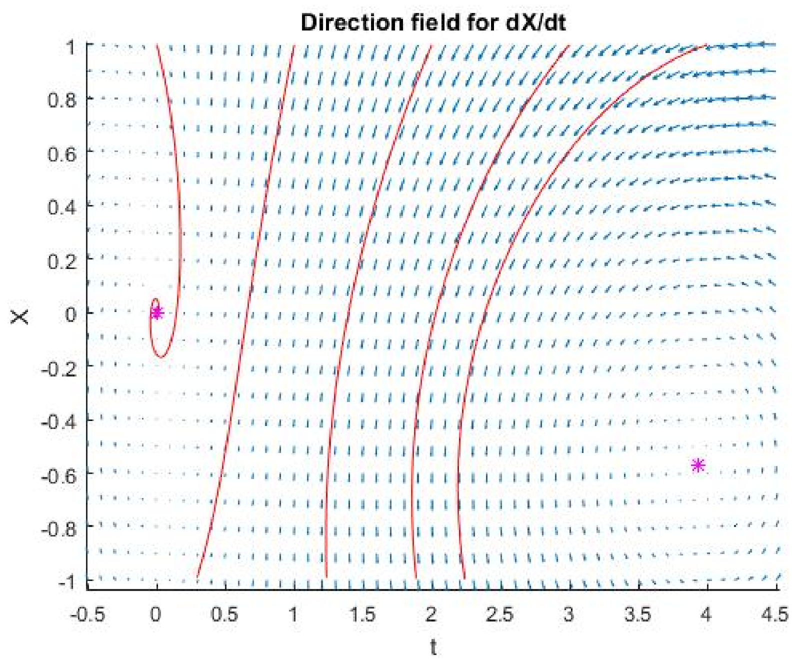
From this we can calculate $\text{tr}(J) = 2.2908$ and $\det(J) = 26.9612$. Furthermore we have

$$\lambda_\pm = 1.1454 \pm 5.0645i.$$

Here we see that $\lambda_\pm \in \mathbb{C}$ and since the real part of $\lambda_\pm$ is positive we have $X_0 = (3.9379, -0.5727)$ is an unstable spiral.

(c) MATLAB was used to produce the following figures.





From the first image we can see that our analysis was correct in determining the systems dynamics. The second image provides some traces with initial starting points following the arrows given in the direction field.

Problem 3

Consider the following system

$$\dot{X} = \begin{pmatrix} 1 & 1 \\ -\varepsilon & 1 \end{pmatrix} X$$

where $|\varepsilon| \ll 1$. Show that the stability of the system chances as ε changes in the value from slightly negative to slightly positive.

Solution: Consider the problem as given. Note that the system can be written in the form $\dot{X} = AX$ where A is the coefficient matrix provided. The first thing we do when determining the systems dynamics is solve for the characteristic function. That is we have

$$\det(A - \lambda\mathbb{I}) = \begin{vmatrix} 1-\lambda & 1 \\ -\varepsilon & 1-\lambda \end{vmatrix} = \lambda^2 - 2\lambda + (\varepsilon + 1).$$

The roots of the resulting quadratic take the form of

$$\lambda_{\pm} = \frac{1}{2}\left(2 \pm \sqrt{4 - 4(\varepsilon + 1)}\right) = 1 \pm 2\sqrt{-\varepsilon}.$$

Now assume that $\varepsilon < 0$ this implies that $-\varepsilon > 0$. Then we have that $\lambda_{\pm} = 1 \pm 2\sqrt{-\varepsilon}$ are two real distinct eigenvalues. Furthermore since we are assuming that ε is small and near zero we have that both eigenvalues are positive. All of this information collectively imply that we have an unstable node.

Now assume that $\varepsilon > 0$ this implies that $-\varepsilon < 0$. Then we have that

$$\lambda_{\pm} = 1 \pm 2i\sqrt{\varepsilon}.$$

Here we see that $\lambda_{\pm} \in \mathbb{C}$ and since the real part is positive we have an unstable spiral. Thus we can conclude as ε goes from slightly negative to slightly positive the system goes from having an unstable node to an unstable spiral.

Problem 4

Consider the following predator-prey model

$$\begin{cases} \dot{x} = x(1 - \sigma x - 0.5y) \\ \dot{y} = y(0.25x - 0.75) \end{cases}$$

where $\sigma > 0$.

(a) Find all the critical points.

(b) How do they change as σ increase from 0? Observe that there is a critical point in the interior of the first quadrant only if $\sigma < \frac{1}{3}$.

(c) Determine the stability of the critical points. item Find the value $\psi < \frac{1}{3}$ where the nature of the critical point in the first quadrant changes. Describe the change that takes place in this critical point as σ passes through ψ.

(d) Draw the slope field for a σ value between 0 and ψ, and another for σ between ψ and $\frac{1}{3}$.

(e) Describe the effect on the populations as σ increases from zero to $\frac{1}{3}$.

Solution: Consider the problem as given.

(a) To solve for all the critical points we we evaluate this by visually analyzing the system. From $\dot{x} = 0$ we can clearly see that $x = 0$ will work. With $x = 0$ we have that $y = 0$. Now looking at $\dot{y} = 0$ we see that $y = 0$ would work and form there we can solve for x in $\dot{x}$ to get $x = \frac{1}{\sigma}$. Also from $\dot{y} = 0$ we can see that $x = 3$ would work and solving for y in $\dot{x}$ we get $y = 2 - 6\sigma$. Thus collectively we have the critical points of the system being

$$X_0 = \left\{ (0,0), \left(\frac{1}{\sigma}, 0\right), (3, 2-6\sigma) \right\} = \{A, B, C\}.$$

(b) Consider A, we see that there is no dependence on σ so it does not change as σ increases from 0. For B we see that as σ increases from 0 the critical points moves from infinity to $(0,0)$, or A, along the x-axis. For C we have the critical point moving along the line $x = 3$ from y being large and positive, through the x-axis and then to negative and large in magnitude. We can note that for $0 < \sigma < \frac{1}{3}$ we have C being in the first quadrant, and for $\sigma > \frac{1}{3}$ the critical points moves into the fourth quadrant. Now for $\sigma = \frac{1}{3}$ we have that $B = (3,0)$ and $C = (3,0)$, that is to say when $\sigma = \frac{1}{3}$ the critical points B and C meet.

(c) For the stability of the critical points we use the same trick as done in Problem 2. So we first compute the Jacobian

$$J = \begin{pmatrix} 1 - 2\sigma x - \frac{1}{2}y & -\frac{1}{2}x \\ 0.25y & 0.25x - 0.75 \end{pmatrix}.$$

So consider the critical point A first. Then we have

$$J(A) = \begin{pmatrix} 1 & 0 \\ 0 & -0.75 \end{pmatrix}$$

Then by computing $\det(J(A) - \lambda\mathbb{I})$ we have

$$\det(J(A) - \lambda\mathbb{I}) = \begin{vmatrix} 1-\lambda & 0 \\ 0 & -0.75-\lambda \end{vmatrix} = \lambda^2 - 0.25\lambda - 0.75.$$

Solving for the roots of the resulting polynomial we get $\lambda_+ = 1$ and $\lambda_- = -0.75$. Since they are both real distinct and opposite sign values we have that A is a saddle node, so unstable.

Now Consider the critical point B. Here we have

$$J(B) = \begin{pmatrix} -1 & -\frac{1}{2\sigma} \\ 0 & \frac{1}{4\sigma} - 0.75 \end{pmatrix}$$

Then by computing $\det(J(B) - \lambda\mathbb{I})$ we have

$$\det(J(B) - \lambda\mathbb{I}) = \begin{vmatrix} -1-\lambda & -\frac{1}{2\sigma} \\ 0 & \frac{1}{4\sigma} - 0.75 - \lambda \end{vmatrix} = \lambda^2 + \left(1.75 - \frac{1}{4\sigma}\right)\lambda + (0.75 - \frac{1}{4\sigma}.$$

Here we can let $\omega = 1.75 - \frac{1}{4\sigma}$ and solve for the roots of the resulting polynomial to get

$$\lambda^{+}\omega\lambda + (\omega - 1) = 0 \implies \lambda_{\pm} = \frac{1}{2}\left(-\omega \pm \sqrt{\omega^2 - 4(\omega - 1)}\right) = \frac{1}{2}\left(-\omega \pm (\omega - 2)\right).$$

From this we see that $\lambda_+ = -1$ and $\lambda_- = -\omega + 1$ will have to be broken down into cases. Taking it back to σ we have $\lambda_- = \frac{1}{4\sigma} - 0.75$ and have the following two cases:

(i) For $0 < \sigma < \frac{1}{3}$ we have $\lambda_- > 0$ which implies we have a saddle node, so unstable.

(ii) For $\sigma \geq \frac{1}{3}$ we have that $\lambda_- < 0$ which implies we have a stable node.

Now consider the critical point C. here we have

$$J(C) = \begin{pmatrix} -3\sigma & -\frac{3}{2} \\ \frac{1}{2} - \frac{3}{2}\sigma & 0 \end{pmatrix}.$$

Then by computing $\det(J(C) - \lambda\mathbb{I})$ we have

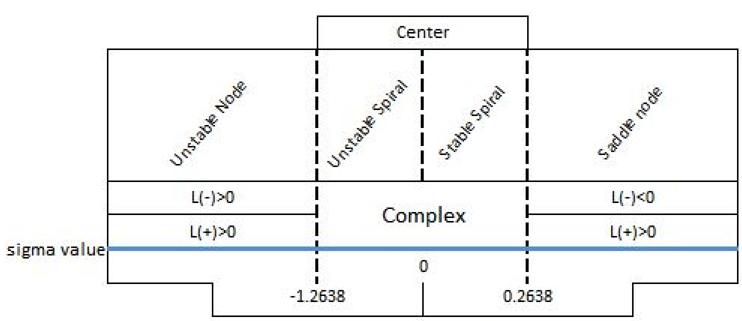
$$\det(J(C) - \lambda\mathbb{I}) = \begin{vmatrix} -3\sigma - \lambda & -\frac{3}{2} \\ \frac{1}{2} - \frac{3}{2}\sigma & -\lambda \end{vmatrix} = \lambda^2 + 3\sigma\lambda + \left(\frac{3}{4} - \frac{9}{4}\sigma\right).$$

Solving for the roots of the resulting polynomial we have

$$\lambda_{\pm} = \frac{1}{2}\left(-3\sigma \pm \sqrt{9\sigma^2 + 9\sigma - 3}\right).$$

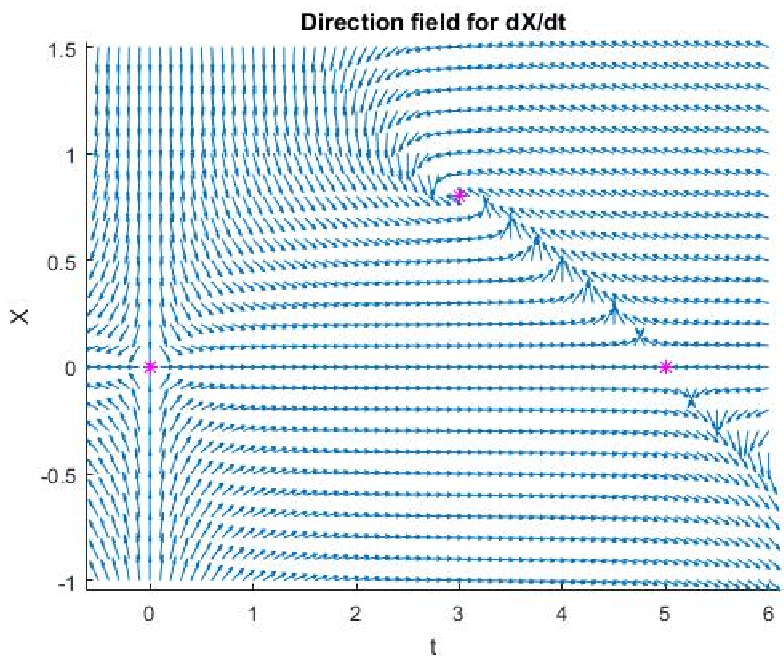
Here we nee to be concerned when the term inside the square root becomes negative because that would be mean $\lambda_{\pm} \in \mathbb{C}$. Thus the first thing we note is the sign of $9\sigma^2 + 9\sigma - 3$. Since it is simply a polynomial of degree two we can look at the roots and check around the roots for the sign. From this we conclude that the term is positive when $\sigma \in (-\infty, -1.2638) \cup (0.2638, \infty)$ and negative when $\sigma \in (-1.2638, 0.2638)$. From this information we can note that if $\sigma \in (-1.2638, 0.2638)$ we have that $\lambda_{\pm} \in \mathbb{C}$, more precisely for $\sigma \in (-1.2638, 0)$ we have an unstable spiral, $\sigma = 0$ we have a center, and $\sigma \in (0, 0.2638)$ we have a stable spiral.

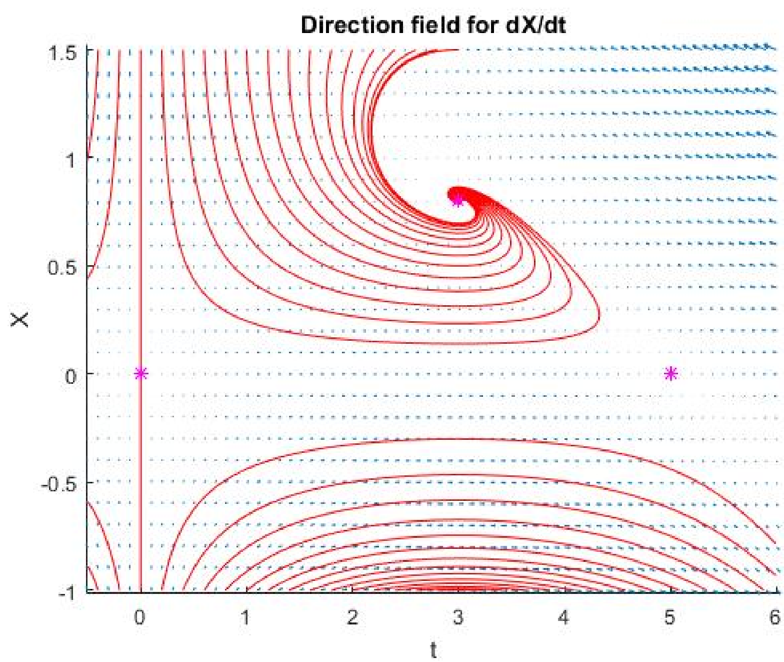
From this we only need to determine the the $\lambda_{\pm}$ values for σ in $(-\infty, -1.2638)$ and $(0.2638, \infty)$. From the formula of $\lambda_{\pm}$ (or looking at a the graph) that $\lambda_+ > 0$ and $\lambda_- > 0$ in $(-\infty, -1.2638)$ which corresponds to an unstable node. Now for $\lambda_{\pm}$ values for σ in $(0, 0.2638)$ we have $\lambda_+ > 0$ and $\lambda_- < 0$ which corresponds to a saddle node. All of this information can be displayed in a table format to make it easier to see.



(d) The σ_1 was actually solved for in the previous part. That is we noticed there was a change in dynamics when σ passed through 0.2638. Thus the σ_1 the questions ask for is $\sigma_1 = 0.2638$.

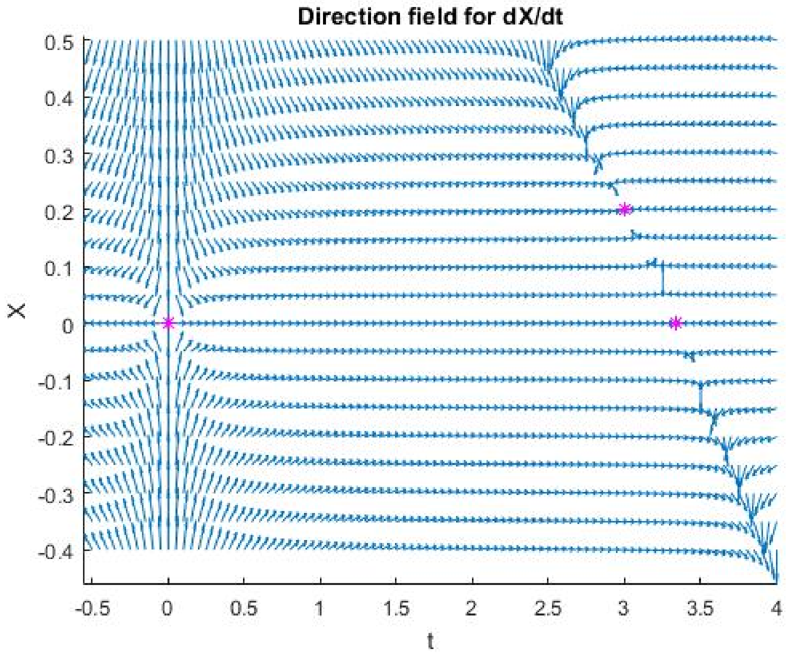
(e) MATLAB was used to generate the following figures. First we consider $\sigma \in (0, \sigma_1)$.

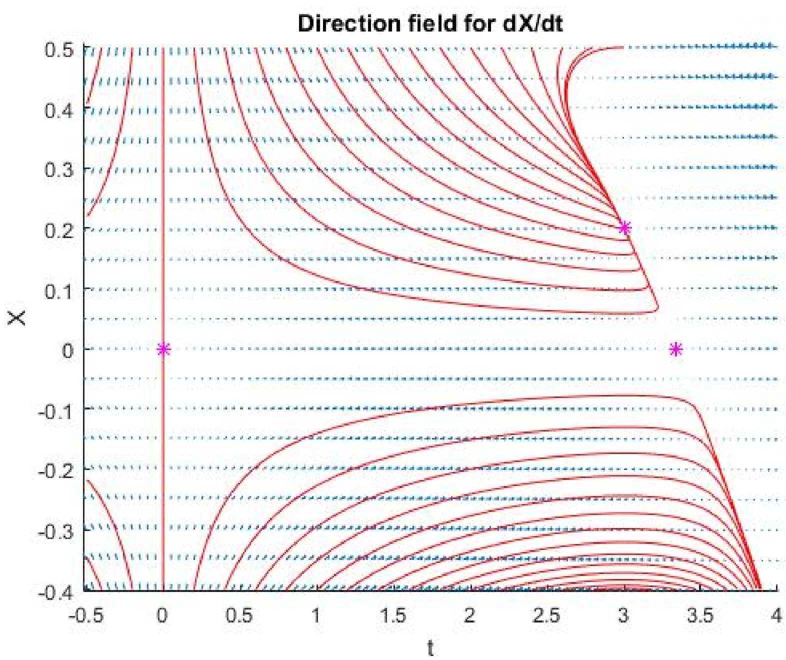




From these figures we can go back and notice that our analysis is correct.

Now we consider $\sigma \in \left(\sigma_1, \frac{1}{3}\right)$.





Again we can use these figures to check our analysis.

(f) As σ increases from 0 to $\frac{1}{3}$ we have that the population will tend towards the critical point $(3, 2-6\sigma)$ up until σ hit σ_1 in a circular fashion, that is to say there is some give in take in the population numbers. After σ passes σ_1 we see that the population will go towards the same fixed point but a lot faster then circling around the fixed point. That is to say there is not that much give and take in the population numbers.

Problem 5

Consider the population model

$$\frac{dP}{dt} = aP\left(E_1 P^{-c} - E_2\right)$$

where $t > 0$, $0 < c \leq 1$ is the "competition constant', P is the number of individuals, a, E_1, and E_2 are constants.

(a) Find the equilibrium populations.

(b) Determine their stability.

(c) Plot the slope field.

(d) Interpret c.

(e) Further dynamics.

Solution: Consider the problem as stated with the assumption that a,E_1, and E_2 are positive constants.

(a) From looking at the system it is clear $P = 0$ is one equilibrium point. By solving $(E_1 P^{-c} - E_2) = 0$ we attain the other equilibrium point being $P = \left(\frac{E_2}{E_1}\right)^{-\frac{1}{c}}$.

(b) To determine the stability we consider a small perturbation to the fixed points and see what the system tells us. First lets consider $P = 0 + \varepsilon = \varepsilon$. We get that

$$\frac{dP}{dt} = a\varepsilon(E_1\varepsilon^{-c} - E_2).$$

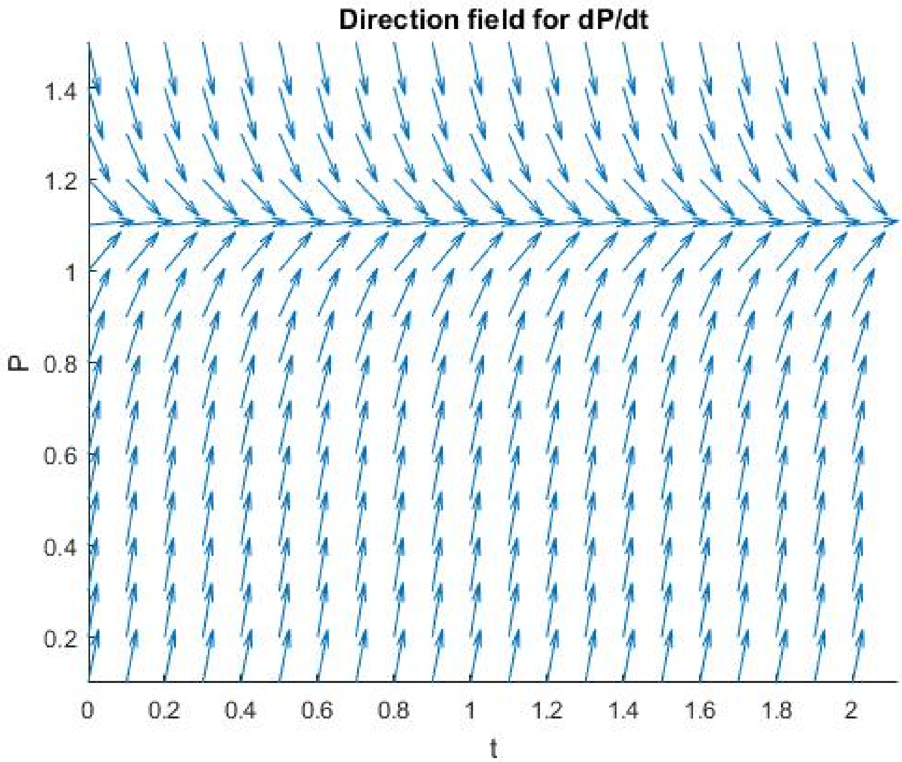
Since we are assuming that $\varepsilon \ll 1$ and since $0 < c \leq 1$ we have that ε^{-c} is going to be really large. With this in mind as long as E_1 and E_2 are comparable, $E_1\varepsilon^{-c}$ will dominate the term in which it is positive. So the fixed points $P = 0$ is unstable.

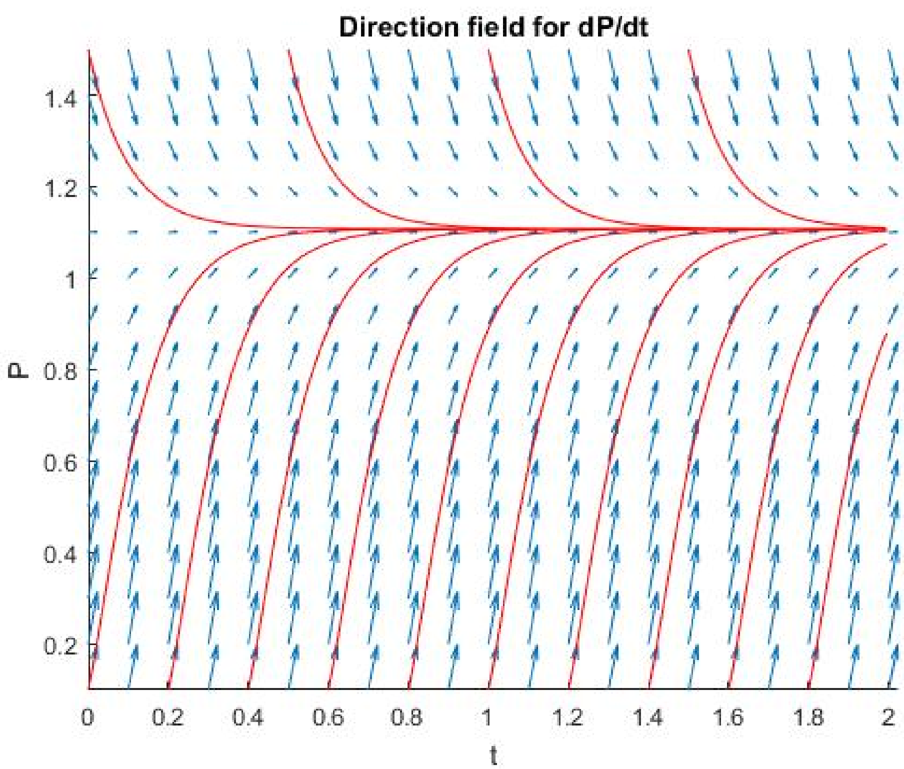
Now we consider a small perturbation for the other fixed point to have $P = \left(\frac{E_2}{E_1}\right)^{-\frac{1}{c}} + \varepsilon$ in which we get

$$\begin{aligned}\frac{dP}{dt} &= a\left(\left(\frac{E_2}{E_1}\right)^{-\frac{1}{c}} + \varepsilon\right)\left(E_1\left(\left(\frac{E_2}{E_1}\right)^{-\frac{1}{c}} + \varepsilon\right)^{-c} - E_2\right) \\ &= E_1 a\left(\left(\frac{E_2}{E_1}\right)^{-\frac{1}{c}} + \varepsilon\right)\left(\left(\left(\frac{E_2}{E_1}\right)^{-\frac{1}{c}} + \varepsilon\right)^{-c} - \frac{E_2}{E_1}\right)\end{aligned}$$

From this we can see that the only term we need to worry about is right handed parenthesis since the whole first portion is positive. Since we are making the assumption that E_1 and E_2 are comparable we can make the argument that $\left(\left(\frac{E_2}{E_1}\right)^{-\frac{1}{c}} + \varepsilon\right)^{-c}$ will be much smaller then $\frac{E_2}{E_1}$. That is to say $\frac{E_2}{E_1}$ is the dominating term in which making $\frac{dP}{dt} < 0$, so this fixed point is stable.

(c) The figure was generated using MATLAB.





(d) To interpret the figures from (c) we note to plot the slopes we let $a = 2$, $E_1 = 10$,

$E_2 = 9.5$ and $c = .5$ which were found through experimentation. From the plots we can see that if the population starts at any point between 0 and approximately 1.1, the population will rise until it hits the carrying capacity of the system, which again is approximately 1.1. Now if the population starts above that, we are exceeding the carrying capacity of the system which means the population needs to die down until we hit the carrying capacity, which can visually be seen by the figures.

(e) For further dynamics into the system we can proceed by scaling the argument. Let ,

$$P^* = \left(\frac{E_1}{E_2}\right)^{\frac{1}{c}}, \qquad \tilde{P} = \frac{P}{P^*}, \text{ and } \qquad \tilde{t} = aE_2 t.$$

Using the chain rule that we have done before we get

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial \tilde{t}}\frac{\partial \tilde{t}}{\partial t} = aE_2 \frac{\partial}{\partial \tilde{t}}.$$

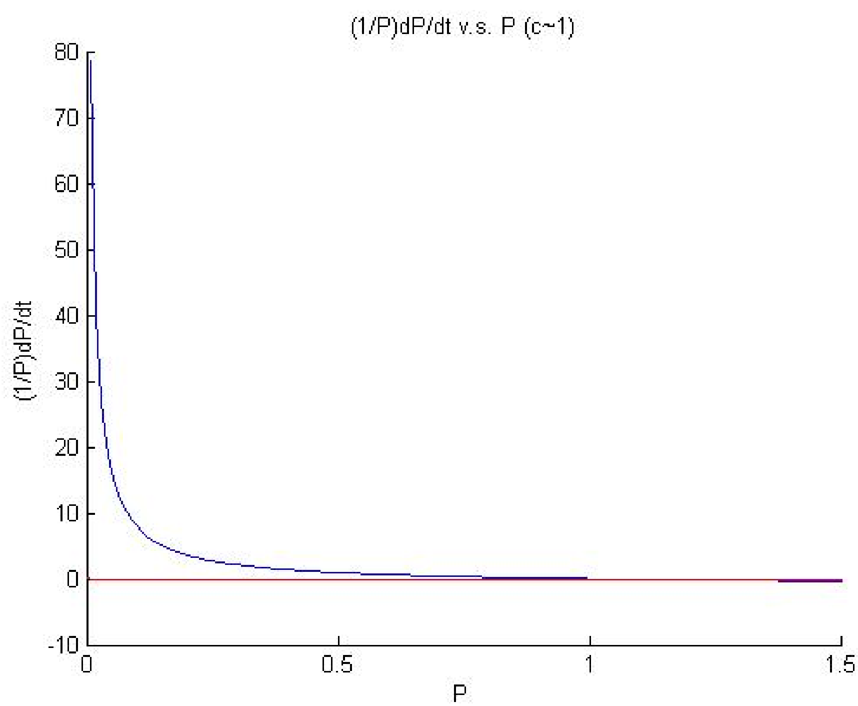
By substituting these values in we get the system to take on the form of

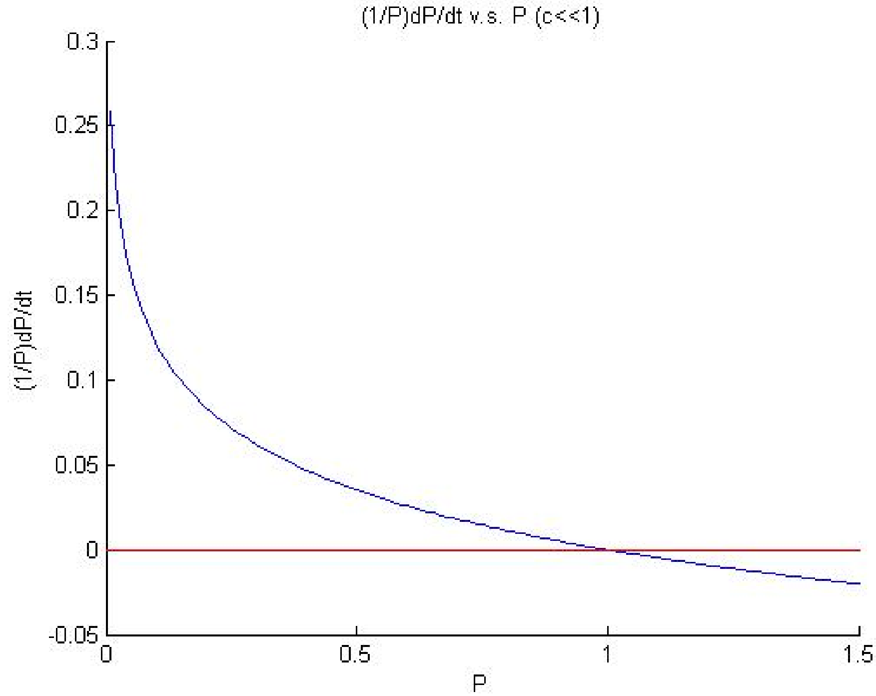
$$\frac{d\tilde{P}}{d\tilde{t}} = \tilde{P}\left(\tilde{P}^{-c} - 1\right).$$

Note that we have the related growth rate to be

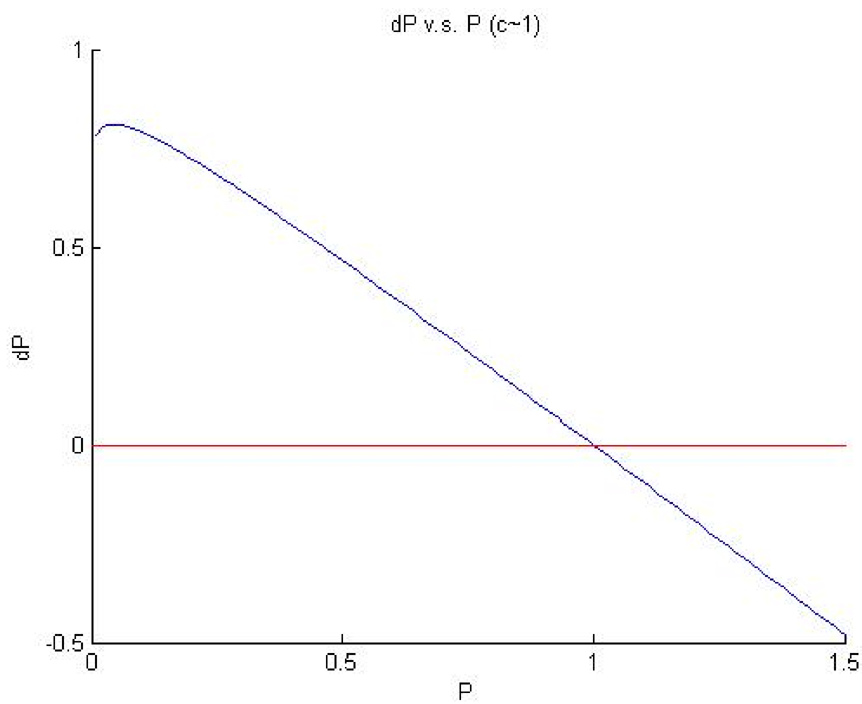
$$\frac{1}{\tilde{P}}\frac{d\tilde{P}}{d\tilde{t}} = \tilde{P}^{-c} - 1.$$

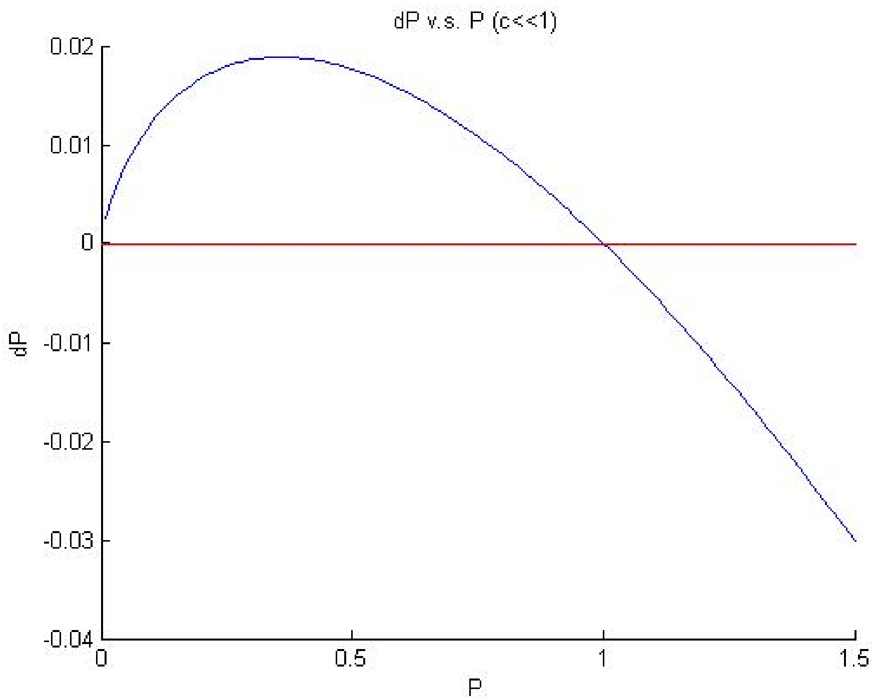
Plotting these equations for $c \sim 1$ and $c \ll 1$ gives some more insight into the system.





From the above two figures we can see that the related rates for $c \sim 1$ decreases rapidly compared to $c \ll 1$. For stability we can look at $\frac{d\tilde{P}}{d\tilde{t}}$ plotted against $\tilde{P}$.





For $c \sim 1$ we see that we have explosive growth away from zero and then see it slow down at $\tilde{P} = 1$. Then from $c \ll 1$ we see the gradual growth away from 0 and slows down at $\tilde{P} = 1$. In other words the different value of c determine the growth away from zero, but still slows down near its carrying capacity.