Dynamical Systems HW

(1) Consider
$$\dot{x} = a_{11}x + a_{12}y \qquad \Rightarrow \quad \dot{\underset{\sim}{x}} = A\underset{\sim}{x} \qquad (*)$$
$$\dot{y} = a_{21}x + a_{22}y$$

$A = \{a_{ij}\}$ are constants

let $p = a_{11} + a_{22}$ let $q = a_{11}a_{22} - a_{12}a_{21}$

for $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ then the "trace" of A

$tr(A) \equiv p$

$\det(A) = q$

let $\Delta = p^2 - 4q \quad = tr(A)^2 - 4\det(A)$

(a) Show that $\underset{\sim}{x}_0 \equiv \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is a critical point of $(*)$.

(b) $\underset{\sim}{x}_0$ is a node if $q > 0$, $\Delta \geq 0$

(c) $\underset{\sim}{x}_0$ is a saddle if $q < 0$

(d) $\underset{\sim}{x}_0$ is a spiral if $p \neq 0$ & $\Delta < 0$

(e) $\underset{\sim}{x}_0$ is a center if $p = 0$ $q > 0$

(f) $\underset{\sim}{x}_0$ is asymptotically stable if $q > 0$ & $p < 0$.

(g) Stable if $q>0$ and $p=0$

(h) unstable if $q<0$ or $p>0$

(2) Consider
$$\begin{cases} \dot{x} = -x+y-2xy = -x+y-2xy \\ \dot{y} = -4x-y+x^2-y^2 \end{cases}$$

(a) Find all critical points

(b) Determine the local linear stability of the critical points.

(c) Draw a slope field for the system

(3) Consider $\dot{\underline{x}} = \begin{pmatrix} +1 & 1 \\ -\varepsilon & +1 \end{pmatrix} \underline{x}$, where $|\varepsilon| \ll 1$

Show how the stability of the system changes as ε changes in value from slightly negative to slightly positive.

(4) Consider the following Predator-prey problem:
$$\dot{x} = x\left(1-\sigma x-\tfrac{1}{2}y\right) \qquad \dot{y} = y(0.25x-0.75)$$

(a) Find all critical pts. $\sigma > 0$

(b) How do they change as σ increases from 0? Observe that there's a critical point in the interior of the first quadrant only if $\sigma < 1/3$

(c) Determine stability.

(d) Find value $\sigma_1 < \frac{1}{3}$ where the nature of the crit. pt. in 1st quadrant changes. Describe the change that takes place in this critical point as σ passes through σ_1

(e) Draw the slope field for a σ value between 0 and σ_1, and another for σ between σ_1 and $1/3$.

(f) Describe the effect on the populations as σ increases from zero to $1/3$.

(5) Consider the population model

$$\frac{dP}{dt} = aP(E_1 P^{-c} - E_2) \qquad t > 0$$

$0 < c \leq 1$ is the "competition constant", P is the number of individuals, a, E_1 & E_2 are constants.

(a) Find equilibrium populations

(b) Determine their stability

(c) Plot slope field

(d) Interpret c.