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MTH 581
11/18/16

Homework 5

Problem 1

Consider

$$\begin{cases} \ddot{u} + \omega_0^2 u = \varepsilon \dot{u}^2 u \\ u(0) = A & \qquad u = u(t) \\ \dot{u}(0) = 0 \end{cases}$$

Find the leading order and first correction approximation to u. Use Poncaré Lindstedt (Optional: try regular perturbation)

Solution: To find the leading order terms and the first approximation to u using the Poincaré Lindstedt we consider the transformation $\tau = \omega t$. From this we can consider the following approximation expansion with the associated derivative operator

$$\begin{aligned} u &= u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + O(\varepsilon^3) \\ \omega &= \omega_0 + \varepsilon \omega_1 + \varepsilon^2 \omega_2 + O(\varepsilon^3) \\ \frac{d}{dt} &= \frac{d}{d\tau}\frac{d\tau}{dt} = \omega \frac{d}{d\tau} \\ \frac{d^2}{dt^2} &= \omega^2 \frac{d^2}{d\tau^2} \end{aligned}$$

Then we can rewrite the system of equation to be the following (written in summations to shorten the notation for this part):

$$\begin{cases} \omega^2 \sum_{i=0}^{\infty} \ddot{u}_i \varepsilon^i + \omega_0^2 \sum_{i=0}^{\infty} u_i \varepsilon^i - \varepsilon \omega^2 \left(\sum_{i=0}^{\infty} \dot{u}_i \varepsilon^i\right)^2 \left(\sum_{i=0}^{\infty} u_i \varepsilon^i\right) = 0 \\ \sum_{i=0}^{\infty} u_i(0) \varepsilon^i = A \\ \omega \sum_{i=0}^{\infty} \dot{u}_i(0) \varepsilon^i = 0 \end{cases}$$

Now considering the ODE we have the following

$$\begin{aligned} 0 = &\left(\omega_0 + \varepsilon \omega_1 + \varepsilon^2 \omega_2 + O(\varepsilon^3)\right)^2 \left(\ddot{u}_0 + \varepsilon \ddot{u}_1 + \varepsilon^2 \ddot{u}_2 + O(\varepsilon^3)\right) \\ &+ \omega_0^2 \left(u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + O(\varepsilon^3)\right) \\ &- \varepsilon \left(\omega_0 + \varepsilon \omega_1 + \varepsilon^2 \omega_2 + O(\varepsilon^3)\right)^2 \left(\dot{u}_0 + \varepsilon \dot{u}_1 + \varepsilon^2 \dot{u}_2 + O(\varepsilon^3)\right)^2 \left(u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + O(\varepsilon^3)\right) \end{aligned}$$

Combining the terms in order to have it easier to pick out the order of ε terms we have

$$\begin{aligned} 0 = &\left[\omega_0^2 \ddot{u}_0 + \varepsilon(\omega_0^2 \ddot{u}_1 + 2\omega_0 \omega_1 \ddot{u}_0) + O(\varepsilon^2)\right] \\ &+ \left[\omega_0^2 u_0 + \varepsilon(\omega_0^2 u_1) + O(\varepsilon^2)\right] \\ &- \left[0 + \varepsilon(u_0 \dot{u}_0^2 \omega_0^2) + O(\varepsilon^2)\right] \end{aligned}$$

Now considering the order of ε's. First we consider ε^0 and solve the system:

$$\begin{cases} \omega^2\ddot{u}_0 + \omega_0^2 u_0 = 0 \\ u_0(0) = A \\ \dot{u}_0(0) = 0 \end{cases} \implies u_0 = A\cos(\tau)$$

Grouping the terms for ε^1, making the substitution with the u_0 and then solving for u_1 we have

$$\begin{cases} \omega_0^2\ddot{u}_1 + 2\omega_0\omega_1\ddot{u}_0 + \omega_0^2 u_1 - u_0\dot{u}_0^2\omega_0^2 = 0 \\ u_1(0) = 0 \\ \dot{u}_1(0) = 0 \end{cases} \implies \begin{cases} \ddot{u}_1 + u_1 = -A^3\cos(\tau)\sin(\tau) + \frac{2\omega_1}{\omega_0}A\cos(\tau) \\ u_1(0) = 0 \\ \dot{u}_1(0) = 0 \end{cases}$$

$$\implies \begin{cases} \ddot{u}_1 + u_1 = \left(\frac{2\omega_1}{\omega_0}A + \frac{1}{4}A^3\right)\cos(\tau) - \frac{1}{4}A^3\cos(3\tau) \\ u_1(0) = 0 \\ \dot{u}_1(0) = 0 \end{cases} .$$

Now in order to eliminate the secular term we can solve for ω_1 to do so. That is we solve

$$\frac{2\omega_1}{\omega_0}A + \frac{1}{4}A^3 = 0 \implies \omega_1 = -\frac{\omega_0 A^2}{8}.$$

With this we get the system to be

$$\begin{cases} \ddot{u}_1 + u_1 = -\frac{1}{4}A^3\cos(3\tau) \\ u_1(0) = 0 \\ \dot{u}_1(0) = 0 \end{cases} .$$

Solving this using the same methods as we have been using, then applying the conditions we have that

$$u_1(\tau) = \frac{A^3}{4}\cos(3\tau)\left(\cos(\tau) - 1\right).$$

Thus we can write down the approximation as

$$u(t) = A\cos(\omega t) + \varepsilon\frac{A^3}{4}\cos(3\omega t)\left(\cos(\omega t) - 1\right) + O(\varepsilon^2)$$
$$\omega = \omega_0 - \varepsilon\frac{\omega_0 A^2}{8} + O(\varepsilon^2)$$

Problem 2

Consider the boundary layer problem

$$\text{BVP} = \begin{cases} \varepsilon y'' + y' + y = 0 & 0 < x < 1 \\ y(0) = \alpha,\ y(1) = \beta & \alpha, \beta > 0 \end{cases}$$

(a) Find the exact solution $y(x)$.

(b) Show BVP is singular.

(c) Find an inner and outer approximation to the solution, to the lowest order, to within constant(s)

(d) Find the matching condition that determines the unknown constants in (c).

(e) Show that the inner and outer solutions approximate the exact solution in each of the regions.

Solution: We will be considering the problem given in the problem.

(a) To solve for the exact solution we note that it is a second order linear equation with constant coefficients. Applying the method of characteristics we have the characteristic equation to be

$$p(\lambda) = \varepsilon\lambda^2 + \lambda + 1.$$

Here we see the roots of the equation are

$$M_{1,2} = \frac{-1 \pm \sqrt{1-4\varepsilon}}{2\varepsilon}.$$

For $\varepsilon \ll 1$ we have $M_{1,2}$ being real and distinct so we have the solution being

$$y = C_1 e^{M_1 t} + C_2 e^{M_2 t}.$$

Applying the boundary conditions we have the following solutions for the unknown constants:

$$C_1 = \alpha - \frac{\beta - \alpha e^{M_1}}{e^{M_2} - e^{M_1}}$$
$$C_2 = \frac{\beta - \alpha e^{M_1}}{e^{M_2} - e^{M_1}}$$

(b) To show that the BVP is singular we first attempt to apply the regular perturbation method and see what goes wrong. We first make the assumption that y can be written as

$$y = y_0 + \varepsilon y_1 + \varepsilon^2 y_2 + O(\varepsilon^3).$$

Thus the system can be re-written as

$$\begin{cases} \sum_{i=0}^{\infty} y_i'' \varepsilon^{i+1} + \sum_{i=0}^{\infty} y_i' \varepsilon^i + \sum_{i=0}^{\infty} y_i \varepsilon^i = 0 \\ \sum_{i=0}^{\infty} y_i(0)\varepsilon^i = \alpha \\ \sum_{i=0}^{\infty} y_i(1)\varepsilon^i = \beta \end{cases}$$

Now considering the cases for the different orders of ε. First we consider ε^0, in which the system and its solution becomes

$$\begin{cases} y_0' + y_0 = 0 \\ y_0(0) = \alpha \\ y_0(1) = \beta \end{cases} \implies \begin{cases} C_1 e^{-x} \\ y_0(0) = \alpha \\ y_0(1) = \beta \end{cases} \implies \begin{cases} C_1 = \alpha \\ C_1 = e^1 \beta \end{cases}.$$

Now unless we have that $\alpha = e^1\beta$ we have that the system is singular. Furthermore if assume this relationship to be true, one can check, that the very next consideration (ε^1) will imply that $\alpha = 0$ and $\beta = 0$. This would mean that the only solution to the system would indeed be the trivial solution

(c) Consider the following notation to simplify the expressions

$$y_{inner} = y_i \qquad y_{outer} = y_o$$

We first start by writing down the outer solution. Since the singularity occurs near $x = 1$ we set $\varepsilon = 0$ and consider the condition near $x = 1$. That is we have and solve

$$\begin{cases} y_o' + y_o = 0 \\ y_o(1) = \beta \end{cases} \implies y_o = \beta e^{1-x}$$

To find the inner approximation we first introduce $\tau = \frac{x}{\delta(\varepsilon)}$ with $Y(\tau) = y_i(x)$ as a scaling. From this we have that

$$\frac{d}{dx} = \frac{d}{d\tau}\frac{d\tau}{dx} = \frac{1}{\delta(\varepsilon)}\frac{d}{d\tau} \implies \frac{d^2}{dx^2} = \frac{1}{\delta^2(\varepsilon)}\frac{d^2}{d\tau^2}.$$

Thus from here we have the system to be

$$\begin{cases} \frac{\varepsilon}{\delta^2(\varepsilon)}Y'' + \frac{1}{\delta(\varepsilon)}Y' + Y = 0 \\ Y(0) = \alpha \\ Y(1) = \beta \end{cases}.$$

From this we have three terms to balance:

$$\frac{\varepsilon}{\delta^2(\varepsilon)} \qquad \frac{1}{\delta(\varepsilon)} \qquad 1$$

Here we can deduce two balances to consider to estimate $\delta(\varepsilon) = \varepsilon^p$ focusing on the inner region.

(i) $\frac{\varepsilon}{\delta^2(\varepsilon)} \sim 1$ in which we need $\delta(\varepsilon) = \frac{1}{2}$. However from this we have the last term $\frac{1}{\delta(\varepsilon)}$ being large.

(ii) $\frac{\varepsilon}{\delta^2(\varepsilon)} \sim \frac{1}{\delta(\varepsilon)}$ in which we need $\delta(\varepsilon) = \varepsilon$. From this we have $1 \ll \frac{1}{\delta(\varepsilon)}$, so this balancing will work.

From the cases above the following system and then solution

$$\begin{cases} \frac{1}{\varepsilon}Y'' + \frac{1}{\varepsilon}Y' + Y = 0 \\ Y(0) = \alpha \\ Y(1) = \beta \end{cases} \implies \begin{cases} Y'' + Y' + \varepsilon Y = 0 \\ Y(0) = \alpha \\ Y(1) = \beta \end{cases}$$

$$\implies \begin{cases} Y = C_1 + C_2 e^{\tau} \\ Y(0) = \alpha \\ Y(1) = \beta \end{cases} \implies Y(\tau) = \alpha + C_2\left(e^{-\tau} - 1\right)$$

$$\implies y(x) = \alpha + C_2\left(e^{-\frac{x}{\varepsilon}} - 1\right)$$

(d) To figure out the matching condition that determines the unknown constants we consider the following.

$$\lim_{\varepsilon\to 0} y_o(\sqrt{\varepsilon}\eta) = \lim_{\varepsilon\to 0} \beta e^{1-\sqrt{\varepsilon}\eta} = \beta e^1$$

$$\lim_{\varepsilon\to 0} y_i(\sqrt{\varepsilon}\eta) = \lim_{\varepsilon\to 0} \alpha + C_2\left(e^{-\frac{\sqrt{\varepsilon}\eta}{\varepsilon}} - 1\right) = \alpha - C_2$$

Now setting the conditions equal to each other we have

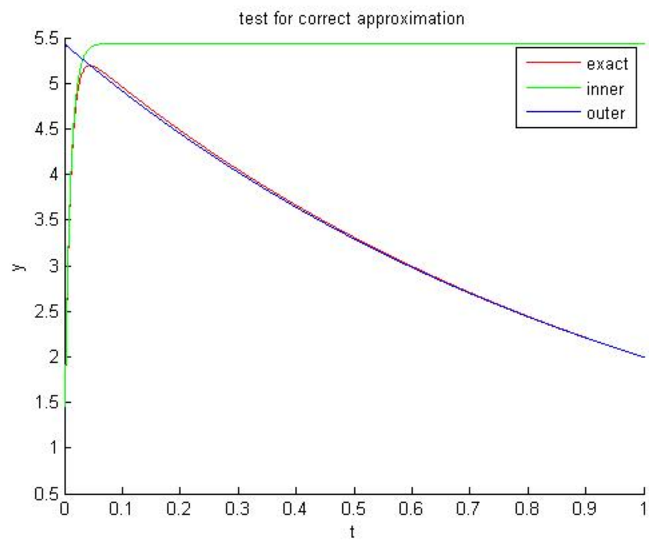
$$C_2 = \alpha - \beta e^1.$$

Since we have in essence constructed the inner and outer approximation we can construct the uniform approximation below

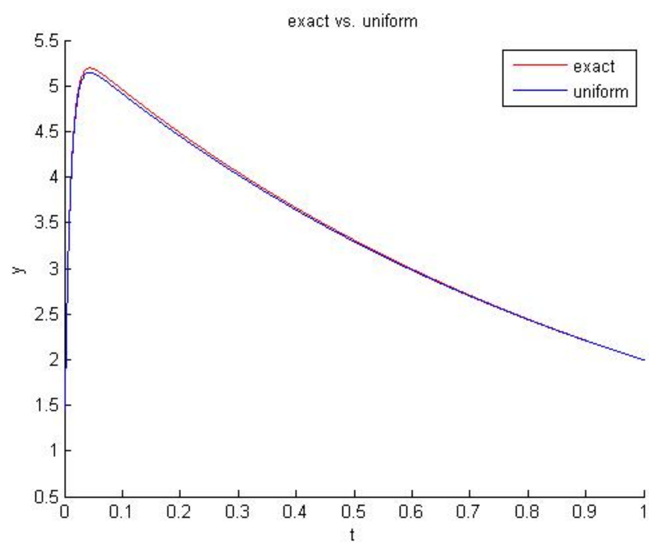
$$\begin{aligned} y_u &= y_i + y_o - \beta e^1 \\ &= \beta e^{1-x} + \alpha + (\alpha - \beta e^1)(e^{-\frac{x}{\varepsilon}} - 1) - \beta e^1 \end{aligned}$$

No simplifying was done due to the programming for the next portion, it just makes it a little simpler to throw in and make some interesting graphs.

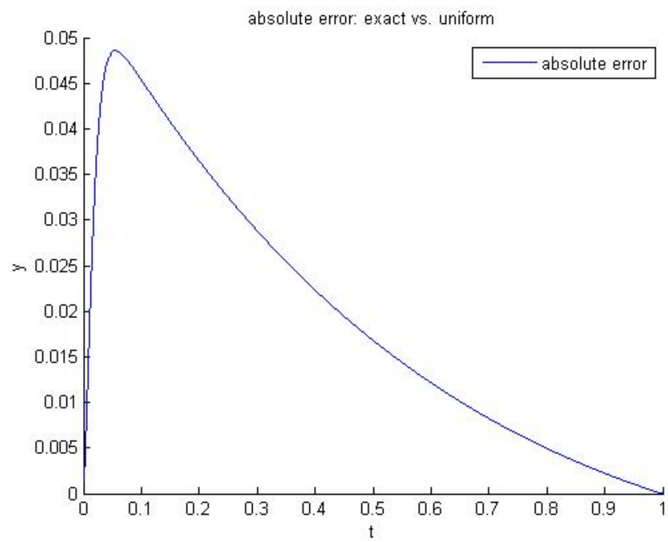
(e) To show that the approximations were good we first can plot them in order to make sure that the approximations are close within each respective region. Here we can arbitrarily choose $\alpha = 1$, $\beta = 2$ and $\varepsilon = .01$. ε was chosen to be large with respect to a realization of the problem so we can clearly see the plots. Decreasing ε in the code will show in increase in accuracy for the approximations. This can be done in a 'for' loop to show different values of ε, but again beyond the scope of the homework.



Here we can see that each approximation does fairly well visually in approximating the exact solution. From this we can move on to looking at the uniform approximation.



From this figure we can see that the uniform also does pretty well, in terms of having such a large ε respectively. However, one way to analyze the approximation is by looking at the absolute error.



From this figure we can see that where the approximation is at its worse is where we have the inner and outer being forced together. The code produced also shows the maximum error which for $\varepsilon = .01$ is 0.0486. The code that produces the graph is located at the appendix for the homework towards the end. The code was produced in order to be understandable for novice MATLAB users, but also have the structure to be adapted into a 'for' or 'while' loop to compare different values of ε for this particular problem.

Problem 3

Consider the boundary layer problem:

$$\text{BVP} = \begin{cases} \varepsilon y'' - y' = 1 & 0 < x < 1 \\ y(0) = \alpha,\ y(1) = \beta & \alpha, \beta > 0 \end{cases}$$

(a) Find the exact solution.

(b) Plot the exact solution.

(c) Argue why we expect a boundary layer close to $x = 1$. (use the plot for guidance, but use the BVP for the argument).

(d) Find the inner and outer approximate solutions, use the transformation $\zeta = \frac{1-x}{\varepsilon^\lambda}$ in the boundary layer and show that $\lambda = 1$ balances the BVP terms in the layer.

(e) Use matching to fully determine constraint(s) on the inner and outer solutions.

Solution:

(a) To find the exact solution we first note that is a 2nd order non-homogeneous boundary value problem. So we have to solve for the homogeneous and non-homogeneous versions of the ODE in the problem. Considering the homogeneous problem

$$\varepsilon y'' - y' = 0,$$

we can note that since ε is just a small number, the ODE has constant coefficients. With that being said we can seek out the characteristic function and solve the characteristic polynomial. That is by letting $\lambda^n = y^{(n)}$ where $y^{(n)}$ is the nth derivative of y with respect to x. Thus we get the following

$$\varepsilon\lambda^2 - \lambda = 0 \implies \lambda(\varepsilon\lambda - 1) = 0.$$

Hence we have that $\lambda = 0, -\frac{1}{\varepsilon}$ being the roots of the characteristic equation. Since they are both real and distinct roots we have the solution to the homogeneous ODE being

$$y_H(x) = C_1 + C_2 e^{\frac{x}{\varepsilon}}.$$

For the particular solution we will simply guess and check our answer. That is, we can guess that the solution to

$$\varepsilon y'' - y' = 1,$$

will take the form $y = Ax$, for some constant A. To solve for A we plug it into the ODE in which we get $-A = 1$ directly. Thus we have

$$y_P(x) = -x.$$

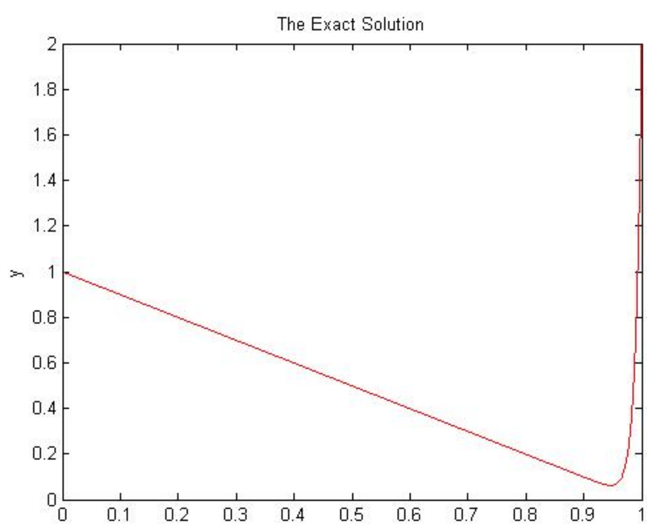
Hence

$$y(x) = y_H(x) + y_P(x) = C_1 + C_2 e^{\frac{x}{\varepsilon}} - x.$$

Now applying the boundary conditions to solve for the unknown coefficients we have that

$$C_1 = \frac{\alpha e^{\frac{1}{\varepsilon}} - \beta - 1}{e^{\frac{1}{\varepsilon}} - 1} \qquad C_2 = \frac{\beta - \alpha + 1}{e^{\frac{1}{\varepsilon}} - 1}$$

(b) The plot of the exact solution is plotted below.



Since the MATLAB code from Problem 2 can be used to produce the same graphs that show the inner and outer approximation, uniform, and absolute error simply by changing the exact, approximation, and matching constant it was done so. The code is included in the Appendix, however the graphs where not produced on this homework to save some space.

(c) If we look at the ODE in our boundary value problem, we can note that by setting $\varepsilon = 0$ we have

$$-y' = 1 \implies y = -x + P.$$

We note that this solution drops linearly, and at if $P > 1$, then it will not be able to pass through $\beta > 0$. In any event, we need to set

$$\alpha = y(0) = P$$

Hence, $y(0))$ can be satisfied by $P = \alpha$.

We reiterate that the assumption is that $\alpha, \beta > 0$. At the other end, at $x = 1$, the constraint $(y(1))$ will not be satisfied unless we restrict $\beta = -1 + \alpha$. Since we have an issue using $y(1) = \beta$ we can argue that there will be a boundary layer near $x = 1$: if $beta > -1 + \alpha$ there is consistency. If $beta \gg -1 + \alpha$ the presumed exact solution to the problem must develop very large derivative in the neighborhood of $x = 1$ in order to pass through β. We thus have enough evidence to suspect that the boundary layer, if present, will be on the $x = 1$.

(d) Note that we will be using the following notation

$$y_{inner} = y_i \qquad\qquad y_{outer} = y_o,$$

to shorten the subscripts dealing with the inner and outer approximation. Considering y_o first, we set $\varepsilon = 0$ in which we are dealing with

$$-y_o' = 1 \qquad\qquad y_o(0) = \alpha.$$

Solving this by integration we have that $y_o = -x + P_1$. Then applying $y_o(0) = \alpha$, we attain

$$y_o = -x + \alpha.$$

Now considering the inner approximation, we consider the transformation given in the problem. With this transformation and the chain rule we have that

$$\frac{d}{dx} = \frac{d}{d\zeta}\frac{d\zeta}{dx} = -\frac{1}{\varepsilon^{\lambda}}\frac{d}{d\zeta} \implies \frac{d^2}{dx^2} = \frac{1}{\varepsilon^{2\lambda}}\frac{d^2}{d\zeta^2}.$$

Thus we have that the problem with $Y(\zeta) = y_i(x)$ and $(\cdot)' = \frac{d}{d\zeta}$ becomes

$$\frac{\varepsilon}{\varepsilon^{2\lambda}}Y'' + \frac{1}{\varepsilon^{\lambda}}Y' = 1.$$

Here we see that we have two balancing terms and make the following arguments.

(i) If we want $\frac{\varepsilon}{\varepsilon^{2\lambda}}$ and 1 to be dominant, it implies that $\lambda = \frac{1}{2}$. However with this implication we have the other term $\frac{1}{\varepsilon^{\lambda}} = \frac{1}{\varepsilon^{\frac{1}{2}}}$ to be dominant. So this balancing will not work.

(ii) If we want $\frac{\varepsilon}{\varepsilon^{2\lambda}}$ and $\frac{1}{\varepsilon^{\lambda}}$ to be dominant, it implies that $\lambda = 1$. With this λ value we see that the other term, 1, is not dominant. In other words this balancing works.

Now that we have $\lambda = 1$, the ODE to be given by

$$\frac{1}{\varepsilon}Y'' + \frac{1}{\varepsilon}Y' = 1 \implies Y'' + Y' = \varepsilon.$$

Solving the ODE, we use the same strategy from part (a). Note that for $y(1) = \beta$, we will have $Y(0) = \beta$ since we are using the designated scaling. Solving the system now we have

$$y_i = Q_1 + Q_2 e^{-\zeta}$$

(e) Using matching to fuller determine constant(s) in the inner solution (since the outer was already solved for), we first use $y(1) = \beta$ to put y_i in terms of one unknown constant. Using this boundary value we have that

$$Q_2 = \beta + Q_1 \implies y_i = \beta - Q_1 + Q_1 e^{-\frac{1-x}{\varepsilon}}.$$

Now trying to match y_i and y_o we have the following limits to consider with $\eta = \frac{1-x}{\sqrt{\varepsilon}}$

$$\lim_{\varepsilon \to 0^+} y_o(\eta) = \lim_{\varepsilon \to 0^+} \alpha + \sqrt{\varepsilon} - 1 = \alpha - 1$$
$$\lim_{\varepsilon \to 0^+} y_i(\eta) = \beta - Q_1$$

Thus we have that $\alpha - 1 = \beta - Q_1$, solving for Q_1 we get $Q_1 = \beta - \alpha + 1$.

Appendix

Problem 2

```
clear
clc
%parameters
a = 1;
b = 2;
e = .01;
t = [0:.00001:1];
%building coefficients
M1 = (-1+sqrt(1-4*e))/(2*e);
M2 = (-1-sqrt(1-4*e))/(2*e);
C1 = (a*exp(M2)-b)/(exp(M2)-exp(M1));
C2 = (b-a*exp(M1))/(exp(M2)-exp(M1));
%solutions
yexact = C1*exp(M1*t)+C2*exp(M2*t);
```

```
youter = b*exp(1-t);
yinner = b*exp(1)+(a-b*exp(1))*exp(-t./e);
%plots to test solutions before making uniform
figure
hold on
plot(t,yexact,'r')
plot(t,yinner,'g')
plot(t,youter,'b')
title ('test for correct approximation')
ylabel ('y')
xlabel ('t')
legend ('exact','inner','outer')
%plot with solution and uniform
yuniform = youter + yinner -b*exp(1);
figure
hold on
plot(t,yexact,'r')
plot(t,yuniform,'b')
title ('exact vs. uniform')
xlabel ('t')
ylabel ('y')
legend ('exact','uniform')
%plot absolute error and out put max error
aberr = abs(yexact - yuniform);
max(abs(yexact - yuniform))
figure
hold on
plot(t,aberr)
title ('absolute error: exact vs. uniform')
xlabel ('t')
ylabel ('y')
legend ('absolute error')
```

Problem 3

```
clear
clc
close all
%parameters
a = 1;
b = 2;
e = .01;
x = [0:.0001:1];
%building coefficients
C1 = (a*exp(1/e)-b-1)/(exp(1/e)-1);
C2 = (b-a+1)/(exp(1/e)-1);
Q1 = b-a+1;
%solutions
yexact = C1+C2*exp(x./e)-x;
youter = a-x;
yinner = b-Q1+Q1*exp(-(1-x)./e);
%plot for (a)
figure
```

```
plot(x,yexact,'r')
title ('The Exact Solution')
xlabel ('x')
ylabel ('y')
%plots to test solutions before making uniform
figure
hold on
plot(x,yexact,'r');
plot(x,youter,'b');
plot(x,yinner,'g');
title ('test for correct approximation')
ylabel ('y')
xlabel ('x')
legend ('exact','outer','inner')
%plot with solution and uniform
yuniform = youter + yinner -a+1;
figure
hold on
plot(x,yexact,'r')
plot(x,yuniform,'b')
title ('exact vs. uniform')
xlabel ('x')
ylabel ('y')
legend ('exact','uniform')
%plot absolute error and out put max error
aberr = abs(yexact - yuniform);
max(abs(yexact - yuniform));
figure
hold on
plot(x,aberr)
title ('absolute error: exact vs. uniform')
xlabel ('x')
ylabel ('y')
legend ('absolute error')
```